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TEMPORAL BEHAVIOR OF QUANTUM MECHANICAL SYSTEMS

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The temporal behavior of quantum mechanical systems is reviewed. We mainly focus our attention on the time development of the so-called "survival" probability of those systems that are initially prepared in eigenstates of the unperturbed Hamiltonian, by assuming that the latter has a continuous spectrum.

The exponential decay of the survival probability, familiar, for example, in radioactive decay phenomena, is representative of a purely probabilistic character of the system under consideration and is naturally expected to lead to a master equation. This behavior, however, can be found only at intermediate times, for deviations from it exist both at short and long times and can have significant consequences.

After a short introduction to the long history of the research on the temporal behavior of such quantum mechanical systems, the short-time behavior and its controversial consequences when it is combined with von Neumann's projection postulate in quantum measurement theory are critically overviewed from a dynamical point of view. We also discuss the so-called quantum Zeno effect from this standpoint.

The behavior of the survival amplitude is then scrutinized by investigating the analytic properties of its Fourier and Laplace transforms. The analytic property that there is no singularity except a branch cut running along the real energy axis in the first Riemannian sheet is an important reflection of the time-reversal invariance of the dynamics governing the whole process. It is shown that the exponential behavior is due to the presence of a simple pole in the second Riemannian sheet, while the contribution of the branch point yields a power behavior for the amplitude. The exponential decay form is cancelled at short times and dominated at very long times by the branch-point contributions, which give a Gaussian behavior for the former and a power behavior for the latter.

In order to realize the exponential law in quantum theory, it is essential to take into account a certain kind of macroscopic nature of the total system, since the exponential behavior is regarded as a manifestation of a complete loss of coherence of the quantum subsystem under consideration. In this respect, a few attempts at extracting the exponential decay form on the basis of quantum theory, aiming at the master equation, are briefly reviewed, including van Hove's pioneering work and his well-known "λ^2T" limit.

In the attempt to further clarify the mechanism of the appearance of a purely probabilistic behavior without resort to any approximation, a solvable dynamical model is presented and extensively studied. The model describes an ultrarelativistic particle interacting with N two-level systems (called "spins") and is shown to exhibit an exponential behavior at all times in the weak-coupling, macroscopic limit. Furthermore, it is shown that the model can even reproduce the short-time Gaussian behavior followed by the exponential law when an appropriate initial state is chosen. The analysis is exact and no approximation is involved. An interpretation for the change of the temporal behavior in quantum systems is drawn from the results obtained. Some implications for the quantum measurement problem are also discussed, in particular in connection with dissipation.

1. Introduction and Summary

We know that radioactive decay is well subject to the exponential law, which can easily be understood on the basis of a purely probabilistic argument. We are also familiar with this kind of exponential behavior in a dissipative classical system, like for example a one-particle motion characterized by *friction* in an atmosphere having a huge number of degrees of freedom. However, an unstable nucleus in a radioactive material is usually in an excited quantum mechanical state, whose behavior is governed not by classical dynamics but by quantum mechanics. In this paper, we review and examine what kind of temporal behavior should take place, and up to what extent we can observe the exponential decay in an unstable quantum mechanical system.

Let us first consider a classical system composed of many particles, in which we are interested in a single-mode behavior (for example, a one-particle motion). On a fine time scale, whose unit is much smaller than the collision time, we should observe a purely dynamical motion governed by the many-body Newton equations. On the other hand, if we observe the phenomenon on a macroscopic time scale, whose unit is much longer than the typical collision time, we can usually decompose the whole motion into two parts, the first being a dissipative systematic motion, and the second a fluctuating one, driven by a random force with a vanishing local time average. Here we are considering the local time average, defined by an averaging procedure over a local time region which is larger than the microscopic relaxation time and smaller than the macroscopic time constant.

In this view, we are dealing with the original many-body problem as a phenomenological stochastic process of a single-mode variable (for example, a one-particle motion), which is described by the so-called *Langevin equation*. The systematic part, i.e. the local time average of the whole motion, is usually subject to the exponential law, which is characterized by a *dissipation* constant (like the *friction* constant in a one-particle motion). We can derive this description from the many-body Newton equations with the help of a coarse-graining procedure. When the number of dynamical degrees of freedom of the atmosphere is infinitely large, we can obtain the Gaussian distribution of the random force through the central limit theorem, and then we can replace, by means of a sort of *ergodic theorem*, the local time average with a statistical ensemble average for the thermal equilibrium with a certain temperature T.

On the other hand, when the microscopic relaxation time is negligibly small, we can assume that the random force is white and then the stochastic process becomes Markoffian. We usually call this kind of process a *Gauss-Markoffian* one. In this case, we can easily derive the Fokker-Planck equation, which justifies the approach to thermal equilibrium. Usually, the Langevin equation is to be formulated for Gauss-Markoffian processes. In this case, the ensemble average of every physical quantity can be written as a sum of products of two-time correlation functions. Note that we can also ascribe the process driven by a colored noise to a Gauss-Markoffian process, if the colored noise can be derived from a white noise through a simple mathematical transformation.

The situation with quantum systems is somewhat delicate for many reasons. In order to understand this situation more clearly, let us start our theoretical considerations from a general discussion on the temporal behavior of quantum-mechanical systems. First imagine, for example, that we have a particle system coupled with a field in free space. The case of a many-body system with a finite (but very large) number of degrees of freedom will be discussed later.

Many years ago, Weisskopf and Wigner[1] first gave a simple formulation of the quantum theory of unstable systems, leading to the famous exponential law. Breit and Wigner[2] introduced an expression for the quantum-mechanical wave functions or S-matrix elements. Such an expression, which is analytic with respect to the *energy* variable conjugated to *time*, is equivalent to the exponential law. Even though these formulas can work well for phenomenological purposes, the theoretical procedure they gave was unfortunately not rigorous. Another important analysis yielding the exponential law was given by Gamow in his seminal paper on the quantum tunnelling problem.[3]

From the point of view of the relationship between the decay phenomena and the energy-time uncertainty relation, Fock and Krylov[4] claimed that the exponential decay could not be theoretically accepted. Later, Khalfin[5] confirmed this argument on the basis of a mathematical theorem given by Paley and Wiener.[6]

In 1953 Hellund[7] strongly claimed that the long-time behavior is not subject to the exponential law, but rather to a power law. By improving this conclusion, Namiki and Mugibayashi[8] showed that the quantum decay should be subject to the following three-step behavior: the Gaussian law at short times, the exponential law at intermediate times, and finally the power law at longer times. Later Araki, Munakata, Kawaguchi and Goto,[9] by making use of the Lee model,[10] remarked that the exponential decay stems from simple poles located on the second Riemannian sheet of the analytic expression of the relevant S-matrix element. A similar but more thorough argument was also given by Schwinger.[11] The quantum decay problem was also considered and reviewed by Fonda, Ghirardi and Rimini,[12] and was recently discussed by Cho, Kasari and Yamaguchi[13,14] by making use of a new solvable model of particle decay. Chiu, Sudarshan and Bhamathi[15] and Horwitz[16] presented other solvable models by extending the Lee model. One of the points of great interest in

Refs. 9 and 15 was the field-theoretical renormalization procedure. Throughout the present paper, however, we shall not enter into the renormalization problem.

As for a general overview of the analytical expressions, we can refer to early work on the damping theory initiated by Heitler:[17] See, for example, the paper by Arnous and Zienau.[18] One should remember that the time-reversal invariance of the whole system is reflected in the fact that the S-matrix element has no singularity on the first Riemannian sheet, except the branch cut running along the real energy axis.

The above-mentioned temporal behavior of unstable quantum systems is closely related to the so-called quantum Zeno paradox or quantum Zeno effect in the quantum measurement problem. The quantum Zeno paradox (QZP), named after the famous Greek philosopher Zeno, states that an unstable quantum system becomes stable (i.e. never decays) in the limit of infinitely frequent measurements. Of course, in practice, we cannot observe this very limit,[a] but can only investigate the quantum Zeno effect (QZE), i.e. a milder version of the QZP, stating that the probability of finding the initial state is increased by a (finite) number of repetitions of a measurement. We are now simply considering the notion of quantum measurement in terms of the naive notion of wave function collapse (WFC), namely a simple projection onto the initial state.[20–22]

The seminal idea of QZP or QZE was introduced under the assumption that the Gaussian short-time behavior can be observed and utilized in the relevant quantum decay and only the naive WFC takes place in quantum measurements.[23–26] In this context, one might think that the observation of this kind of phenomenon is a clear-cut experimental evidence in support of the naive WFC. Usually, the experimental observation of the Gaussian short-time behavior in a decay process is so difficult to perform that Cook[27] proposed to use atomic transitions of the oscillatory type, and inspired an important experiment[28] and then an interesting debate.[29,30] As was discussed in the latter papers, we are led to the conclusion that the experimental observation of the QZE does *not* necessarily support the naive WFC. In order to explain the situation in detail, we have to examine one of the central questions in the quantum measurement problem: What is the wave function collapse? In this context, the issue of the temporal development of quantum systems, by which we have obtained the idea of QZP or QZE, is closely related to the quantum measurement problem. Two of the present authors have already formulated a reasonable theory of quantum measurements without resorting to the naive WFC.[31,32] In this paper, however, we shall not enter into this kind of problem.

Let us now consider and discuss a dynamical system composed of a finite number of particles put in a finite box. The energy spectrum of this many-body system is discrete, and correspondingly, the elastic collision S-matrix element has a series of simple poles running from the minimum energy (say, E_g) to infinity along the real axis on the complex E plane. As the number of degrees of freedom and the box size

[a]Even in principle, this limit cannot be observed, because of the uncertainty principle. See Ref. 19.

become very (infinitely) large in an appropriate way, the energy spectrum becomes asymptotically continuous and the series of simple poles approaches a branch cut running from E_g to infinity along the real axis, just like in the field-theoretical case mentioned above. This is a result of the asymptotic procedure, supplemented by an appropriate coarse-graining procedure, that endowes the off-diagonal matrix elements of the interaction Hamiltonian with random phases and then provokes the appearance of the so-called *diagonal singularity*.[33] A similar procedure was used in the theory of nuclear reactions.[34] As a consequence, the temporal behavior of the system is shown to be classified in three steps, the first being Gaussian, the second exponential, the third power-like. Usually, the Gaussian period is very short and the power decay is very weak, so that the exponential decay seems to dominate over the whole process. If we perform a sort of time-scale transformation in the weak-coupling case, following van Hove's procedure,[33] only the exponential decay part remains in the whole process. Being left with the exponential decay only, we can regard the whole process as a probabilistic one. Within the theoretical scheme given by van Hove, in fact, we can derive the master equation, which is a manifestation of irreversible processes, from the quantum-mechanical Schrödinger equation.[33]

At the beginning of this introductory section, we gave an outline of the derivation of the so-called Langevin equation from the many-body Newton equation in a classical system. Even in a quantum system, we can follow a similar kind of procedure by deriving a quantum Langevin equation from the basic quantum mechanical Schrödinger or Heisenberg equations. Strictly speaking, however, we have to notice that all observables in quantum mechanics are represented by operators, and their temporal evolution should obey an operator-valued Heisenberg equation as a first principle. Therefore, we should first naturally derive an operator-valued Langevin equation. When and only when the operator nature can be neglected, we can talk about a c-number Langevin equation as an approximation. Even in this case, the random force never rigorously satisfies the Gauss-Markoffian property, so that we cannot calculate all the expectation values of the physical quantities by means of a sum of products of two-point correlation functions. When and only when the colored property of the quantum random force, and consequently all the important quantum properties, can be neglected, we can use a c-number quantum Langevin equation, which can be compared with the classical one.

If we want to formulate a quantum Langevin equation rigorously, we also have to take into account the quantum properties of spin statistics, i.e. the well-known distinction between fermions and bosons. In order to do this, we should first formulate an operator-valued Langevin equation for quantum fields, as was already proposed by Mizutani, Muroya and Namiki.[35] More details will be published in a forthcoming paper.

It is true that actual physical systems are so complicated that we cannot easily obtain exact solutions of the fundamental equation. In this context, it is meaningful to examine the temporal evolution of quantum systems by means of exactly solvable models. In this paper, we shall describe in detail some attempts of this kind.[36-38] This is one of the important purposes of the present paper.

We organize this paper in the following way. In Sec. 2, the short-time behavior of quantum mechanical systems is derived, and its relations with the quantum Zeno paradox and quantum Zeno effect are critically discussed. After a short introduction to the classical exponential law and the quantum mechanical deviations from it at short times, the difference between QZE and QZP is explained in Sec. 2.1. The seminal formulation of the paradox is reviewed in Sec. 2.2, and an explicit example is considered in Sec. 2.3, where it is also shown that both QZE and QZP can be given a purely dynamical explanation. It is then shown in Sec. 2.4 that the QZP is only a mathematical idealization and is physically unattainable; however, its milder version, i.e. the quantum Zeno effect (QZE), can be observed in practice. Section 3 deals with the long-time behavior of quantum systems. There, the general expression for the survival probability amplitude is given for quantum systems with a huge (ideally infinite) number of degrees of freedom, where the so-called random-phase approximation seems most plausible. We see that the Fourier or Laplace transforms of the amplitude can clarify its analytical structure, which in turn determines its temporal behavior. The following subsections aim at giving a deeper insight into the profound connection between the analytic structure and the time dependence of the amplitude. We also confirm the general statements on the basis of a concrete, but still rather general Hamiltonian. After a naive (textbook) derivation of the the short-time Gaussian behavior and the long-time exponential law, a counter theorem claiming the unattainability of the exponential form at very long times is introduced in Sec. 3.1, and the analytical structure of the survival amplitude is reinvestigated carefully and explicitly in Sec. 3.2. It is shown that the exponential behavior is due to a simple pole located on the second Riemannian sheet, and turns out to be dominated by a power decay at very long times. These exponential and power-like behaviors are a manifestation of the analytic property of the amplitude and the role played by the branch cut in the complex energy plane is further analyzed in Sec. 3.3 on the basis of a concrete Hamiltonian. Section 4 is devoted to an explicit demonstration of the exponential decay form on the basis of a solvable dynamical model. The model is solved exactly and a propagator, representative of the survival amplitude in this case, shows the exponential form at all times in the weak-coupling macroscopic limit. The temporal behavior is shown to be sensitive to the choice of the initial state and the short-time Gaussian as well as the following exponential behaviors are derived in the case of a wave-packet initial state. A possible relation between the notions of dissipation and decoherence is mentioned, in relation to the quantum measurement problem. The last section, Sec. 5, is devoted to conclusions and additional comments.

2. Short Time Behavior and Quantum Zeno Effect

We shall start by giving a brief outline of the problem. In classical physics, an expression for the decay probability of an unstable system is easily obtained by a heuristic approach (see, for example, Ref. 3): One assumes that there is a *decay probability per unit time* Γ that the system will decay according to a certain specific process. Such a probability per unit time is constant, and does not depend, for instance, on the total number N of unstable systems, on their past history, or on the environment surrounding them. If the number of systems at time t is $N(t)$ (a very large number), the number of systems that will decay in the time interval dt is

$$-dN = N\Gamma dt \quad \text{or} \quad \frac{dN}{dt} = -\Gamma N\,, \tag{2.1}$$

which yields

$$N(t) = N_0 e^{-\Gamma t}\,, \tag{2.2}$$

where $N_0 = N(0)$. One defines then the "survival" or "nondecay probability"

$$P(t) = \frac{N(t)}{N_0} = e^{-\Gamma t}\,. \tag{2.3}$$

The (positive) quantity Γ is interpreted as the inverse of the "lifetime" τ. Notice the short-time expansion

$$P(t) = 1 - \Gamma t + \cdots\,. \tag{2.4}$$

Even though the above derivation can be found in elementary textbooks at undergraduate level, the assumptions underpinning it are delicate. The reader will recognize the essential basic features of a Markoffian process, in which memory effects are absent. More to this, one is excluding *a priori* the possibility that cooperative effects take place, making Γ and P environment-dependent. It is not surprising, then, that the resultant solution (2.3) exhibits all the basic ingredients of a dissipative behavior.

Let us now turn to a quantum mechanical description of the phenomenon. Let $|a\rangle$ be the wave function of a quantum system Q at time $t = 0$. The evolution of Q is governed by the unitary operator $U(t) = \exp(-iHt)$, where H is the Hamiltonian. We define the survival or nondecay probability at time t as the square modulus of the survival amplitude

$$P(t) = |\langle a|e^{-iHt}|a\rangle|^2\,. \tag{2.5}$$

A naive and elementary expansion at short times yields

$$\begin{aligned} P(t) &= 1 - t^2(\langle a|H^2|a\rangle - (\langle a|H|a\rangle)^2) + \cdots \\ &\equiv 1 - t^2(\Delta H)^2 + \cdots\,, \end{aligned} \tag{2.6}$$

which is quadratic in t and therefore yields a vanishing decay rate for $t \to 0$. Observe that we are implicitly assuming that $|a\rangle$ be normalizable and $(\Delta H)^2$ nonvanishing, or, in other words, that $|a\rangle$ is not an eigenstate of H. (Note that one simply gets $P(t) = 1$ if $|a\rangle$ is an eigenstate of H.) We also assumed that all moments of H in the state $|a\rangle$ be finite. Under these assumptions, we can easily infer that the survival probability at short times is of the Gaussian type:

$$P(t) \simeq \exp\left(-\frac{t^2}{\tau_{\mathrm{G}}^2}\right): \quad \tau_{\mathrm{G}}^{-2} \equiv (\Delta H)^2 . \tag{2.7}$$

The temporal behavior of $P(t)$ is closely related to the famous issue of the time-energy uncertainty relation, which has been a long-standing argument of discussion. In this context, we can refer to old papers, like for instance those by Fock and Krylov[4] and Khalfin.[24] The former claimed that the exponential decay was not theoretically acceptable, and the latter confirmed this conclusion on the basis of a mathematical theorem given by Paley and Wiener.[6]

Needless to say, the above result is in manifest contradiction with the exponential law (2.3) that predicts a constant, nonvanishing initial decay rate Γ. More to this, there are some other important differences between the classical and the quantum mechanical results. First of all, observe that the derivation of (2.5)–(2.7) holds for individual systems, while that of (2.3) makes use of an ensemble made up of a huge number N of identically prepared systems. This is because probabilities are ontological in quantum mechanics, unlike in classical physics. Second, in the quantum case the initial state plays an important role and directly enters in the definition of P, raising subtle questions about the problem of the initial state preparation, as is widely known: See, for example, Refs. 33 and 12. By contrast, in the classical case, the initial state plays no crucial role, and simply characterizes the constant N_0 in (2.2). This is obviously related to the underlying Markoffian assumptions. Finally, notice that the quantum mechanical analysis holds true for any initial state $|a\rangle$, not necessarily unstable.

The requirements that the initial state $|a\rangle$ be normalizable and its energy finite are very important: If these two conditions are not met, the behavior of the survival probability can be different from (2.6) or (2.7). A typical example is the Breit-Wigner spectrum, as was discussed in Refs. 5 and 39. The above two requirements will play an important role in the model we shall study in Sec. 4.

We shall postpone our considerations on the temporal behavior at longer times to Sec. 3. Now we analyze one of the most striking and famous consequences of the quadratic short-time behavior: The quantum Zeno effect.

2.1. *Quantum Zeno paradox and quantum Zeno effect*

Zeno was born in Elea, a southern Italian town not far from Naples, about 490 B.C. At those times, that region was part of the so-called *Magna Graecia*, and was culturally Greek. Zeno was a disciple of Parmenides, the most prominent figure of the so-called Eleatic school of philosophers. Parmenides firmly believed in a unique, entire, total *Truth* ("being"), and could not accept the idea that this Truth could change ("becoming") or be composed of smaller entities ("multiplicity"). His dialogue with Socrates on these matters is one of the most famous debates in the history of Greek philosophy.[40]

Even though Zeno's contribution to the philosophy of "being" were not as important as Parmenides's, the former was an unequaled orator, and is believed to have invented dialectic, that was subsequently so widely used by Socrates. Two arguments given by Zeno in support of Parmenides's philosophical viewpoints are most famous. In the first one, Achilles cannot reach a turtle because when the former has arrived at the position previously occupied by the latter, the turtle has already moved away from it, and so on *ad infinitum*. In the second one, that concerns us more directly, a sped arrow never reaches its target, because at every instant of time, by looking at the arrow, we clearly see that it occupies a definite position in space. At every moment the arrow is therefore immobile, and by summing up so many "immobilities" it is clearly impossible, according to Zeno, to obtain motion. If he lived nowadays, Zeno would probably consider the existence of photographs (in which all moving objects are still) as the best proof in support of his ideas.

In the light of his paradoxical arguments, Zeno can be considered a forerunner of sophism. We know today that the two above-mentioned paradoxes can be solved by infinitesimal calculus. Nevertheless, one should not overlook the fact that Zeno basically aimed at giving very provocative arguments against the concept of "becoming", in order to ridicule the critics who tried to deride the philosophy of "being".

It is somewhat astonishing that a conclusion similar to Zeno's holds in quantum theory: It is indeed possible to exploit the quantum mechanical vanishing decay rate at short times (2.6) in a very interesting way, by slowing down (and eventually halting) the decay process. The essential features of this effect were already known to von Neumann[41] and were investigated by several authors in the past.[23–26] Misra and Sudarshan first named this phenomenon after the Greek philosopher.

We shall first give an elementary derivation of this effect, and present a more rigorous result in the following subsection. Let us start from Eq. (2.6) or (2.7), that we rewrite as

$$P(t) = 1 - \frac{t^2}{\tau_{\mathrm{G}}^2} + \cdots \simeq e^{-t^2/\tau_{\mathrm{G}}^2}\,, \tag{2.8}$$

where τ_{G} is the characteristic time of the Gaussian evolution. Suppose we perform N measurements at equal time intervals t, in order to ascertain whether the system is still in its initial state. After each measurement, the system is "projected" onto the quantum mechanical state representing the result of the measurement, and the evolution starts anew. The total duration of the experiment is $T = Nt$. The probability of observing the initial state at time T, after having performed the N above-mentioned measurements, reads

$$P^{(N)}(T) = [P(t)]^N = [P(T/N)]^N \simeq \left(1 - \frac{1}{\tau_{\mathrm{G}}^2}\left(\frac{T}{N}\right)^2\right)^N \overset{N\,\mathrm{large}}{\sim} e^{-T^2/\tau_{\mathrm{G}}^2 N}. \tag{2.9}$$

Notice that both T and N are finite, in the above. This is the *quantum Zeno effect*: Repeated observations "slow down" the evolution and increase the probability that the system is still in the initial state at time T. In the limit of continuous observation ($N \to \infty$) one obtains the *quantum Zeno paradox*

$$P^{(N)}(T) \simeq \left(1 - \frac{1}{\tau_G^2}\left(\frac{T}{N}\right)^2\right)^N \overset{N\to\infty}{\longrightarrow} 1\,. \tag{2.10}$$

Infinitely frequent observations halt the evolution, and completely "freeze" the initial state of the quantum system. Zeno seems to be right, after all: The quantum "arrow", although sped under the action of the Hamiltonian, does not move, if it is continuously observed.

What is the cause of this peculiar conclusion? Mathematically, the above result is ascribable to the general property

$$P^{(N)}(T) \simeq \left(1 - O\left(\frac{1}{N^2}\right)\right)^N \overset{N\to\infty}{\longrightarrow} 1\,. \tag{2.11}$$

However, the above limit is unphysical, for several reasons. Indeed, notice that there is a profound difference between the N-finite and the N-infinite cases: In order to perform an experiment with N finite one must only overcome practical problems, from the physical point of view. (Of course, this can be a very difficult task, in practice.) On the other hand, the $N \to \infty$ case is *physically unattainable*, and is rather to be regarded as a mathematical limit (although a very interesting one). In this sense, we shall say that the quantum Zeno *effect*, with N finite, becomes a quantum Zeno *paradox* when $N \to \infty$.[30]

It must also be emphasized that in the above analysis the observations (measurements) are *instantaneous*, and this is why it is possible to consider the $N \to \infty$ limit: No time is spent in measurements.[30] This is in line with the Copenhagen school of thought, and can be regarded as a common and general feature of von Neumann-like descriptions of a measurement process. Even though such a picture is often accepted among physicists, it is very misleading, in our opinion, and hides some very important aspects of the problem: Indeed a measurement process, if analyzed (as it should) as a concrete physical process, takes place during a *very long time* on a microscopic scale, although we can regard it as if it happened *instantaneously* on a macroscopic scale.

The physical unrealizability of the above-mentioned limit will be thoroughly discussed in Sec. 2.4, where it will be shown that, as a consequence of the uncertainty relations, the $N \to \infty$ limit turns out to be impossible, *even in principle.*[19]

2.2. *Misra and Sudarshan's formulation of the quantum Zeno paradox*

Let us now look at the seminal formulation of the quantum Zeno paradox as given by Misra and Sudarshan.[26] The derivation of the paradox is entirely based on von Neumann's projection rule and therefore hinges upon the concept of quantum measurement process. The reader should notice the fundamental role played by the projection operator $\mathcal{O}$ in the following, and should remember that its effect on the state of the system is supposed to take place *instantaneously*. We shall argue, in the next subsection, that the quantum Zeno effect and paradox can be derived on a purely dynamical basis, without making use of von Neumann's projections.[30]

Q is our unstable quantum system, whose states belong to the Hilbert space $\mathcal{H}$ and whose evolution is described by the unitary operator $U(t) = \exp(-iHt)$, where H is a semi-bounded Hamiltonian. The initial density matrix of system Q is assumed to be an undecayed state ρ_0, and let $\mathcal{O}$ be the projection operator over the subspace of the undecayed states. By definition,

$$\rho_0 = \mathcal{O}\rho_0\mathcal{O}\,, \quad \mathrm{Tr}[\rho_0\mathcal{O}] = 1\,. \tag{2.12}$$

Assume that we perform a measurement at time t, denoted by the projection operator $\mathcal{O}$, in order to check whether Q is still undecayed. Accordingly, the state of Q changes into

$$\rho_0 \to \rho(t) = \mathcal{O}U(t)\rho_0 U^\dagger(t)\mathcal{O}\,, \tag{2.13}$$

so that the probability of finding the system undecayed is given by

$$P(t) = \mathrm{Tr}[U(t)\rho_0 U^\dagger(t)\mathcal{O}]\,. \tag{2.14}$$

The process (2.13) will be referred to as "naive wave function collapse". As already stressed, the observations (measurements) schematized via the operator $\mathcal{O}$ take place *instantaneously*.

The formulation of the QZP proceeds as follows: We prepare Q in the initial state ρ_0 at time 0 (this is formally accomplished by performing an initial, "preparatory" measurement of $\mathcal{O}$) and perform a series of observations at times $T/N, 2T/N, \ldots, (N-1)T/N, T$. The state $\rho^{(N)}(T)$ of Q after the preparation and the above-mentioned N measurements reads

$$\rho^{(N)}(T) = V_N(T)\rho_0 V_N^\dagger(T)\,, \quad V_N(T) \equiv [\mathcal{O}U(T/N)\mathcal{O}]^N \tag{2.15}$$

and the probability of finding the system undecayed is given by

$$P^{(N)}(t) = \mathrm{Tr}[V_N(T)\rho_0 V_N^\dagger(T)]\,. \tag{2.16}$$

So far, the derivation is straightforward[b] and does not involve, in our opinion, any new concept. By contrast, much more delicate is the set of assumptions leading to the "paradox": One considers the so-called limit of continuous observation, in which $N \to \infty$, and defines

$$\mathcal{V}(T) \equiv \lim_{N\to\infty} V_N(T)\,, \tag{2.17}$$

[b]Except, of course, that one is led to wonder about the physical meaning of the operator $\mathcal{O}$, and about the reason why a "projection", unlike all other physical phenomena, should occur instantaneously. These questions are left to the reader. The von Neumann formalism was critically analyzed in Refs. 31 and 32.

provided the above limit exists in the strong sense. The final state and the probability of observing the undecayed system read then

$$\tilde{\rho}(T) = \mathcal{V}(T)\rho_0\mathcal{V}^\dagger(T)\,, \tag{2.18}$$

$$\mathcal{P}(T) \equiv \lim_{N\to\infty} P^{(N)}(T) = \mathrm{Tr}[\mathcal{V}(T)\rho_0\mathcal{V}^\dagger(T)]\,. \tag{2.19}$$

Moreover one assumes, on physical grounds, the strong continuity of $\mathcal{V}(t)$ in $t=0$,

$$\lim_{t\to 0^+} \mathcal{V}(t) = \mathcal{O}\,. \tag{2.20}$$

Misra and Sudarshan then proved that under general conditions[c] the operators $\mathcal{V}(T)$ (exist for all real T and) form a semigroup labeled by the time parameter T. Moreover, $\mathcal{V}^\dagger(T) = \mathcal{V}(-T)$, so that $\mathcal{V}^\dagger(T)\mathcal{V}(T) = \mathcal{O}$. This implies, by virtue of Eq. (2.12), that

$$\mathcal{P}(T) = \lim_{N\to\infty} P^{(N)}(T) = \mathrm{Tr}[\rho_0\mathcal{O}] = 1\,. \tag{2.21}$$

If the particle is continuously observed (to check whether it decays or not), it is "frozen" in its initial state and will never be found to decay! This is the essence of the "quantum Zeno paradox".

Two important problems are still open, at this point. First: Is the present technique, that strongly hinges upon the action of projection operators *à la* von Neumann, really *necessary* in order to derive the QZP or the QZE? Second: Is the $N\to\infty$ limit physically sensible?

We shall tackle the first question in Sec. 2.3, where a purely dynamical derivation of the QZP and QZE is presented, that makes use of unitary evolution and does not involve any projection (provided a final measurement is carried out). We shall come back to the second question in Sec. 2.4, where the $N\to\infty$ limit will be shown to be in contradiction with the Heisenberg uncertainty principle.

2.3. *Dynamical quantum Zeno effect: A simple model*

Let us show that it is possible to obtain both the QZE (2.9) or (2.16) and the QZP (2.10) or (2.21) by making use of a purely dynamical process. The general case is treated in Ref. 30. Here, we shall not give a general proof of the above statement, because the best proof is by inspection of an explicit example: Consider the experimental setup sketched in Fig. 1. An incident neutron, travelling along the y-direction, interacts with several identical regions in which there is a static magnetic field B, oriented along the x-direction. We neglect unnecessary complications and describe the interaction by the Hamiltonian $H = \mu B\sigma_1$ (μ being the (modulus of the) neutron magnetic moment, and σ_1 the first Pauli matrix.) Let the initial neutron state be $\rho_0 = \rho_{\uparrow\uparrow} \equiv |\uparrow\rangle\langle\uparrow|$, where $|\uparrow\rangle$ and $|\downarrow\rangle$ are the spin states of the neutron along the z axis, which can be identified with the undecayed and decayed

[c]Some technical requirements are needed for the proof.

states of the previous subsections, respectively. Assume that there are N regions in which B is present and that the interaction between the neutron and the magnetic fields has a total duration $T = Nt$ ($t = \ell/v$, where ℓ is the length of each region where B is present and v the neutron speed). It is then straightforward[30] to show that if T is chosen so as to satisfy the "matching" condition $T = (2m+1)\pi/\omega$, where m is an integer and $\omega = 2\mu B$, the final state and the probability that the neutron spin is found down at time T read respectively

$$\rho(T) = \rho_{\downarrow\downarrow}\,, \tag{2.22}$$

$$P_{\downarrow}(T) = 1\,. \tag{2.23}$$

The experimental setup in Fig. 1(a) is such that if the system is initially prepared in the up state, it will evolve to the down state after time T (π-pulse).

The situation outlined in Fig. 1(b) is very different. The experiment has been modified by inserting at every step a device able to select and detect the down component of the neutron spin. This is accomplished by a magnetic mirror M and a detector D. The magnetic mirror yields a spectral decomposition[42,31,32] by splitting a neutron wave with indefinite spin (a superposed state of up and down spins) into two branch waves (each of which is in a definite spin state along the z axis) and then forwarding the down component to a detector. The action of the magnetic mirror can be compared to the inhomogeneous magnetic field in a typical Stern-Gerlach experiment. It is very important, in connection with the QZE, to bear in mind that the magnetic mirror does *not* destroy the coherence between the two branch waves: Indeed, the two branch waves corresponding to different spin states can be split coherently and brought back to interfere.[43] The global action of

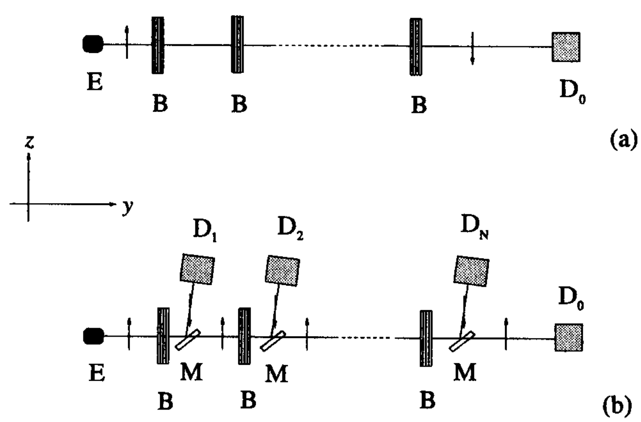


Fig. 1. (a) "Free" evolution of the neutron spin under the action of a magnetic field. An emitter E sends a spin-up neutron through several regions where a magnetic field B is present. The detector D_0 detects a spin-down neutron: no Zeno effect occurs. (b) The neutron spin is "monitored" at every step, by selecting and detecting the spin-down component. D_0 detects a spin-up neutron: the Zeno effect takes place.

M and D can be formally represented by the operator $\mathcal{O} \equiv \rho_{\uparrow\uparrow}$. If the initial Q state and the "matching" condition for $T = Nt$ are the same as before, the density matrix and the probability that the neutron spin is up at time $T = (2m+1)\pi/\omega$ read respectively

$$\rho^{(N)}(T) = V_N(T)\rho_0 V_N^\dagger(T) = \left(\cos^2 \frac{\omega t}{2}\right)^N \rho_{\uparrow\uparrow} = \left(\cos^2 \frac{\pi}{2N}\right)^N \rho_{\uparrow\uparrow}\,, \quad (2.24)$$

$$P_\uparrow^{(N)}(T) = \left(\cos^2 \frac{\pi}{2N}\right)^N . \quad (2.25)$$

This discloses the occurrence of a QZE: Indeed, $P_\uparrow^{(N)}(T) > P_\uparrow^{(N-1)}(T)$ for $N \geq 2$, so that the evolution is "slowed down" as N increases. In the limit of infinitely many observations

$$\rho^{(N)}(T) \stackrel{N\to\infty}{\longrightarrow} \tilde{\rho}(T) = \rho_{\uparrow\uparrow}\,, \quad (2.26)$$

$$P_\uparrow^{(N)}(T) \stackrel{N\to\infty}{\longrightarrow} \mathcal{P}_\uparrow(T) = 1\,. \quad (2.27)$$

Frequent observations "freeze" the neutron spin in its initial state, by inhibiting ($N \geq 2$) and eventually hindering ($N \to \infty$) transitions to other states. Compare Eqs. (2.26) and (2.27) with (2.22) and (2.23): The situation is completely reversed.

It must be observed, however, that it is possible to obtain the *same* result without making use of projection operators, by simply performing a different analysis involving *only unitary processes.* Observe first that the preceding analysis involves only the Q states. If the state of the total (neutron+detectors) system is duely taken into account, the final total density matrix, in the channel representation, reads

$$\tilde{\Xi}_{ij} \equiv \begin{pmatrix} c^{2N} & & & & 0 \\ & s^2c^{2N-2} & & & \\ & & s^2c^{2N-4} & & \\ & & & \ddots & \\ 0 & & & & s^2 \end{pmatrix}, \quad i,j = 0,1,\ldots,N, \quad (2.28)$$

where $c = \cos(\pi/2N)$ and $s = \sin(\pi/2N)$. This corresponds to the case of frequent observations, in which the neutron route was observed at every step. The $i = j = 0$ component corresponds to detection by D_0, while the $i = j = n$ ($n = 1, \ldots, N$) component corresponds to detection in channel $N - n + 1$. Observe that in the above expression the total density matrix $\tilde{\Xi}_{ij}$ has no off-diagonal components as a consequence of the "wave function collapse" by measurement.

Remove now $D_1, \ldots, D_N$ in Fig. 1(b): In other words, *no* observation of the neutron route is carried out (except the final one performed by D_0). A straightforward calculation yields the following final density matrix for the Q system

$$\Xi_{ij} \equiv \begin{pmatrix} c^{2N} & isc^{2N-1} & isc^{2N-2} & \dots & isc^{N} \\ -isc^{2N-1} & s^2c^{2N-2} & s^2c^{2N-3} & \dots & s^2c^{N-1} \\ -isc^{2N-2} & s^2c^{2N-3} & s^2c^{2N-4} & \dots & s^2c^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -isc^{N} & s^2c^{N-1} & s^2c^{N-2} & \dots & s^2 \end{pmatrix}, \quad i,j = 0,1,\dots,N. \tag{2.29}$$

The two expressions above clearly show that we have the *same* probability $P_{\uparrow}^{(N)} = [\cos^2(\pi/2N)]^N$ of detecting a spin-up neutron at D_0 *irrespectively* of the presence of detectors $D_1, \dots, D_N$ in Fig. 1(b). It appears therefore that *no projection rule is necessary* in this context. The quantum Zeno effect can be given a purely dynamical explanation.

If $N \to \infty$, both density matrices tend to the limiting expression

$$\Xi_{ij}^{\infty} \equiv \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad i,j = 0,1,\dots. \tag{2.30}$$

This is the quantum Zeno paradox. It can be obtained both by making use of projection operators *à la* von Neumann and by means of a purely dynamical process. The above conclusions can be generalized to an arbitrary quantum system undergoing a Zeno-type dynamics, and it can even be shown that it is possible to "mimic" the instantaneous action of a projection operator by making use of the impulse approximation in quantum mechanics.[30]

It must also be stressed that an idea analogous to the one described in this subsection was outlined by Peres,[44] who made use of photons, rather than neutrons. His proposal inspired an interesting experiment.[45]

2.4. *The $N \to \infty$ limit and its physical unrealizability*

Let us now discuss the meaning of the $N \to \infty$ limit and show that it is unphysical.[19] We start by observing that the condition $\omega T = \omega N t = (2m+1)\pi$, which is to be met at every step in Fig. 1(b), implies (by setting $m = 1$ for simplicity and without loss of generality)

$$B\ell = \frac{\pi \hbar v}{2\mu N} = O(N^{-1}), \tag{2.31}$$

where all quantities were defined before Eq. (2.22). Obviously, as N increases in the above equation, the *practical* realization of the experiment becomes increasingly difficult. But close scrutiny of Eq. (2.25) shows that $P_{\uparrow}^{(N)}(T)$ cannot tend to 1, even *in principle*, in the $N \to \infty$ limit, because of the uncertainty relations. Indeed, let

ϕ be the argument of the cosine in Eq. (2.25):

$$\phi \equiv \frac{\mu B \ell}{\hbar v} = \frac{\pi}{2N} . \tag{2.32}$$

Mathematically, the above quantity is of order N^{-1}. On the other hand, from a *physical* point of view, it is impossible to avoid uncertainties in the neutron position Δx and speed Δv. As a consequence, ϕ cannot vanish, because it is lower bounded as follows

$$\phi \simeq \phi_0 = \frac{\mu B \ell}{\hbar v_0} > \frac{\mu B \Delta x}{\hbar v_0} > \frac{\mu B}{2 M v_0 \Delta v} , \tag{2.33}$$

where M is the neutron mass and we assumed that the size ℓ of the interaction region (where the neutron spin undergoes a rotation under the action of the magnetic field) is larger than the longitudinal spread Δx of the neutron wave packet. An accurate analysis[19] shows that the same bound holds in the opposite situation ($\ell < \Delta x$) as well.

By defining the magnetic energy gap $\Delta E_m = 2\mu B$ and the kinetic energy spread of the neutron beam $\Delta E_k = \Delta(Mv^2/2)|_{v=v_0}$, the above inequality reads

$$\phi > \frac{1}{4} \frac{\Delta E_m}{\Delta E_k} . \tag{2.34}$$

It is now straightforward to obtain an expression for the value of the probability that a spin-up neutron is observed at D_0 when N is large:

$$P_{\uparrow}^{(N)}(T) \simeq (\cos\phi_0)^{2N} \simeq \left(1 - \frac{1}{2}\phi_0^2\right)^{2N} \simeq \left[1 - \frac{1}{32}\left(\frac{\Delta E_m}{\Delta E_k}\right)^2\right]^{2N} . \tag{2.35}$$

Notice that not only the above quantity does *not* tend to 1, but it *vanishes* in the $N \to \infty$ limit. In other words, in the experiment outlined in Fig. 1(b), *no* spin-up neutron would be observed at D_0 in the $N \to \infty$ limit!

What is reasonable to expect in practice? Even though the analysis of the previous subsections does not take into account the limits imposed by the uncertainty principle, it must be considered that, in practice, N cannot be made arbitrarily large. In order to evaluate how big N can be in order that the QZE be observable in the above experiment, set $P_{\uparrow}^{(N)}(T) \sim 1/2$. We get

$$N \simeq \frac{64 \ln 2}{(\Delta E_m / \Delta E_k)^2} \sim 10^4 , \tag{2.36}$$

where we assumed reasonable values for the energies of a thermal neutron. In conclusion, N turns out to be large enough in order that the QZE be experimentally observable, at least up to a certain approximation.

Criticisms against the physical meaning and realizability of the $N \to \infty$ limit were put forward some years ago by Ghirardi et al.[46] Although different from ours, these criticisms were based on the time-energy uncertainty relations. Our argument, outlined for neutrons, holds true in general, and even in the recent experiment performed with photons in Innsbruck.[45] An exhaustive analysis is given in Ref. 19.

There are in fact other strong arguments against the $N \to \infty$ limit, from a physical point of view. For instance, the above calculation considers only the time spent by the neutron in the magnetic field B. In practice, however, one cannot neglect the time elapsed during the interaction between the neutron and a magnetic mirror M, which is of the order of 10^{-6}–10^{-7} s.

The above discussion should have made it clear that although the short-time behavior is essentially governed by an evolution of the Gaussian type, the limit of continuous observation ($N \to \infty$) should never be lightheartedly taken for granted. It is safer and much more reasonable to start one's considerations from formulas like (2.9), where both T and N are finite.

On the other hand, we shall soon see that there are physical situations in which the temporal behavior of a quantum system can be considered approximately exponential (the mathematical and physical conditions of validity of such an assumption will be thoroughly discussed in the following section). In such a case, the survival probability at time T after N measurements would read

$$P_{\mathrm{E}}^{(N)}(t) \simeq \left(1 - \frac{1}{\tau_{\mathrm{E}}}\left(\frac{T}{N}\right)\right)^{N} \simeq e^{-T/\tau_{\mathrm{E}}}, \tag{2.37}$$

where the subscript $_{\mathrm{E}}$ stands for "exponential", and τ_{E} is the characteristic time of the exponential law. Notice that, unlike in (2.9), in this case the final result *does not strongly depend on* N.

If there exist finite values of T and N such that

$$\frac{T}{\tau_{\mathrm{E}}} \simeq \frac{T^2}{\tau_{\mathrm{G}}^2 N} \quad \Longleftrightarrow \quad \frac{T}{N\tau_{\mathrm{G}}} \simeq \frac{\tau_{\mathrm{G}}}{\tau_{\mathrm{E}}}, \tag{2.38}$$

then the exponential and the Gaussian regions are comparable with each other. Which temporal behavior is actually to be observed depends therefore on the relative magnitude of the parameters τ_{G} and τ_{E}. For this reason we may not completely exclude the possibility of observing a sort of quantum Zeno effect within the framework of the exponential decay. An analogous point was raised by Schulman, Ranfagni and Mugnai,[47] within the context of the WKB approximation.

It should be emphasized that it is very difficult to obtain general estimates of the two characteristic times mentioned above. For example, Khalfin, as well as other authors, even considered and critically discussed the possibility that the proton decay has never been observed because its Gaussian characteristic time $\tau_{\mathrm{G}}^{\mathrm{prot}}$ might be longer than the lifetime of the Universe.[48]

3. Quantum Behavior at Longer Times

In the first part of the preceding section, we have seen that the temporal behavior of the survival probability at short times is well described by a Gaussian form.

Our main task in this section is to investigate the temporal behavior of quantum systems at longer times and to understand the status of the familiar exponential law in quantum theory.

Since an articulate and complete analysis of the temporal behavior of quantum mechanical systems, especially at very long times, is a rather complex issue, we shall organize our discussion in the following way. First, in Sec. 3.1, we consider a particular interaction Hamiltonian and discuss the temporal behavior both at short and long times. Then, in Sec. 3.2, we reconsider our conclusions in the most general case and discuss which problems arise in the very long time region as a consequence of the analytic properties of the survival amplitude. Finally, for the sake of clarity, in Sec. 3.3, we return to the particular case and thoroughly discuss the very long time behavior.

Suppose that the total Hamiltonian is decomposed into two parts, $H = H_0 + H'$, where H_0 and H' are the unperturbed and the interaction Hamiltonians, respectively. Let $|n\rangle$ be an eigenstate of the former belonging to the eigenvalue E_n

$$H_0|n\rangle = E_n|n\rangle, \quad 1 = \sum_n |n\rangle\langle n| . \tag{3.1}$$

Even though we use the discrete notation for notational simplicity, it is to be understood that the possibility of continuous spectrum is included as well. At $t = 0$, the interaction is turned on and the system starts to evolve from an initial state, say $|\alpha\rangle$, into the other states. The initial state is not an eigenstate of H, and is prepared as an appropriate wave packet, that can be represented by a superposition of $|n\rangle$'s. For the sake of simplicity, we shall denote the initial state by an eigenstate of H_0, say $|a\rangle$. The interaction Hamiltonian H' is chosen so that the condition

$$\langle n|H'|n\rangle = 0 \tag{3.2}$$

is satisfied $\forall n$. This ensures that the Lippmann-Schwinger equation

$$|\psi_a\rangle = |a\rangle + \frac{1}{E_a - H_0} H'|\psi_a\rangle \tag{3.3}$$

can be solved. Notice that the condition (3.2) corresponds to mass renormalization in field theories. As already mentioned, we shall start by discussing the temporal behavior of the survival probability amplitude in a simple case.

3.1. *Derivation of temporal behaviors: Standard textbook method*

Here we shall choose a particular interaction Hamiltonian H' subject to the condition that it has nonvanishing matrix elements only between the initial state $|a\rangle$ and the other states $|n\rangle$ $(n \neq a)$:

$$\langle a|H'|a\rangle = 0, \quad \langle n|H'|n'\rangle = 0, \quad \langle a|H'|n\rangle \neq 0, \quad (n,\, n' \neq a) . \tag{3.4}$$

We shall see later that the second relation can be considered as a stronger version of the so-called random phase approximation [see (3.28) below]. Observe that these conditions are peculiar to some solvable models. Historically, the Lee model was based on a similar idea.[10] Nevertheless, the above can still be considered as a rather general form of the interaction Hamiltonian, as can be seen in some standard textbooks.[49]

We are interested in the temporal behavior of the survival probability amplitude $\langle a|U_I(t)|a\rangle$. The evolution operator in the interaction picture $U_I(t)$ satisfies the Schrödinger equation

$$i\frac{d}{dt}U_I(t) = H_I'(t)U_I(t), \quad H_I'(t) = e^{iH_0t}H'e^{-iH_0t}, \tag{3.5}$$

where $H_I'(t)$ is the interaction Hamiltonian in the interaction picture. The solution $U_I(t)$ can be obtained iteratively from the recursion relation

$$\begin{aligned} U_I(t) &= 1 - i\int_0^t dt' H_I(t')U_I(t') \\ &= 1 - i\int_0^t dt' H_I'(t') + (-i)^2\int_0^t dt_1\int_0^{t_1} dt_2 H_I'(t_1)H_I'(t_2)U_I(t_2), \end{aligned} \tag{3.6}$$

under the initial condition $U_I(0) = 1$. Notice that the conditions (3.2) and (3.4) for the interaction Hamiltonian imply

$$\langle a|H_I'(t_1)H_I'(t_2)U_I(t_2)|a\rangle = \langle a|H_I'(t_1)H_I'(t_2)|a\rangle\langle a|U_I(t_2)|a\rangle. \tag{3.7}$$

This relation enables us to obtain a closed equation for the diagonal matrix element of the evolution operator $\langle a|U_I(t)|a\rangle$ (survival amplitude)

$$\langle a|U_I(t)|a\rangle = 1 - \int_0^t dt_1\int_0^{t_1} dt_2 e^{iE_a(t_1-t_2)}f(t_1-t_2)\langle a|U_I(t_2)|a\rangle, \tag{3.8}$$

or an integro-differential equation

$$\frac{d}{dt}\langle a|U_I(t)|a\rangle = -\int_0^t dt_1 e^{iE_a(t-t_1)}f(t-t_1)\langle a|U_I(t_1)|a\rangle \tag{3.9}$$

with the initial condition $\langle a|U_I(0)|a\rangle = 1$. Here the function $f(t)$ is defined as

$$f(t) = \langle a|H'e^{-iH_0t}H'|a\rangle. \tag{3.10}$$

Note that (3.8) is a series of repeated convolutions of $e^{iE_a(t_1-t_2)}f(t_1-t_2)$, and all other higher-order terms disappear, due to the particular conditions (3.4). From this point of view, it is convenient to reduce (3.8) or (3.9) to an algebraic equation by means of a Laplace transform, which is quite appropriate in order to incorporate the initial condition. The solution can now be written as an inverse Laplace transform

$$\langle a|U_I(t)|a\rangle = \frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon}\frac{e^s}{g(s,t)}ds, \tag{3.11}$$

where the function g is given by

$$g(s,t) = s + t\int_0^\infty e^{-\frac{s}{t}u} e^{iE_a u} f(u)du\,. \tag{3.12}$$

(The second term in $g(s,t)$ will turn out to correspond to the self-energy part $\Sigma_a(E)$ in the general case. See the next subsection.)

Having obtained the solution in a closed form, we can discuss its temporal behavior and extract its explicit time dependence both at short and long times. For small t, the function g is expanded around $t = 0$

$$\begin{aligned} g(s,\,t) &= s + t^2\int_0^\infty e^{-su} e^{iE_a tu} f(tu)du \\ &= s + t^2\int_0^\infty e^{-su}\left(f(0) + \frac{d}{dx}\left[e^{iE_a x}f(x)\right]\Big|_{x=0} tu + \cdots\right)du \\ &\simeq s + \frac{f(0)}{s}t^2\,, \end{aligned} \tag{3.13}$$

which determines the behavior at small t

$$\begin{aligned} \langle a|U_I(t)|a\rangle &\simeq \frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} \frac{se^s}{s^2 + f(0)t^2}ds = \cos\sqrt{f(0)}t \\ &\simeq 1 - \frac{f(0)}{2}t^2 \simeq e^{-\frac{f(0)}{2}t^2}\,. \end{aligned} \tag{3.14}$$

Notice that $f(0) = \langle a|H'^2|a\rangle$ is positive definite. The system exhibits the Gaussian behavior for small t, in accordance with the general argument stated in Sec. 2. [Specialize Eq. (3.6) in Sec. 2 to the case in which $|a\rangle$ is an eigenstate of H_0, under the condition (3.2).]

On the other hand, for large t, we can see that the amplitude $\langle a|U_I(t)|a\rangle$ behaves quite differently. In this case, the function g is expanded as a power series of $1/t$

$$\begin{aligned} g(s,t) &= s + t\int_0^\infty e^{-\frac{s}{t}u} e^{iE_a u} f(u)du \\ &= s + t\int_0^\infty \left(1 - \frac{s}{t}u + \cdots\right) e^{iE_a u} f(u)du \\ &\simeq s\left(1 - \int_0^\infty u e^{iE_a u} f(u)du\right) + t\int_0^\infty e^{iE_a u} f(u)du\,. \end{aligned} \tag{3.15}$$

Therefore we obtain the following exponential behavior at longer times

$$\langle a|U_I(t)|a\rangle \sim e^{-\Lambda t}\,, \tag{3.16}$$

as long as the exponent Λ, given by

$$\Lambda = \frac{\displaystyle\int_0^\infty e^{iE_a u} f(u)du}{1 - \displaystyle\int_0^\infty u e^{iE_a u} f(u)du}\,, \tag{3.17}$$

exists. The derivation explained here is similar to that given by Messiah,[49] even though we endeavoured to bridge the gap with the general theory to be sketched in the following subsection. Notice that the above form (3.16) does *not* necessarily mean that the survival probability *decays exponentially.* In order to obtain such a behavior we have to show that the real part of the exponent Λ is positive.

The positivity of the real part of the exponent Λ cannot always be shown, as can be seen in some simple solvable models, where only the oscillating behavior appears at all times. Consider, for example, the Hamiltonian $H = H_0 + H' = \mu B_0 \sigma_3 + \mu B \sigma_1$, where $H_0 = \mu B_0 \sigma_3$ is regarded as the free part and $H' = \mu B \sigma_1$ as the interaction (a similar example was considered in the context of the quantum Zeno effect, in the paper by Inagaki et al.[29]). If we start from the initial state $|a\rangle = |\uparrow\rangle$, the above analysis yields [see (3.10) for the definition of f] $f(t) = (\mu B)^2 e^{i\mu B_0 t}$ and a purely imaginary $\Lambda = i\mu B_0 B^2/(2B_0^2 + B^2)$. As was to be expected, Eq. (3.16) yields an oscillatory behavior at all times and no exponential law can be seen. This is ascribable to the finiteness of the system considered: We cannot expect any exponential behavior if the number of degrees of freedom of the system is finite. As a matter of fact, the numerator appearing in Λ in (3.17) is purely imaginary unless the initial state $|a\rangle$ is continuosly degenerated with respect to energy. We have

$$\begin{aligned}\int_0^\infty e^{iE_a u} f(u)du &= \lim_{\epsilon\to+0} \left\langle a \middle| H' \frac{i}{E_a - H_0 + i\epsilon} H' \middle| a \right\rangle \\ &= \left\langle a \middle| H' \mathrm{P} \frac{i}{E_a - H_0} H' \middle| a \right\rangle + \pi \langle a|H'\delta(E_a - H_0)H'|a\rangle \\ &= i \sum_{E_n \neq E_a} \frac{|\langle n|H'|a\rangle|^2}{E_a - E_n} + \pi \sum_{E_n = E_a, n\neq a} |\langle n|H'|a\rangle|^2 . \end{aligned} \tag{3.18}$$

This expression implies that if $|a\rangle$ is not degenerated the last term gives no contribution, which makes the above quantity and therefore the exponent Λ purely imaginary. The degeneracy of the initial state is necessary for the survival probability to decay exponentially. Remember that we are mainly concerned with a particle system coupled with fields in free space or with an interacting system made up of a particle and a many-body system with a huge number of degrees of freedom, put in a finite but very large box, as was mentioned in the introductory section. For such systems, we naturally expect the continuous energy spectrum to be always degenerated.

It is worth noting that the above expression is a realization of the famous Fermi Golden Rule,[50] according to which the decay rate is given by

$$\Gamma = 2\pi \sum_{E_n = E_a, n \neq a} |\langle n|H'|a\rangle|^2 \tag{3.19}$$

by first-order perturbation theory. We understand that twice the real part of Λ, standing for our decay rate, coincides with the above Γ in the weak-interaction limit (so that the denominator in (3.17) is approximately equal to one). Needless to say, this formula can be improved by including higher-order perturbation terms in a well-known way.

Although the conclusion we have obtained seems reasonable and its derivation sound and standard,[49] it is known that at very long times quantum systems never show the exponential behavior, a fact in contradiction with the above result as well as with our naive expectation. In fact, the unattainability of the exponential law at very long times is a mathematically unavoidable consequence. In order to see this, recall that the evolution operator in the interaction picture $U_I(t)$ is expressed as

$$U_I(t) = e^{iH_0t}e^{-iHt}\,, \tag{3.20}$$

in terms of the total Hamiltonian H. By introducing a complete orthonormal set of eigenstates of the latter, we easily arrive at the following form for the survival probability amplitude

$$\langle a|U_I(t)|a\rangle = e^{iE_at}\int \omega_a(E)e^{-iEt}dE\,, \tag{3.21}$$

where $\omega_a(E)$ is the energy density of the initial state

$$\omega_a(E) \equiv \sum_{E_n=E} |\langle n|a\rangle|^2\,. \tag{3.22}$$

Here the summation is taken over all the quantum numbers, except energy, that are necessary for the specification of a complete orthonormal set. We see that the survival probability amplitude is the Fourier transform of the energy density $\omega_a(E)$, a fact first pointed out by Fock and Krylov.[4] Now let us assume, on physical grounds, that the spectrum of the total Hamiltonian is bounded from below so that the vacuum state is stable: There exists a certain finite energy E_g below which the function $\omega_a(E)$ vanishes. Khalfin showed[5] that this condition on ω_a requires that its Fourier transform (the survival probability amplitude) satisfies the inequality

$$\int_{-\infty}^{\infty} \frac{|\ln|\langle a|U_I(t)|a\rangle||}{1+t^2}dt < \infty\,, \tag{3.23}$$

as a consequence of the fundamental Paley-Wiener Theorem.[6] The inequality (3.23) implies that the survival probability does not decay exponentially: The decay process proceeds more slowly than exponentially at large times. Notice that the only assumption made in the above argument is the existence of a finite E_g: Its very value is irrelevant. Therefore the conclusion is quite general.[d]

At this point, we may feel somewhat puzzled with the above results, especially with that at long times, because we are familiar with the exponential decay law in classical theory and naturally expect the same form to be valid also in quantum

[d] A full account of this theorem, related topics and further references are given in Ref. 51.

theory. Indeed, the exponential law for decay processes has been well confirmed experimentally: No deviation from it at long times has ever been observed. A list of experimental results, examining the exponential law, can be found in Ref. 13. Then how can we reconcile quantum theory with the exponential law?

The above-mentioned theorem (3.23) requires that the amplitude decays more slowly than exponentially in the large-time limit $t \to \infty$. Therefore we may expect it to decay exponentially at intermediate times, even if not at very long times. At any rate, something must have been overlooked in our derivation of (3.16). In the next subsection, we shall reanalyze the temporal behavior in full generality and shall focus our attention on the analytic properties of the amplitude: Indeed, in the above analysis, we have assumed that the order of the integrations over s and u and the $t \to 0$ and $t \to \infty$ limits can be interchanged, which is licit only when the integrals converge uniformly and the initial state is a well-behaved wave packet. This is by no means trivial, and is where subtleties might come into play.

3.2. *General arguments*

Let us analyze in full generality the temporal behavior of the survival probability amplitude $\langle a|U_I(t)|a\rangle$. The evolution operator in the interaction picture $U_I(t)$ is related to its counterpart $U(t)$ in the Heisenberg picture and to the S-matrix in the usual way:

$$U(t) = e^{-iHt} = e^{-iH_0t}U_I(t) \stackrel{t\to\infty}{\longrightarrow} e^{-iH_0t}S, \tag{3.24}$$

where H and H_0 stand for the total and the free Hamiltonians, respectively. It is important to notice that the unitary operator $U(t)$ is expressed as a Fourier transform

$$U(t) = \frac{i}{2\pi}\int_C e^{-iEt}G(E)dE; \quad G(E) = \frac{1}{E-H}, \tag{3.25}$$

and a similar relation holds for its free counterpart $U_0(t)$, with $G_0(E)$.

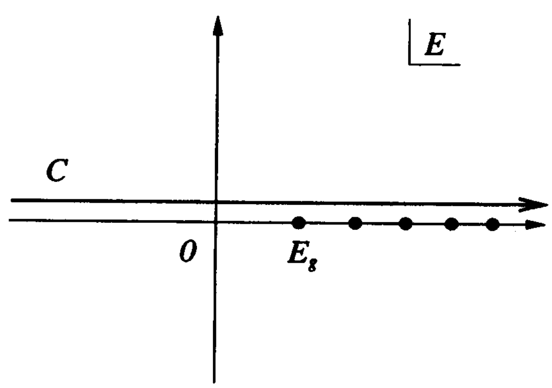


Fig. 2. Singularities of $G(E)$ or $G_0(E)$ for the case of discrete spectrum and the integration contour C for the initial-value problem.

Note that the analytic structure of $G(E)$ $[G_0(E)]$ in the complex E-plane is symmetric with respect to the real E axis, as a reflection of the time-reversal invariance of the whole process. Because of the relation $|e^{-iEt}| = \exp[|E|t\sin(\arg E)]$, we know that $U(t)$ $[U_0(t)]$ vanishes for $t < 0$ if the integration contour C, running from $-\infty$ to ∞ along the real E axis, lies a little above all the singularities of $G(E)$ $[G_0(E)]$. For the initial-value problem (Cauchy problem), therefore, we have to choose the integration contour C in the way shown in Fig. 2.

Let us start our discussion from the case of a system composed of a finite number of particles (say, N) put in a finite box (of volume V). Then H $[H_0]$ has a discrete spectrum and, correspondingly, the singularities of $G(E)$ $[G_0(E)]$ appear as a series of simple poles, running from the lowest-energy point (i.e. the lower bound of the spectrum) to infinity on the real E axis. We denote the lowest-energy point in both cases by E_g, for simplicity. See Fig. 2.

We can now start the perturbation theory on the basis of the following expansion

$$\begin{aligned} G(E) = {} & \frac{1}{E-H_0} + \frac{1}{E-H_0}H'\frac{1}{E-H_0} \\ & + \frac{1}{E-H_0}H'\frac{1}{E-H_0}H'\frac{1}{E-H_0} \\ & + \frac{1}{E-H_0}H'\frac{1}{E-H_0}H'\frac{1}{E-H_0}H'\frac{1}{E-H_0} \\ & + \frac{1}{E-H_0}H'\frac{1}{E-H_0}H'\frac{1}{E-H_0}H'\frac{1}{E-H_0}H'\frac{1}{E-H_0} \\ & + \cdots . \end{aligned} \tag{3.26}$$

The survival probability amplitude (in the Heisenberg picture) is given by[e]

$$\langle a|U(t)|a\rangle = \frac{i}{2\pi}\int_C e^{-iEt}G_a(E)dE\,, \quad G_a(E) \equiv \langle a|G(E)|a\rangle\,. \tag{3.27}$$

We introduce now the plausible assumption that the phases of the off-diagonal matrix elements of the interaction Hamiltonian H' are randomized in the infinite N and V limit (keeping N/V finite). That is,

$$\langle n|H'|n'\rangle \quad \text{have random phases for } n \neq n'\,. \tag{3.28}$$

Note that the discrete energy spectrum becomes continuous in this limit. Consequently, the above-mentioned simple discrete poles of $G_0(E)$ get closer and closer in this limit, and finally merge into a continuous line, distributing over $[E_g, \infty)$ along

[e]The quantity $G_a(E)$ is essentially the same as the energy density $\omega_a(E)$ in (3.22), though the meaning of E is different: The latter E stands for the spectrum of the total Hamiltonian H while the former E is just an integration variable running from $-\infty$ to ∞ in the Fourier transform, irrespectively of the spectrum of H.

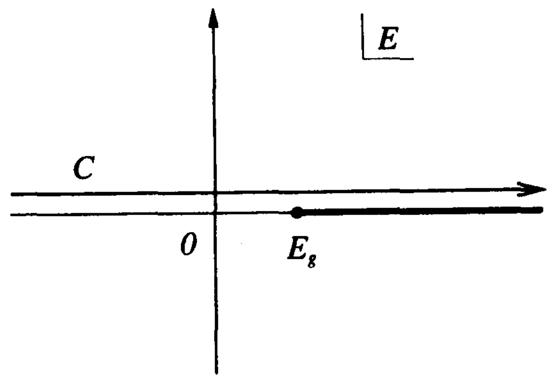


Fig. 3. Singularities of $G_a(E)$ for the continuous spectrum and the integration contour for the initial-value problem. The branch cut reflects the continuous spectrum of the free Hamiltonian and runs along the real E axis.

the real axis, where E_g is the lowest-energy point of the free Hamiltonian. In this case, there is a branch point at E_g and a branch cut on $[E_g, \infty)$ for $G(E)$ and, in particular, for the self-energy part $\Sigma_a(E)$ to be introduced shortly. See Fig. 3.

Under the above random phase assumption, it is easy to show that $G_a(E)$, which is nothing but the Fourier transform of the survival amplitude, is written as

$$\begin{aligned} G_a(E) &= \frac{1}{E-E_a} + \left(\frac{1}{E-E_a}\right)^2 \left\langle a \middle| \left(H' + H'\frac{1}{E-H_0}H' + \cdots\right) \middle| a \right\rangle \\ &= \frac{1}{E-E_a}\left[1 + \left(\frac{\Sigma_a(E)}{E-E_a}\right) + \left(\frac{\Sigma_a(E)}{E-E_a}\right)^2 + \cdots\right] \\ &= \frac{1}{E-E_a-\Sigma_a(E)} \,. \end{aligned} \tag{3.29}$$

The "self-energy part" $\Sigma_a(E)$ is composed of all the even-order contributions

$$\Sigma_a(E) = \Sigma_a^{(2)}(E) + \Sigma_a^{(4)}(E) + \cdots \,, \tag{3.30}$$

where

$$\Sigma_a^{(2)}(E) = \left\langle a \middle| H'\frac{1}{E-H_0}H' \middle| a \right\rangle = \sum_{n\neq a} \frac{|\langle a|H'|n\rangle|^2}{E-E_n} \,, \tag{3.31}$$

$$\begin{aligned} \Sigma_a^{(4)}(E) &= \left\langle a \middle| H'\frac{1}{E-H_0}H'\frac{1}{E-H_0}H'\frac{1}{E-H_0}H' \middle| a \right\rangle - \frac{1}{E-E_a}[\Sigma_a^{(2)}(E)]^2 \\ &= \sum_{n\neq a}\sum_{n'\neq a,n} \frac{|\langle a|H'|n'\rangle|^2 \cdot |\langle n'|H'|n\rangle|^2}{(E-E_n)(E-E_{n'})^2} \,, \end{aligned} \tag{3.32}$$

and so on. Observe that these quantities correspond to the "proper" self-energy parts in field theory (the "improper" parts are subtracted at every step of the expansion). In general, we obtain

$$\Sigma_a(E) = \sum_{n \neq a} \frac{1}{E - E_n} \left[|\langle a|H'|n\rangle|^2 + \sum_{n' \neq a,n} \frac{|\langle a|H'|n'\rangle|^2 \cdot |\langle n'|H'|n\rangle|^2}{(E - E_{n'})^2} + \cdots \right]. \tag{3.33}$$

In the infinite N and V limit, of course, the summation in (3.33) becomes the following integral:

$$\Sigma_a(E) = \int_{E_g}^{\infty} dE' \rho_a(E') \frac{1}{E - E'} K_a(E'), \tag{3.34}$$

where $\rho_a(E')$ stands for the state number density and $K_a(E')$ for the quantity inside the square brackets $[\cdots]$ in (3.33). This expression implies that $G_a(E)$ has a branch point at E_g and the corresponding branch cut on the real axis, and has no other singularities on the first Riemannian sheet, while on the second Riemannian sheet possible singularities can appear. See Fig. 3. This procedure, based on the above-mentioned random phase approximation, is the same as that originally introduced in the theory of nuclear reactions.[34] It is also very close to the *diagonal singularity* formulated by van Hove,[33] although we have not yet used his limit concerning the time-scale transformation.

The survival probability amplitude (3.29), with the above $\Sigma_a(E)$, has been derived under the random phase assumption (3.28) for many-body systems having a huge but finite number of degrees of freedom. However, as is well known, it is an exact relation in field-theoretical cases where we have a particle system coupled to fields in free space or two (or more) fields coupled to each other. Of course, in such cases the number of degrees of freedom is infinite. This is the reason why the quantity $\Sigma_a(E)$ has been called the "self-energy part". Investigation of the analytic properties of the amplitude or the relevant S-matrix elements on the basis of quantum field theory can be traced back to the damping theory initiated by Heitler:[17] See the review paper given by Arnous and Zienau.[18] Therefore, we can discuss the temporal evolution of the survival probability amplitude, in the wider framework of field theory and many-body problems, on the basis of the expression we have just obtained.

It is worth pointing out that the familiar Breit-Wigner[2] (or Weisskopf-Wigner[1]) form does neither match nor follow from the above analysis, since the Breit-Wigner spectrum, which is just assumed on a phenomenological basis, extends from $-\infty$ to ∞. Also, notice that if we put $E = E_a$ in $\Sigma_a(E)$ (which is in general a complex number), Eq. (3.29) yields the Weisskopf–Wigner formula[1] of the exponential decay. Of course, this cannot be justified even in the weak-coupling limit, since the exponent should be given by the simple pole of $G(E)$, say $\bar{E}$, satisfying

$$\bar{E} = E_a - \Sigma_a(\bar{E}), \tag{3.35}$$

which is clearly different from the former, $\bar{E} \neq E_a - \Sigma_a(E_a)$.

In order to observe the temporal behavior of the survival probability amplitude at longer times, the original contour C, running along the real E axis from $-\infty$ to ∞, is now deformed into a new contour. Recalling again that $G_a(E)$ has a branch cut along the real E axis from E_g to ∞ and a simple pole on the second Riemannian sheet, we understand that the (E_g, ∞) portion of C is equivalent to the sum of a path running just below the real E axis on the second Riemannian sheet from E_g to $-\infty$ and a circle turning clockwise around the simple pole. The new contour, equivalent to the original one, is thus composed of the path C' and the circle on the second sheet, as shown in Fig. 4. We therefore obtain from Eqs. (3.27), (3.29) and (3.35)

$$\begin{aligned}\langle a|U(t)|a\rangle &= \frac{i}{2\pi}\int_C e^{-iEt}G_a(E)dE \\ &= Ze^{-i\bar{E}t} + \frac{i}{2\pi}\int_{C'} e^{-iEt}G_a(E)dE\,, \end{aligned} \tag{3.36}$$

where C' is the new contour shown in Fig. 4 and $Z^{-1} = [\partial G_a^{-1}(E)/\partial E]_{E=\bar{E}}$. Since the imaginary part of $\bar{E}$ is negative, the first term, representing the contribution from the simple pole on the second Riemannian sheet, yields the exponential decay form. As the decay process proceeds further, the exponential term becomes very small and the last integral on the contour C', given by

$$\frac{i}{2\pi}\int_{-\infty}^{E_g} e^{-iEt}[G_a(E+i\epsilon) - G_a(E-i\epsilon)]dE \tag{3.37}$$

eventually dominates. In the above, the second function in the integrand, $G_a(E-i\epsilon)$, must be evaluated on the second sheet. Note that $[G_a(E+i\epsilon) - G_a(E-i\epsilon)] \propto [\Sigma_a(E+i\epsilon) - \Sigma_a(E-i\epsilon)] \propto \lambda^2$, where λ represents the strength of interaction.

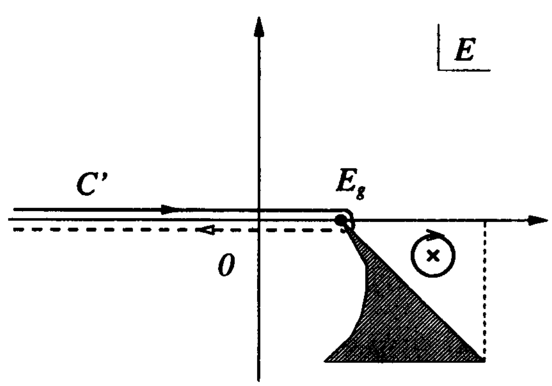


Fig. 4. The deformed integration contour, composed of a path C' and a circle on the second Riemannian sheet. The path C' starts from $-\infty$, runs just above the real axis on the first sheet, goes down to the second sheet at E_g and runs back to $-\infty$ just below the real axis. The circle encloses the simple pole clockwise on the second sheet.

The power law can be derived from this expression, under appropriate assumptions, as an asymptotic formula for very large t. This is in accordance with the general theorem (3.23) stated before. The behavior for large t, however, depends on the details of $\rho_a(E')$ and $K_a(E')$: For this reason, in the next subsection, we shall examine the above asymptotic formula in the particular case discussed in the preceding subsection.

It is important here to mention van Hove's limiting procedure concerning the time-scale transformation. In spite of their crucial role in the derivation of the closed equations (3.8) and (3.9) for the survival probability amplitude in the particular case studied in Sec. 3.1, the conditions (3.4) on the interaction Hamiltonian do not seem quite natural for systems with many (ideally infinite) degrees of freedom (roughly called "macroscopic systems"): The initial state $|a\rangle$ (which is now considered to be a direct product of a quantum subsystem and the remaining part of the whole system), would be given a rather special status if (3.4) were to be satisfied. On the other hand, the random phase approximation (3.28), introduced at the beginning of this subsection, seems more plausible in such a case. Pauli[52] first derived a master equation in the quantum theory of many-body systems along this line of thought.

Almost three decades later, van Hove showed that the repeated use of the random phase approximation, which is so crucial for the derivation of Pauli's master equation, can be avoided by taking the so-called van Hove limit[33]

$$\lambda \to 0 \quad \text{with} \quad \lambda^2 t \quad \text{kept constant}, \tag{3.38}$$

where λ represents the strength of the interaction, as mentioned above. He showed that the survival amplitude of some suitably chosen initial state decays exponentially in this limit, provided that the diagonal-singularity assumption holds. This assumption, which appears reasonable for systems with many degrees of freedom, requires the predominance of the diagonal matrix elements $\langle\alpha|H'H'|\alpha\rangle$, which turns out to be of order N (N representing the number of degrees of freedom of the system), over the remaining off-diagonal terms. Observe that in our terminology, the limit (3.38) is equivalent to making the time constant τ_{G}, defined in Eq. (2.7), and the integral on C' in (3.36) vanish, for the former gives a vanishing contribution $\sim O(e^{-\frac{1}{\lambda^2}}$ and the latter is $\sim O(\lambda^2)t^{-\delta}(\delta > 0)$ [remember Eq. (3.37) and the subsequent discussion]: In conclusion, only the exponential decay, characterized just by the lowest-order contribution $\Sigma_a^{(2)}$, remains as a purely probabilistic law. By means of this procedure, van Hove succeeded in deriving the master equation from the quantum-mechanical Schrödinger equation. It is also important to notice that in order to realize the exponential law, and consequently the master equation, a wave packet must be taken as an initial state. It was shown[53] that consideration of wave packets in quantum mechanical scattering processes is equivalent to the extension of scattering amplitudes to the complex E plane. The weak-coupling limit (3.38) has also been investigated from a mathematical point of view.[54–56]

Finally, before closing our general discussion, it is worth showing another method to derive temporal behaviors, without resorting to Fourier or Laplace transforms. It is not difficult to derive the following differential equation for the survival amplitude

$$
\begin{aligned}
\frac{d}{dt}\langle a|U_I(t)|a\rangle &= -\int_0^t dt_1 \langle a|H_I'(t)H_I'(t_1)|a\rangle\langle a|U_I(t_1)|a\rangle \\
&+ \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \langle a|H_I'(t)H_I'(t_1)H_I'(t_2)H_I'(t_3)|a\rangle_p \langle a|U_I(t_3)|a\rangle \\
&- \int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \int_0^{t_3} dt_4 \int_0^{t_4} dt_5 \\
&\times \langle a|H_I'(t)H_I'(t_1)H_I'(t_2)H_I'(t_3)H_I'(t_4)H_I'(t_5)|a\rangle_p \langle a|U_I(t_5)|a\rangle \\
&+ \cdots , \qquad (3.39)
\end{aligned}
$$

on the basis of the iterative solution of the Schrödinger equation

$$
\begin{aligned}
U_I(t) &= 1 - i\int_0^t dt' H_I(t')U_I(t') \\
&= 1 - i\int_0^t dt' H_I'(t') + (-i)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 H_I'(t_1)H_I'(t_2) + \cdots , \qquad (3.40)
\end{aligned}
$$

under the conditions (3.2) ("mass renormalization") and (3.28) (random phase approximation). Observe that due to these conditions, all terms of odd order give no contribution to the survival amplitude. The "proper" matrix elements $\langle a|\cdots|a\rangle_p$ in (3.39) are defined by ($E_{ab} \equiv E_a - E_b$, etc.)

$$
\begin{aligned}
&\langle a|H_I'(t)H_I'(t_1)H_I'(t_2)H_I'(t_3)|a\rangle_p \\
&\quad \equiv \sum_{n\neq a} \langle a|H_I'(t)H_I'(t_1)|n\rangle\langle n|H_I'(t_2)H_I'(t_3)|a\rangle \\
&\quad = \sum_{n\neq a}\sum_{n'\neq a,n} e^{iE_{an'}(t-t_3)} e^{iE_{n'n}(t_1-t_2)} |\langle a|H'|n'\rangle|^2 \cdot |\langle n'|H'|n\rangle|^2 , \qquad (3.41)
\end{aligned}
$$

$$
\begin{aligned}
&\langle a|H_I'(t)H_I'(t_1)H_I'(t_2)H_I'(t_3)H_I'(t_4)H_I'(t_5)|a\rangle_p \\
&\quad \equiv \sum_{n\neq a}\sum_{n'\neq a} \langle a|H_I'(t)H_I'(t_1)|n\rangle\langle n|H_I'(t_2)H_I'(t_3)|n'\rangle\langle n'|H_I'(t_4)H_I'(t_5)|a\rangle \\
&\quad = \sum_{n\neq a}\sum_{n'\neq a,n}\sum_{n''\neq a,n,n'} e^{iE_{an'}(t-t_5)} e^{iE_{n'n''}(t_1-t_4)} e^{iE_{n''n}(t_2-t_3)} \\
&\quad \times |\langle a|H'|n'\rangle|^2 \cdot |\langle n'|H'|n''\rangle|^2 \cdot |\langle n''|H'|n\rangle|^2 , \qquad (3.42)
\end{aligned}
$$

and so on. It is evident that these terms correspond to the self-energy parts of the 4th order $\Sigma_a^{(4)}(E)$, 6th order $\Sigma_a^{(6)}(E)$ and so on, in Eq. (3.30), in Fourier space.

If "memory effects" are neglected so that the amplitudes $\langle a|U_I(t_i)|a\rangle$ appearing on the right hand side of (3.39) can be evaluated at time t instead of t_i, the differential equation (3.39) is solved to give an exponential form for the survival amplitude[57]

$$\langle a|U_I(t)|a\rangle \simeq e^{-\int_0^t d\tau_1 (t-\tau_1) f_2(\tau_1) + \int_0^t d\tau_1 \int_0^{t-\tau_1} d\tau_2 \int_0^{t-\tau_1-\tau_2} d\tau_3 (t-\tau_1-\tau_2-\tau_3) f_4(\tau_1,\tau_2,\tau_3) + \cdots} . \tag{3.43}$$

The functions in the exponent are given by

$$f_2(\tau_1) = \langle a|H' e^{i(E_a - H_0)\tau_1} H'|a\rangle , \tag{3.44}$$

$$f_4(\tau_1, \tau_2, \tau_3) = \langle a|H' e^{i(E_a - H_0)\tau_1} H' e^{i(E_a - H_0)\tau_2} H' e^{i(E_a - H_0)\tau_3} H'|a\rangle_p , \tag{3.45}$$

and so on. These expressions allow us to extract some of the typical temporal behaviors of the amplitude (except at very large times). To this end, let us recall again that the energy spectrum is asymptotically very dense in many-body systems and becomes continuous in field-theoretical cases so that the summations appearing in (3.41) and (3.42), for example, are to be understood as integrations over energy. Here we consider the following two cases[57]:

- When the matrix elements of the interaction Hamiltonian H' have a relatively wide energy range over which their variations are small, the phase factors appearing in the functions $f_2, f_4, \ldots$ [see (3.44) and (3.45)] oscillate rapidly except for very small τ's. Therefore, after summing over the intermediate states, the functions f_i can be considered to be proportional to δ functions with respect to the integration variables τ's, which results in a linear t behavior

  $$\langle a|U_I(t)|a\rangle \simeq e^{-t[\int_0^\infty d\tau_1 f_2(\tau_1) - \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 \int_0^\infty d\tau_3 f_4(\tau_1,\tau_2,\tau_3) + \cdots]} , \tag{3.46}$$

  provided the integrals in the exponent converge. Observe that the upper limits of integration can be safely taken equal to ∞ under such conditions and therefore the above form becomes a relatively good approximation at large t. The time t has been assumed to be much larger than the average energy spacing.
- On the contrary, if the matrix elements have relatively narrow bands, the phase factors in the functions $f_2, f_4, \ldots$ are safely replaced with unity. This means that we can replace the arguments of f_i in the integrals by zeros and arrive at

  $$\langle a|U_I(t)|a\rangle \simeq e^{-\frac{1}{2} f_2(0) t^2 + \frac{1}{4!} f_4(0,0,0) t^4 + O(t^6)} . \tag{3.47}$$

  Since we have assumed that the time t is small enough so that the above-mentioned replacements are allowed, this approximate form is valid at short times. We infer that the dominant term at short times is of second order in t, in accordance with the argument presented in the preceding section.

It is worth stressing that the behavior at both short and large times has been derived from one and the same expression for the amplitude $\langle a|U_I(t)|a\rangle$ under different physical conditions. At the same time, however, we should keep in mind that the behavior that really dominates in the particular time region under consideration does depend on the relative magnitudes of the Gaussian and the exponential (linear t) decays, as well as on the characteristic time constants of the exponents.

3.3. *Reanalyzing the particular case*

In this subsection, we shall reconsider the particular interaction Hamiltonian of Sec. 3.1 and study in detail the analytic properties of the survival amplitude. This subsection is mainly written in an peolagogical spirit, and therefore some parts of it are a specific version of the general argument presented in Sec. 3.2. Nevertheless, we believe that it is very instructive to explicitly exhibit the temporal behavior at long times, and to perform an analysis in terms of Laplace transforms.

Let us go back to the definition (3.12) of $g(s,t)$ and see how and to what extent the exponential law can be realized. Since the amplitude $\langle a|U_I(t)|a\rangle$ is given by the inverse Laplace transform (3.11), we understand that the exponential decay occurs only when the complex function $g(s,t)$ has zeros in the left-half complex s plane. In order to perform the inverse Laplace transform, the function g has to be analytically continued into the left-half complex s plane. By performing the integration over u in (3.12) we obtain

$$\begin{aligned} g(s,t) &= s + t\left\langle a\left|H'\frac{i}{E_a - H_0 + i\dfrac{s}{t}}H'\right|a\right\rangle \\ &= s + it\int_{C_0}\frac{\sum_r |\langle E_0, r|H'|a\rangle|^2}{E_a - E_0 + i\dfrac{s}{t}}dE_0\,. \end{aligned} \tag{3.48}$$

Here $|E_0, r\rangle$ are eigenstates of H_0, and form a complete orthonormal set, with r being quantum numbers describing possible degeneracies. Observe that the integrand has a simple pole at $E_0 = E_a + i\frac{s}{t}$. Since $t > 0$ and s is taken to have a positive real part (in order to assure the convergence of the integration over t in the Laplace transform), the pole lies above the integration contour C_0 which extends along the real E_0 axis. Therefore in order to extend the function $g(s,t)$ into the left-half complex s plane, where the real part of s is negative, the integration contour should be deformed in the complex E_0 plane so that its relative configuration with respect to the singularity is maintained.[9] See Fig. 5. This gives us the following change for the last term in (3.48)

$$\begin{aligned} it\int_{C_0}\frac{\sum_r |\langle E_0, r|H'|a\rangle|^2}{E_a - E_0 + i\dfrac{s}{t}}dE_0 \longrightarrow\; & it\int_{C_0}\frac{\sum_r |\langle E_0, r|H'|a\rangle|^2}{E_a - E_0 + i\dfrac{s}{t}}dE_0 \\ & + 2\pi t\sum_r |\langle E_0, r|H'|a\rangle|^2\Big|_{E_0 = E_a + i\frac{s}{t}}\,, \end{aligned} \tag{3.49}$$

as the pole moves into the lower-half E_0 plane crossing the real E_0 axis. Notice that the second term of the right hand side, which represents the contribution from the

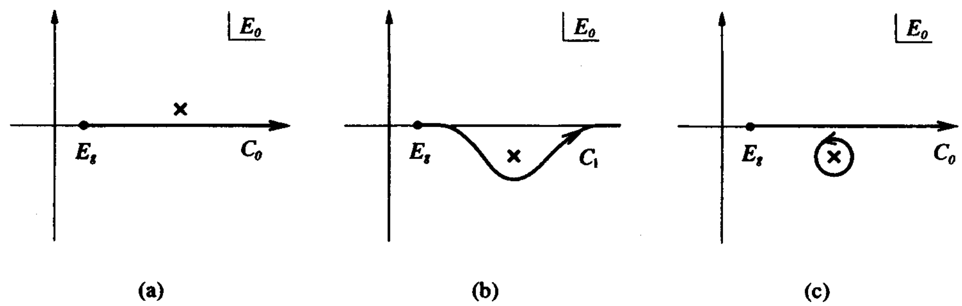


Fig. 5. Integration contours (a) C_0 for $\Re(s) > 0$ and (b) C_1 for $\Re(s) < 0$. The former extends along the real E_0 axis starting from the lowest energy E_g. The pole at $E_0 = E_a + i\frac{s}{t}$ is located above the contours C_0 and C_1. (c) The contour C_1 can further be deformed and decomposed into the contour C_0 along the real E_0 axis and a circle surrounding the pole.

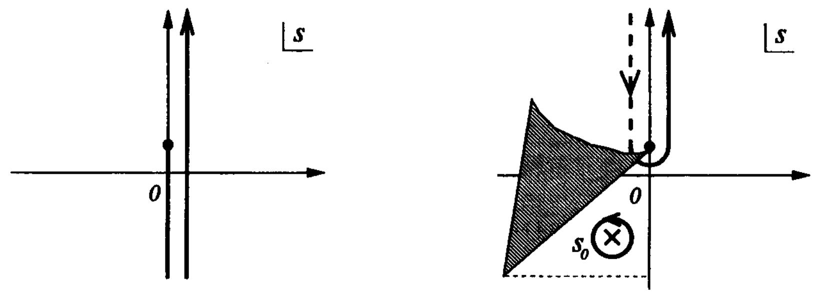
simple pole now appearing in the lower-half E_0 plane, has in general a nonvanishing real part. We see that this term just expresses the discontinuity of the left hand side over the imaginary s axis, thus ensuring the analyticity of the right hand side as a whole.

It is now clear that the analytically continued function $1/g(s,t)$ has the following properties as a function of the complex variable s:

- Under the assumption that H_0 (like H) has a bounded spectrum, a branch cut exists along the imaginary s axis, extending to $-i\infty$ from the branch point at $s = it(E_a - E_g)$, E_g being the lowest value of the spectrum of H_0.
- On the first Riemannian sheet, into which the analytic continuation must be done without crossing the branch cut, the last term in (3.49) does not show up. The function $1/g(s, t)$ has no singularity and is analytic in this plane.
- The last term in (3.49) appears only when the function $g(s,t)$ is continued into the second Riemannian sheet through the cut. This term allows the function $1/g(s, t)$ to have a simple pole in this plane where $\Re(s) < 0$.

These properties of the function $1/g(s,t)$ are enough to understand that the survival probability amplitude $\langle a|U_I(t)|a\rangle$, being nothing but the inverse Laplace transform of $1/g(s,t)$, exhibits two distinct behaviors at large times. Observe that the integration contour over s in (3.11) can be deformed in the left-half plane as in Fig. 6(b). There is a close similarity between the deformed coutours in the Fourier (Fig. 4) and in the Laplace transforms (Fig. 6). Let $s_0 = -\frac{\gamma}{2}t - i\delta Et$ $(\gamma > 0)$ be the zero of $g(s,t)$ in the second Riemannian sheet, so that

$$-\frac{\gamma}{2}\left[1+\int_{E_g}^{\infty}\frac{\sum_r|\langle E_0,r|H'|a\rangle|^2}{\left|E_a-E_0+\delta E-i\dfrac{\gamma}{2}\right|^2}dE_0\right]+2\pi\sum_r|\langle E_0,r|H'|a\rangle|^2\Big|_{E_0=E_a+\delta E-i\frac{\gamma}{2}}=0, \tag{3.50}$$



(a) (b)

Fig. 6. Integration contours over s. (a) The original one in (3.11) and (b) the deformed one. The latter is composed of a circle surrounding the simple pole at s_0 on the second Riemannian sheet and a path starting from $i\infty$ on the second sheet, turning around the branch point at $s = it(E_a - E_g)$ and extending back to $i\infty$ on the first sheet.

$$-\,\delta E\left[1-\int_{E_g}^{\infty}\frac{\sum_r |\langle E_0,r|H'|a\rangle|^2}{\left|E_a-E_0+\delta E-i\dfrac{\gamma}{2}\right|^2}dE_0\right]+\int_{E_g}^{\infty}\frac{(E_a-E_0)\sum_r |\langle E_0,r|H'|a\rangle|^2}{\left|E_a-E_0+\delta E-i\dfrac{\gamma}{2}\right|^2}dE_0=0. \tag{3.51}$$

Notice that both γ and δE, which are obtained as solutions of these equations, are t-independent constants. Close scrutiny[9] shows that in the weak-coupling limit, the second term in the square brackets of (3.50) gives a finite contribution and we exactly reobtain Fermi's Golden Rule (3.19) with the right factor. We see again that the presence of the second term in the left hand side of (3.50) is crucial for $g(s,t)$ to have a zero in the left-half s plane. This simple-pole contribution to $1/g(s,t)$ provides us with the exponential decay form for the survival amplitude, while the integration along the imaginary s axis cancels the former to yield the Gaussian behavior at short times according to the argument exposed in the first part of Sec. 2. As we shall see, the latter contribution is also responsible for the deviation from the exponential law at very long times, as expected on the basis of the above-mentioned theorem (3.23).

To estimate the remaining integration along the imaginary s axis for very large times, we choose the integration contour that starts from $i\infty$ on the second Riemannian sheet, goes down along the imaginary s axis, turns around the branch point at $s = it(E_a - E_g)$, appears into the first Riemannian sheet and runs back to $i\infty$ again. See Fig. 6(b). The contribution along this contour, which we call X_C, is explicitly written as[f]

$$X_C = \frac{1}{2\pi i}\int_{itE_{ag}}^{i\infty}\left\{\frac{e^s}{g(s,t)} - \left[(s-itE_{ag}) \to (s-itE_{ag})e^{-2\pi i}\right]\right\}ds\,. \tag{3.52}$$

[f]The contribution at the branch point is easily shown to vanish.

Here $E_{ag} \equiv E_a - E_g$, and the second term is obtained from the first just by replacing $s - itE_{ag}$ with $(s - itE_{ag})e^{-2\pi i}$, thus representing the integration on the second Riemannian sheet. With the change of the integration variable $s = i(y + tE_{ag})$, X_C is explicitly written as

$$X_C = \frac{e^{iE_{ag}t}}{2\pi i}\int_0^\infty$$
$$\times\left\{e^{iy}\left[y + tE_{ag} + t\int_{E_g}^\infty \frac{\sum_r |\langle E_0, r|H'|a\rangle|^2}{E_g - E_0 - \frac{y}{t}}dE_0\right]^{-1} - (y \to ye^{-2\pi i})\right\}dy\,. \tag{3.53}$$

Let us first calculate the difference between the two integrands. It arises from the multi-valuedness of the denominator in the integrand, the relevant part of which is rewritten, after integration by parts,

$$\int_{E_g}^\infty \frac{\sum_r |\langle E_0, r|H'|a\rangle|^2}{E_g - E_0 - \frac{y}{t}}dE_0 = -\ln\left(E_0 - E_g + \frac{y}{t}\right)\sum_r |\langle E_0, r|H'|a\rangle|^2\Big|_{E_g}^\infty$$
$$+ \int_{E_g}^\infty \ln\left(E_0 - E_g + \frac{y}{t}\right)\frac{d}{dE_0}\sum_r |\langle E_0, r|H'|a\rangle|^2 dE_0 \equiv A\,. \tag{3.54}$$

Notice that the argument of the logarithm is positive because $y > 0$. Since the logarithmic function gains an extra imaginary part $-2\pi i$ when its argument turns clockwise once around the origin, the above logarithmic function changes into

$$\ln\left(E_0 - E_g + \frac{y}{t}\right) \longrightarrow \ln\left(E_0 - E_g + \frac{y}{t}\right) - 2\pi i\theta\left(\frac{y}{t} - |E_0 - E_g|\right) \tag{3.55}$$

when y is replaced with $ye^{-2\pi i}$. Therefore by inserting the above expression for the logarithm into formula (3.54), the denominator in the second term of the right hand side in (3.53) is shown to differ from its counterpart in the first term by

$$-2\pi it\sum_r \left|\left\langle E_g + \frac{y}{t}, r|H'|a\right\rangle\right|^2 \equiv -2\pi itB\,. \tag{3.56}$$

In this derivation, we have assumed that the function $\sum_r |\langle E_0, r|H'|a\rangle|^2$ vanishes rapidly enough at both boundaries in order to make the integration over E_0 converge:

$$\sum_r |\langle E_0, r|H'|a\rangle|^2 \sim \begin{cases} (E_0 - E_g)^\delta & \text{for } E_0 \simeq E_g\,, \quad \delta > 0\,, \\ (E_0)^{-\delta'} & \text{for } E_0 \to \infty\,, \quad \delta' > 0\,. \end{cases} \tag{3.57}$$

We are now in a position to write down the explicit form of X_C. After a few manipulations, we arrive at $(u = y/E_{ag}t)$

$$X_C = -E_{ag}e^{iE_{ag}t}\int_0^\infty \frac{B_u e^{itE_{ag}u}du}{[E_{ag}(1+u) + A_u][E_{ag}(1+u) + A_u - 2\pi iB_u]}\,, \tag{3.58}$$

where

$$\begin{aligned} A_u &= \int_{E_g}^{\infty} \ln(E_0 - E_g + E_{ag}u) \frac{d}{dE_0} \sum_r |\langle E_0, r|H'|a\rangle|^2 dE_0 \,, \\ B_u &= \sum_r |\langle E_g + E_{ag}u, r|H'|a\rangle|^2 \,. \end{aligned} \tag{3.59}$$

In the large t limit, i.e. $tE_{ag} \gg 1$, the integration over u can be estimated by the behavior of the integrand for small $u < u_0 \sim 1/tE_{ag}$. Recall that the function B_u has been assumed to vanish at the boundary $E_0 = E_g$, i.e. at $u = 0$. As a consequence of (3.57), the function B_u behaves, for small u, like

$$B_u = \sum_r |\langle E_g + E_{ag}u, r|H'|a\rangle|^2 \sim E_{ag}C_0 u^\delta \,, \quad \delta > 0 \,, \tag{3.60}$$

where the first factor adjusts the dimension of B_u, while C_0 is a positive constant. We realize that the function X_C exhibits a power decay at very large times

$$\begin{aligned} X_C &\sim - \frac{C_0 e^{iE_{ag}t} \int_0^1 u^\delta e^{iu} du}{[tE_{ag}]^{1+\delta} \left[1 + \frac{1}{E_{ag}} \int_{E_g}^{\infty} \ln(E_0 - E_g) \frac{d}{dE_0} \sum_r |\langle E_0, r|H'|a\rangle|^2 dE_0 \right]^2} \\ &\equiv -Ce^{iE_{ag}t} \frac{1}{t^{1+\delta}} \,. \end{aligned} \tag{3.61}$$

Therefore we have shown that the survival amplitude decays as a function of the inverse power $t^{-(1+\delta)}$ at very long times, which eventually takes over the exponential behavior.

To summarize, the survival probability amplitude $\langle a|U_I(t)|a\rangle$ is shown to have the following large-time behavior

$$\langle a|U_I(t)|a\rangle \sim \mathcal{Z}e^{-\frac{\gamma}{2}t - i\delta Et} - Ce^{iE_{ag}t}\frac{1}{t^{1+\delta}} \,, \tag{3.62}$$

where $\mathcal{Z}^{-1} = (\partial g(s,t)/\partial s)|_{s=-\frac{\gamma}{2}-i\delta E}$. It is very instructive to compare this expression with the general formula (3.36) of the preceding subsection. It is evident from the above derivation that, as was already mentioned, the very large time behavior is in fact governed by the state number density $\rho_a(E)$ and the matrix elements $K_a(E)$. See Eq. (3.34). (The correspondence $B \leftrightarrow \rho_a Ka$ is manifest.)

The power behavior of quantum survival amplitudes at long times has attracted the attention of several investigators in the past. Höhler[58] showed explicitly a power behavior $\sim t^{-\frac{3}{2}}$ for the Lee model.[10] Knight and Milonni[59] investigated the two-level Weisskopf-Wigner model of atomic spontaneous emission under the dipole and the rotating-wave approximations. In this model, only two 'essential states' are considered, that is, an excited state and no photons present, and the ground state and one photon present. They calculated the nondecay amplitude using the Laplace transform and found a non-exponential tail $\sim t^{-2}$ at long times. The former behavior corresponds to the case $\delta = 1/2$ and the latter to $\delta = 1$ in (3.62). It is worth mentioning that these behaviors are in complete agreement with our conclusions, since the quantity corresponding to the function B_u in (3.59) is explicitly given and behaves like $\sim \sqrt{u}$ (i.e. $\delta = 1/2$) in the former case and $\sim u$ ($\delta = 1$) in the latter.

4. Exponential Law in a Solvable Model

4.1. *The AgBr Hamiltonian*

In order to give a concrete idea of how it is possible to obtain the exponential decay law for a quantum system in interaction with another quantum system endowed with a large number of degrees of freedom (a "macroscopic system"), we shall consider the so-called AgBr model,[37] that is exactly solvable and has played an important role in the quantum measurement problem. We present this model in the spirit of Havliček's remark[60]: "Please try to illustrate your assertion on an example which would involve 2 × 2 matrices only." Our model involves only Pauli matrices and some reasonably simple algebra.

We shall focus our attention on the modified version of the AgBr model,[38] that is able to take into account energy-exchange processes, and in particular on its weak-coupling, macroscopic limit.[36] Our exact calculation will show that the model realizes the so-called diagonal singularity[33] and can display the occurrence of an exponential regime *at all times.*

The modified AgBr Hamiltonian[38] describes the interaction between an ultra-relativistic particle Q and a one-dimensional N-spin array (D-system). The array is a caricature of a linear "photographic emulsion" of AgBr molecules, when one identifies the *down* state of the spin with the undivided molecule and the *up* state with the dissociated molecule (Ag and Br atoms). The particle and each molecule interact via a spin-flipping local potential. The total Hamiltonian for the $Q + D$ system reads

$$H = H_0 + H', \quad H_0 = H_Q + H_D\,, \tag{4.1}$$

where H_Q and H_D, the free Hamiltonians of the Q particle and of the "detector" D, respectively, and the interaction Hamiltonian H' are written as[g]

$$\begin{aligned} H_Q = c\hat{p}\,, \quad H_D = \frac{1}{2}\hbar\omega\sum_{n=1}^{N}\left(1+\sigma_3^{(n)}\right), \\ H' = \sum_{n=1}^{N} V(\hat{x}-x_n)\left[\sigma_+^{(n)}\exp\left(-i\frac{\omega}{c}\hat{x}\right)+\sigma_-^{(n)}\exp\left(+i\frac{\omega}{c}\hat{x}\right)\right], \end{aligned} \tag{4.2}$$

where $\hat{p}$ is the momentum of the Q particle, $\hat{x}$ its position, V a real potential, x_n ($n = 1, \ldots, N$) the positions of the scatterers in the array ($x_n > x_{n-1}$) and $\sigma_{i,\pm}^{(n)}$ the Pauli matrices acting on the nth site. An interesting feature of the above Hamiltonian, as compared to the original one,[37] is that we are not neglecting the

[g]In this section, we explicitly write the Planck constant $\hbar$.

energy H_D of the array, namely the energy gap $\hbar\omega$ between the two states of each molecule. This enables us to take into account energy-exchange processes between Q and the spin system D. The original Hamiltonian is reobtained in the $\omega = 0$ limit.

The evolution operator in the interaction picture

$$U_I(t) = e^{iH_0 t/\hbar} e^{-iHt/\hbar} = e^{-i\int_0^t H_I'(t')dt'/\hbar}, \tag{4.3}$$

where $H_I'(t)$ is the interaction Hamiltonian in the interaction picture, can be computed exactly as

$$U_I(t) = \prod_{n=1}^{N} \exp\left(-\frac{i}{\hbar}\int_0^t V(\hat{x} + ct' - x_n)dt' \left[\sigma_+^{(n)} \exp\left(-i\frac{\omega}{c}\hat{x}\right) + \text{h.c.}\right]\right), \tag{4.4}$$

and a straightforward calculation yields the S-matrix

$$S^{[N]} = \lim_{t\to\infty} U_I(t) = \prod_{n=1}^{N} S_{(n)} : \quad S_{(n)} = \exp\left(-i\frac{V_0\Omega}{\hbar c}\sigma^{(n)} \cdot \boldsymbol{u}\right), \tag{4.5}$$

where $\boldsymbol{u} = (\cos(\omega x/c), \sin(\omega x/c), 0)$ and $V_0\Omega \equiv \int_{-\infty}^{\infty} V(x)dx$ is assumed to be finite. The above expression enables us to compute the "spin-flip" probability, i.e. the probability of dissociating one AgBr molecule:

$$q = \sin^2\left(\frac{V_0\Omega}{\hbar c}\right). \tag{4.6}$$

If the initial D state is taken to be the ground state $|0\rangle_N$ (N spins down), and the initial Q state is a plane wave $|p\rangle$, the final state reads

$$S^{[N]}|p, 0\rangle_N = \sum_{j=0}^{N} \binom{N}{j}^{1/2} (-i\sqrt{q})^j \left(\sqrt{1-q}\right)^{N-j} |p_j, j\rangle_N, \tag{4.7}$$

where $|p_j, j\rangle_N$ represents the (spin-symmetrized) state in which Q has energy $p_j = p - j\hbar\omega/c$ and j spins are up.

This enables us to compute several interesting quantities, such as the visibility of the interference pattern obtained by splitting an incoming Q wave function into two branch waves (one of which interacts with D), and the energy "stored" in D after the interaction with Q, as well as the fluctuation around the average. The final results are

$$\begin{aligned} \mathcal{V} &= (1-q)^{N/2} \longrightarrow e^{-\overline{n}/2}, \\ \langle H_D\rangle_F &= qN\,\hbar\omega \longrightarrow \overline{n}\,\hbar\omega, \\ \langle \delta H_D\rangle_F = \sqrt{\langle (H_D - \langle H_D\rangle_F)^2\rangle_F} &= \sqrt{pqN}\,\hbar\omega \longrightarrow \sqrt{\overline{n}}\,\hbar\omega, \end{aligned} \tag{4.8}$$

where $_F$ stands for the final state (4.7), $p = 1 - q$, and the trivial trace over the Q particle states is suppressed. The arrows show the weak-coupling, macroscopic limit

$N \to \infty$, $qN = \bar{n}=$ finite.[38] All results are exact. It is worth stressing that $qN = \bar{n}$ represents the average number of excited molecules, so that interference, and relative energy fluctuations "gradually" disappear as $\bar{n}$ increases. The limit $N \to \infty$, $qN < \infty$ is physically very appealing, in our opinion, because it corresponds to a *finite* energy loss of the Q particle after interacting with the D system. Observe also that (4.7) is a generalized [SU(2)] coherent state and becomes a Glauber coherent state in the $N \to \infty$, $qN =$ finite limit.

4.2. *The exponential law*

Our next (and main) task is to study the behavior of the propagator, in order to analyze the temporal evolution of the system. Observe that the only nonvanishing matrix elements of the interaction Hamiltonian H' are those between the eigenstates of H_0 whose spin-quantum numbers differ by one, so that the conditions (3.4) in Sec. 3 are not satisfied. It is also important to note that the Q particle state $|cp\rangle$, characterized by the energy cp, is changed by H' into the state $|cp \pm \hbar\omega\rangle$, if $\omega \neq 0$. We can therefore expect a dissipation effect and the appearance of the diagonal singularity, which leads to the master equation. Following van Hove's pioneering work,[33] one could calculate the propagator perturbatively in this model. However, the solvability of the present model enables us to perform a nonperturbative treatment and yield an exact expression for the propagator. Define

$$\alpha_n \equiv \alpha_n(\hat{x}, t) \equiv \int_0^t V(\hat{x} + ct' - x_n)dt'/\hbar\,, \tag{4.9}$$

which can be viewed as a "tipping angle" of the nth spin if one identifies V with a magnetic field B,[61] and

$$\sigma_\pm^{(n)}(\hat{x}) \equiv \sigma_\pm^{(n)} \exp\left(\mp i\frac{\omega}{c}\hat{x}\right), \tag{4.10}$$

which satisfy, together with $\sigma_3^{(n)}$, the SU(2) algebra

$$[\sigma_-^{(n)}(\hat{x}), \sigma_+^{(n)}(\hat{x})] = -\sigma_3^{(n)}, \quad [\sigma_\pm^{(n)}(\hat{x}), -\sigma_3^{(n)}] = \pm 2\sigma_\pm^{(n)}(\hat{x})\,. \tag{4.11}$$

We can now return to the Schrödinger picture by inverting Eq. (4.3). We disentangle the exponential[62] in U_I by making use of (4.11) and obtain

$$e^{-iHt/\hbar} = e^{-iH_0t/\hbar} \prod_{n=1}^{N} \left(e^{-i\tan(\alpha_n)\sigma_+^{(n)}(\hat{x})} e^{-\ln\cos(\alpha_n)\sigma_3^{(n)}} e^{-i\tan(\alpha_n)\sigma_-^{(n)}(\hat{x})}\right). \tag{4.12}$$

Notice that the evolution operators (4.4) and (4.12), as well as the S-matrix (4.5) are expressed in a factorized form. This is a property of a rather general class of similar Hamiltonians.[63]

We shall now concentrate our attention on the situation in which the Q particle is initially located at $x' < x_1$, where x_1 is the position of the first scatterer in the linear array, and is moving towards the array with speed c. The spin system is initially set in the ground state $|0\rangle_N$ of the free Hamiltonian H_D (all spins down). This choice of the ground state is meaningful from a physical point of view, because the Q particle is initially outside D.

The propagator, defined by

$$G(x,x',t) \equiv \langle x| \otimes_N \langle 0|e^{-iHt/\hbar}|0\rangle_N \otimes |x'\rangle\,, \tag{4.13}$$

is easily calculated from Eq. (4.12) and we obtain

$$\begin{aligned} G(x,x',t) &= \langle x| \otimes_N \left\langle 0 \left| e^{-ic\hat{p}t/\hbar} \prod_{n=1}^{N} (e^{-\ln\{\cos[\alpha_n(\hat{x},t)]\}\sigma_3^{(n)}}) \right| 0 \right\rangle_N \otimes |x'\rangle \\ &= \langle x|x'+ct\rangle \prod_{n=1}^{N} \left(e^{\ln\{\cos[\alpha_n(x',t)]\}}\right) \\ &= \delta(x-x'-ct) \prod_{n=1}^{N} \cos\alpha_n(x',t)\,. \end{aligned} \tag{4.14}$$

Observe that, due to the choice of the free Hamiltonian H_Q in (4.2), the Q wave packet does not disperse, and moves with constant speed c. We place the spin array at the far right of the origin ($x_1 > 0$) and consider the case where potential V has a compact support and the Q particle is initially located at the origin $x' = 0$, i.e. well outside the potential region of D. The above equation shows that the evolution of Q occurs only along the path $x = ct$. Therefore we obtain

$$G(x,0,t) = \delta(x-ct) \prod_{n=1}^{N} \cos\tilde{\alpha}_n(t)\,, \quad \tilde{\alpha}_n(t) \equiv \int_0^{ct} V(y-x_n)dy/\hbar c\,. \tag{4.15}$$

This result is *exact.* Note that

$$\sin^2\tilde{\alpha}_n(\infty) = \sin^2\left(\frac{V_0\Omega}{\hbar c}\right) = q \tag{4.16}$$

is the spin-flip probability (4.6). We shall now consider the weak-coupling, macroscopic limit[38]

$$q \simeq \left(\frac{V_0\Omega}{\hbar c}\right)^2 = O(N^{-1})\,, \tag{4.17}$$

which is equivalent to the requirement that $\overline{n} = qN$ be finite. We shall see that the limit is equivalent to the van Hove limit, as given by Eq. (3.38). Notice that if we set

$$x_n = x_1 + (n-1)\Delta, \quad L = x_N - x_1 = (N-1)\Delta\,, \tag{4.18}$$

the scaled variable $z_n \equiv x_n/L$ can be considered as a continuous one z in the above limit, for $\Delta/L \to 0$ as $N \to \infty$. Therefore, a summation over n is to be replaced by a definite integration

$$q\sum_{n=1}^{N} h(x_n) \to q\frac{L}{\Delta}\int_{x_1/L}^{x_N/L} h(Lz)dz \simeq \overline{n}\int_{x_1/L}^{x_N/L} h(Lz)dz\,. \tag{4.19}$$

This type of integration gives a finite result if the function h is scale invariant, because the integration volume is considered to be finite from the physical point of view; in fact, the quantities x_1/L and x_N/L should be of the order of unity even in the $L \to \infty$ limit. It will be shown below [Eq. (4.20)] that in the present case the function h is indeed scale invariant.

For the sake of simplicity, we shall restrict our attention to the case of δ-shaped potentials, by setting $V(y) = (V_0\Omega)\delta(y)$. This hypothesis is in fact too restrictive. In the following, we shall see that the requirement that V has a compact support (local potentials) would suffice. We obtain

$$\begin{aligned} G &\propto \exp\left(\sum_{n=1}^{N}\ln\left\{\cos\int_{-x_n}^{ct-x_n}(V_0\Omega/\hbar c)\delta(y)dy\right\}\right) \\ &= \exp\left(\sum_{n=1}^{N}\ln\{\cos[(V_0\Omega/\hbar c)\theta(ct-x_n)]\}\right) \\ &\to \exp\left(-\frac{\overline{n}}{2}\int_{x_1/L}^{x_n/L}\theta(ct-Lz)dz\right) \\ &= \exp\left(-\frac{\overline{n}}{2}\left[\frac{ct-x_1}{L}\theta(x_N-ct)\theta(ct-x_1)+\theta(ct-x_N)\right]\right), \end{aligned} \tag{4.20}$$

where the arrow denotes the weak-coupling, macroscopic limit (4.17).

This brings about an exponential regime *as soon as the interaction starts.* Indeed, if $x_1 < ct < x_N$,

$$G \propto \exp\left(-\overline{n}\frac{c(t-t_0)}{2L}\right), \tag{4.21}$$

where $t_0 = x_1/c$ is the time at which the Q particle meets the first potential. On the other hand, if $ct > x_N$ (that corresponds to the case in which Q has gone through D and the interaction is over) we have

$$G \propto e^{-\overline{n}/2}\,. \tag{4.22}$$

This could also be obtained directly from (4.15) and is in complete agreement with the first formula in (4.8), because $|G|^2$ is nothing but the probability that Q goes through the spin array *and* leaves it in the ground state.

Notice that there is *no* Gaussian behavior at short times. As was discussed in Sec. 2 and Ref. 39, deviations from the exponential behavior at short times are a consequence of the finiteness of the mean energy of the initial state. Observe that the mean energy of the position eigenstates is not well-defined in this model. If the position eigenstates in Eq. (4.13) are substituted with wave packets of size a, the mean energy aquires a well-defined meaning. It is shown below that in this case the exponential regime is attained a short time after $t_0 = x_1/c$, of the order of a/c.

Let $|\tilde{0}\rangle$ be a wave packet of size a, initially distributed around the origin $x' = 0$

$$|\tilde{0}\rangle = \int dx C(x)|x\rangle \otimes |0\rangle_N \,. \tag{4.23}$$

For simplicity, we may choose the following expression for the wave packet

$$C(x) = \frac{1}{\sqrt{a}}\theta(a/2 - |x|)e^{ip_0 x/\hbar} \,. \tag{4.24}$$

(One should choose a smooth C function in order to avoid singularities, however, in such a case the final expression would be slightly more involved and nothing essential would change.) We assume here that the initial wave packet has no overlap with the potential region $[x_1, x_N]$ and that the size of the former is much smaller than that of the latter

$$a/2 < x_1, \quad a \ll L = x_N - x_1 \,. \tag{4.25}$$

Observe that the mean energy of the initial state is finite, for one obtains

$$\langle\tilde{0}|c\hat{p}|\tilde{0}\rangle = cp_0 < \infty \,. \tag{4.26}$$

We calculate the state vectors

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\tilde{0}\rangle, \quad |\psi_0(t)\rangle = e^{-iH_0 t/\hbar}|\tilde{0}\rangle \,, \tag{4.27}$$

evolved under the action of the total Hamiltonian H and the free Hamiltonian H_0, respectively, and define the propagator

$$G(t) \equiv \langle\psi_0(t)|\psi(t)\rangle \,, \tag{4.28}$$

which extracts the net effect of the interaction. A straightforward calculation shows that the propagator G has the following temporal behavior in the weak-coupling, macroscopic limit (4.17)

$$G(t) = \begin{cases} 1 & \text{for } 0 < ct < x_1 - a/2 \,, \\ g_1(t) & \text{for } x_1 - a/2 < ct < x_1 + a/2 \,, \\ g_2(t) & \text{for } x_1 + a/2 < ct < x_N - a/2 \,, \\ g_3(t) & \text{for } x_N - a/2 < ct < x_N + a/2 \,, \\ e^{-\bar{n}/2} & \text{for } x_N + a/2 < ct \,, \end{cases} \tag{4.29}$$

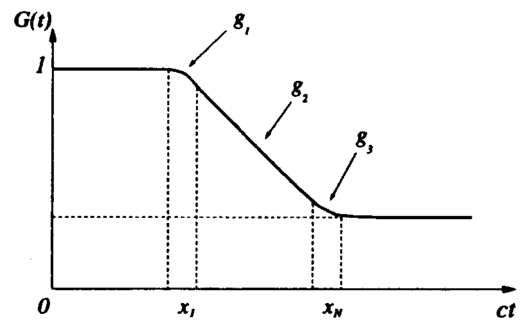


Fig. 7. Temporal behavior of the propagator $G(t)$ defined in (4.28). See (4.29).

where the functions $g_i(t)$ $(i = 1, 2, 3)$ are given by

$$g_1(t) = 1 + \frac{2L}{\bar{n}a}\left[1 - \exp\left(-\frac{\bar{n}}{2L}(ct - x_1 + a/2)\right)\right] - \frac{1}{a}(ct - x_1 + a/2),$$

$$g_2(t) = \frac{2L}{\bar{n}a}[1 - e^{-\bar{n}a/2L}]\exp\left(-\frac{\bar{n}}{2L}(ct - x_1 - a/2)\right),$$

$$g_3(t) = e^{-\bar{n}/2}\left\{1 + \frac{2L}{\bar{n}a}\left[\exp\left(-\frac{\bar{n}}{2L}(ct - x_N - a/2)\right) - 1\right] + \frac{1}{a}(ct - x_N - a/2)\right\}. \tag{4.30}$$

See Fig. 7.

We realize that only the function $g_2(t)$ exhibits the exponential form and that in the time intervals $x_1/c - a/2c < t < x_1/c + a/2c$ and $x_N/c - a/2c < t < x_N/c + a/2c$, the propagator G does not follow an exponential behavior. In the first interval the wave packet has not yet completely entered the potential region and the interaction between the Q particle and the spin system D has not yet attained its full strength. Analogously, in the second region, the wave packet is going out of the array. Notice that in this model the exponential law holds if the whole wave packet is merged in the potential region so that the interaction keeps its full strength; see $g_2(t)$ in (4.30). This suggests an equivalent, maybe more "suggestive" interpretation: one could say that in the first region $x_1/c - a/2c < t < x_1/c + a/2c$, the "initial state" has not been "prepared", yet. (By "initial", in the above, we do *not* mean an eigenstate of the free Hamiltonian, but rather, intuitively, the state that will undergo the exponential decay.) This alternative viewpoint sheds some light on the importance of what is usually referred to as "state preparation" in quantum mechanics: It is *first* necessary to *prepare* the right initial state, in order to observe a certain temporal (e.g. exponential) development of this state.

The region $x_1/c - a/2c < t < x_1/c + a/2c$ may be viewed as a possible residuum of the short-time Gaussian behavior. In fact, in this region, the function $g_1(t)$

behaves like

$$g_1(t) \sim 1 - \frac{1}{4\bar{n}aL}(ct - x_1 + a/2)^2 , \tag{4.31}$$

and a similar behavior of $g_3(t)$ is found in the final region of the linear array. It is worth stressing that the temporal behaviors obtained in (4.21) and in (4.29) are in complete agreement with some general theorems.[8,12,26]

The absence of the power behavior at very long times in this model may be traced back to the special character of our free Hamiltonian H_Q in (4.2): The spectrum of the operator $c\hat{p}$ is *not* bounded from below, so that the total Hamiltonian H does not possess a lower-bounded spectrum. Of course we can safely consider, on physical grounds, only those incident Q particles that have enough (positive) energy to go through the spin system, in order to assure the positivity of the energy of the total system. Nevertheless, it should be recalled that the boundedness of spectrum of the total Hamiltonian is the only condition required for the Paley-Wiener Theorem (3.23). Our model system is out of the range of applicability of the theorem and therefore its exponential behavior at very long times is not in contradiction with it. Similarly, no ground state of the free Hamiltonian H_0 in (4.1) exists owing to the presence of the $c\hat{p}$ operator and the analysis developed in Sec. 3.3 is not applicable to this case either. In this respect, it would be interesting to consider the lower-bounded Hamiltonian $H_Q = c|\hat{p}|$, but, unfortunately, this operator raises serious problems that are closely related to the early attempts at defining a time operator in quantum mechanics.[64]

One might suspect that the approximation of δ-shaped potential is playing an important role in the derivation of the exponential law. As a matter of fact, this is not the case. A detailed calculation, making use of square potentials of strenght V_0 and width Ω, centered at each x_n, yields, for $x_1 + \frac{\Omega}{2} < ct < x_N - \frac{\Omega}{2}$,

$$G \propto \exp\left(-\bar{n}\frac{ct - x_1}{2L} + \frac{\bar{n}\Omega}{12L} \right). \tag{4.32}$$

In this case, the exponential regime is attained a short time after t_0, of the order of the width of the potential V, which, in the present model, can be made arbitrarily small. Also in the present context, the region $t \sim t_0 + O(\Omega/c)$ may be viewed as a residuum of the short-time Gaussian behavior. However, it is clear from Eq. (4.32) that once the particle has *completely* penetrated into the potential region, the exponential law holds *exactly* in the weak-coupling, macroscopic limit. This is, in our opinion, one of the most interesting features of the AgBr Hamiltonian.

It is very interesting to bring to light the profound link between the weak-coupling, macroscopic limit $qN = \bar{n} =$ finite considered in this model and van Hove's "$\lambda^2 T$" limit.[33] First, it is important to note that in the weak-coupling, macroscopic limit, van Hove's diagonal singularity naturally appears in the present model. It is easy to check that for each diagonal matrix element of H'^2, there are

N intermediate-state contributions; indeed, for example

$$\langle 0,\dots,0|H'^2|0,\dots,0\rangle = \sum_{j=1}^{N} |\langle 0,\dots,0|H'|0,\dots,0,1_{(j)},0,\dots,0\rangle|^2 . \tag{4.33}$$

On the other hand, at most two states can contribute to each off-diagonal matrix element of H'^2. This ensures that only the diagonal matrix elements are kept in the weak-coupling, macroscopic limit, $N \to \infty$ with $qN < \infty$, which is the realization of diagonal singularity in our model.

The link between the weak-coupling, macroscopic limit $qN = \overline{n}$ = finite considered above and van Hove's "$\lambda^2 T$" limit can be also intuitively justified as follows: The free part of the particle Hamiltonian is $H_Q = c\hat{p}$, so that the particle travels with constant speed c, and interacts with the detector during the total time $T = L/c \simeq N\Delta/c$. Since the coupling constant $\lambda \propto V_0\Omega$, one gets $\lambda^2 T \propto (V_0\Omega)^2 N\Delta/c \propto qN$. Notice that the "lattice spacing" Δ, the inverse of which corresponds to a density in our one-dimensional model, can be kept finite in the limit. (In such a case, we have to express everything in terms of scaled variables, that is, $\tau \equiv t/L$, z_1 and z_N, introduced just after Eq. (4.18) and $\zeta \equiv a/L$, where a is the size of the wave packet.)

The role played by the energy gap ω need also be clarified. First of all, ω plays a very important role to guarantee the consistency of the physical framework: If $\omega = 0$, all spin states would be energetically degenerated and the choice of the σ_3-diagonal representation would be quite *arbitrary*. In other words, only a nonvanishing ω (or H_D) logically enables us to use the eigenstates of σ_3 in order to evaluate the relevant matrix elements. Although ω does not appear in our final results (4.21) or (4.29), it certainly does in other propagators, involving an initial state of the spin system different from $|0\rangle_N$.

Finally, we would like to shortly comment on other possible causes for the occurrence of the exponential behavior displayed by our model. This is a delicate problem. Our analysis suggests that the exponential behavior (4.21) is mainly due to the locality of the potentials V and the factorized form of the evolution operator (4.12), which shows that the interactions between Q and adjacent spins of the array are independent, and the evolution "starts anew" at every step. This suggest the occurrence of a sort of Markovian process, which would be in agreement with the purely dissipative behavior (4.21). A preliminary calculation shows that the operator

$$\hat{W}(t) = c\hat{p}(t) - \langle c\hat{p}(t)\rangle , \tag{4.34}$$

where $\hat{p}(t)$ is in the Heisenberg picture and the average is taken over a wave packet of Q and the ground state of D, has several interesting features. In particular, some properties of $\hat{W}(t)$ closely resemble those of a Wiener process. In this context, the connection between the exponential "probability dissipation" (4.21) and the (practically irreversible) energy-exchange between the Q particle and the "environment" D is a very open problem and should be investigated in detail. Leggett's remark,[65] about the central relevance of the problem of dissipation to decoherence and quantum measurement theory, makes the above topic very interesting. There is, in our opinion, a profound link between decoherence, irreversibility, "probability dissipation" and genuine energy dissipation. This link is not yet fully understood. Work is in progress to clarify the above issues.

5. Conclusions and Additional Comments

We have analyzed the temporal behavior of quantum mechanical systems at all times, by taking into account seminal as well as more recent results. As we have seen, the temporal development of the survival probability of a suitable initial state undergoes an evolution that can be roughly decomposed into three regions: a Gaussian region at short times, an approximately exponential region at intermediate times and a power region at very long times.

The short-time Gaussian behavior is ascribable to a very general property of the temporal development engendered by the Schrödinger equation. As we have seen, this leads to the quantum Zeno effect and has very interesting spin-offs on the quantum measurement problem. In particular, we have argued that the QZE is by no means a proof in support of the Copenhagen interpretation, based on von Neumann's projecion postulate: Indeed, a purely dynamical explanation of the QZE was put forward, completely based on unitary evolutions. We have also discussed the important difference between the QZE and the QZP, and have argued that the latter is in contradiction with Heisenberg's uncertainty principle.

One of our primary concerns has been to understand under which physical conditions we can observe an exponential behavior, since, as was already stressed, no deviation from the exponential law has ever been observed.[13] The exponential form is representative of a totally incoherent nature of the system under consideration. Remember that the notion of coherence is one of the essential features of the quantum world, where the evolution (of the total system) is always unitary. When we focus our attention on one part (subsystem) of the total system, if the latter is endowed with a huge number of degrees of freedom and can be treated as a kind of reservoir, we can expect the former to behave in a stochastic manner. The interaction between the subsystem and the remaining part of the total system may, under such circumstances, cause dissipation on the former. It is then natural to expect the appearance of irreversible phenomena, governed by master equations, for such subsystems. This is, however, by no means a trivial matter. Because quantum dynamics, i.e. the Schrödinger equation, is symmetric in time, we obviously expect the survival probability to go back to unity after a certain lapse of time, which is of the order of Poincaré time. However, a purely exponential decay of the amplitude would actually imply an actual ("eternal") loss of probability, that would never be recovered. Rigorously speaking, this is at variance with the underlying unitarity of the temporal evolution. An appropriate limiting procedure is required in order to explicitly derive the exponential law, as was repeatedly remarked.

Since it is *mathematically* true that quantum systems never behave exponentially at very long times, it is clear that in order to realize the exponential law at long times on the basis of quantum theory, some additional manipulation is necessary. By considering the exponential law as a manifestation of the classical character of the system, we can recognize the importance of such a notion as the transition from a quantum to a classical regime.

Fermi's Golden Rule,[50] according to which we obtain the decay constant (3.19), is a lowest-order approximation. We believe that the quantum mechanical lowest-order perturbation theory cannot be considered completely satisfactory, in this context, unless it is appropriately supported by some additional limiting procedure that accounts for the macroscopic nature of the total system with which the decaying system is interacting.

We have also seen that van Hove's procedure, based on the weak-coupling, macroscopic limit considered in Sec. 3, yields a satisfactory answer to many of the above problems. In particular, his procedure dispenses with Pauli's random phase assumptions and yields an exponential behavior at all times, by cancelling both the short-time Gaussian region and the long-time power tail. Of course, one is led to wonder about the meaning of the "$\lambda^2 T$" limit, and in particular about the very reasons why Nature should follow this prescription. Even though no clear-cut answer can be given to this very deep question, we believe that there is much more to be understood, on this issue.

What we are actually looking for is a consistent way of deriving the classical nature (loss of coherence) from quantum theory in a certain limit which should properly characterize the classical and macroscopic nature. At this point one may see a close connection with the quantum measurement problem,[20–22,32] where the realization of the classical nature of experimental results, namely the appearance of exclusive events after a measurement, has been one of the central issues of controversy since the birth of quantum mechanics.

As a general rule, explicit examples can be often more useful than general abstract considerations. As Berry recently put it, "Only wimps specialize in the general case. Real scientists pursue examples".[66] In the attempt to follow the same, general philosophy, we have endeavoured to clarify the above issues by considering a particular solvable model, known as modifed AgBr Hamiltonian: This model enjoys several interesting features: It is solvable and relatively simple. It also yields very useful insights into general topics like quantum measurements and dissipation. Finally, it discloses the occurrence of van Hove's limit in a very direct way.

As we have seen, if the initial state of the spin array in the AgBr model is chosen to be the ground state of the free Hamiltonian, in which all spins are down, the survival probability of such a state, once the interaction with an incoming external particle has been switched on, behaves in a purely exponential way. The causes of this behaviour, as well as the physical approximations they reflect, have been analyzed in detail, also in the light of general theorems.

In conclusion, we believe that further understanding in all the above-mentioned topics can be achieved by analyzing the general structure of quantum mechanical evolutions and, at the same time, by looking at interesting models and particular limiting procedures.

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