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# On the inversion of the Radon transform: standard versus $M^2$ approach

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We compare the Radon transform in its standard and symplectic formulations and argue that the analytical inversion of the latter is easier to perform.

**Keywords:** Radon transform; tomography; state reconstruction

## 1. Introduction

The Radon transform [1] is a key mathematical tool in tomography. Its inverse enables one to reconstruct a function if some of its integrals are known. The whole subject has been recently revived by quantum mechanical applications. The possibility of reconstructing the tomographic map of the Wigner quasidistribution function [2–4] associated with a given quantum state [5–8] has motivated experiments [9–11], triggered novel proposals [12] and boosted innovative theoretical techniques [13]. In a classical context, the Radon transform is widely used in target and sample discrimination methods and medical applications. The extension of these well-established applications and techniques to a quantum mechanical context is of great interest and sometimes quite straightforward, the only differences lying in the properties of the functions to be reconstructed and/or discriminated. Quantum applications are widespread and diverse. The entire field, driven by a blizzard of technical advances, is attracting increasing attention and is growing at a lively pace. Good reviews on the subject can be found in [14].

The Radon transform was originally introduced as an integral transform defined over submanifolds of $\mathbb{R}^n$, that may be viewed as a 'configuration space'. However, if $n$ is even, one may think of $\mathbb{R}^n$ as a phase space and consider the integrals over its Lagrangian submanifolds. One may then associate the tomographic map with the symplectic transform on the phase space [15]. In this context, motion is instrumental for the identification of the phase space and its Lagrangian variables: the Hamilton equations do not appear in the definition of the Radon transform and this interpretation differs from the original one. Nevertheless, the approach is prolific and enables one to identify different types of tomograms [16], extend tomography to curved surfaces [17] and consider more general problems and applications. This is in line with previous historical developments, by Radon himself [1], John [18], Helgason [19] and Strichartz [20], and paves the way towards so-far unearthed quantum mechanical applications.

In this article we shall focus on the theoretical and analytical characteristics of the Radon transform and its inversion. The application of these ideas and techniques to experimental data is left for future investigation. We shall compare the standard Radon approach with that based on the aforementioned symplectic identification and shall argue that, although mathematically equivalent, they may differ in practice. In particular, the inversion may be far from trivial and may turn out to be simpler in the symplectic framework.

## 2. Symplectic tomography

Let us focus on the two-dimensional case for the sake of concreteness. The Radon transform, in its original formulation, solves the following problem: reconstruct a function of two variables, say $f(p,q)$, if its integrals over arbitrary lines are given. The Radon transform (or homodyne tomogram) reads

$$f^\sharp(\vartheta, X) = \int_{\mathbb{R}^2} f(q,p)\delta(X - q\cos\vartheta - p\sin\vartheta)\mathrm{d}q\,\mathrm{d}p, \quad (1)$$

where $\delta$ is the Dirac distribution, $\vartheta \in [0, 2\pi)$, $\omega = (\cos\vartheta, \sin\vartheta) \in S$ (the unit sphere in 1D) and $X \in \mathbb{R}$. In order to obtain a symplectic formulation, a central observation is the following: it is possible to express the Radon

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transform in affine language (the so-called tomographic map) [1,21] and write

$$f^{M^2}(\mu,\nu,X)=\int_{\mathbb{R}^2} f(q,p)\delta(X-\mu q-\nu p)\mathrm{d}q\,\mathrm{d}p, \qquad (2)$$

where $\mu,\nu,X\in\mathbb{R}$, $\mu^2+\nu^2>0$. We have named '$M^2$' the tomographic map (Equation (2)) after Man'ko and Marmo, who gave a seminal contribution towards its significance [22–25]. Clearly

$$f^{\sharp}(\vartheta,X)=f^{M^2}(\cos\vartheta,\,\sin\vartheta,X). \qquad (3)$$

Consider now a particle moving on the line $q\in\mathbb{R}$ and a function $f(q,p)$ on its phase space $(q,p)\in\mathbb{R}^2$. Since

$$\begin{aligned}&\begin{pmatrix}\mu & \nu\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}q\\ p\end{pmatrix}\\ &\quad=\mu q+\nu p=\begin{pmatrix}-\nu & \mu\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}q\\ p\end{pmatrix},\end{aligned} \qquad (4)$$

the argument in the Dirac delta in Equation (2) may be considered either as a Euclidean product or as a symplectic product. The two interpretations are completely equivalent and one can equivalently solve the inversion problem by using the Euclidean or symplectic Fourier transform. We shall see in the next section that the two procedures can vastly differ in complexity.

Note that the Radon transform is defined in an equivalent way by

$$f^{\sharp}(\vartheta,X)=\int_{\mathbb{R}} f(X\cos\vartheta+s\sin\vartheta,X\sin\vartheta+s\cos\vartheta)\mathrm{d}s. \qquad (5)$$

The inversion formula, as given by Radon, amounts to considering first the average value of $f^{\sharp}$ on all lines tangent to the circle of center $(q,p)$ and radius $r$, namely,

$$F_{(q,p)}(r)=\frac{1}{2\pi}\int_0^{2\pi} f^{\sharp}(\vartheta,q\cos\vartheta+p\sin\vartheta+r)\mathrm{d}\vartheta \qquad (6)$$

and then compute

$$f(q,p)=-\frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\int_{\varepsilon}^{+\infty}F'_{(q,p)}(r)\frac{\mathrm{d}r}{r}, \qquad (7)$$

where $F'_{(q,p)}(r)$ denotes the derivative with respect to $r$. The Radon transform maps a (suitable) function on the plane into a function on the cylinder. Some conditions that guarantee the invertibility and continuity of the map were studied by Radon himself [1], John [18], Helgason [19] and Strichartz [20].

On the other hand, the inverse transform of Equation (2) reads [22,23]

$$f(q,p)=\int_{\mathbb{R}^3} f^{M^2}(\mu,\nu,X)e^{i(X-\mu q-\nu p)}\frac{\mathrm{d}X\,\mathrm{d}\mu\,\mathrm{d}\nu}{(2\pi)^2}. \qquad (8)$$

## 3. The inverse transform: an explicit example

We now compare the inversions (6), (7) and (8) by looking at a very simple example: the ground state of a one-dimensional quantum harmonic oscillator,

$$f(q,p)=\frac{\alpha^2}{\pi}e^{-\alpha^2(q^2+p^2)}. \qquad (9)$$

Its $M^2$-transform reads

$$\begin{aligned}f^{M^2}(\mu,\nu,X)&=\frac{\alpha^2}{\pi}\int_{\mathbb{R}^2}e^{-\alpha^2(q^2+p^2)}\delta(X-\mu q-\nu p)\mathrm{d}q\,\mathrm{d}p\\ &=\frac{\alpha^2}{|\mu|\pi}\int_{\mathbb{R}}e^{-\alpha^2\left[\left(\frac{X-\nu p}{\mu}\right)^2+p^2\right]}\mathrm{d}p\\ &=\frac{\alpha^2}{|\mu|\pi}\int_{\mathbb{R}}e^{-\frac{\alpha^2}{\mu^2}\left[\left(\sqrt{\mu^2+\nu^2}p-\frac{\nu X}{\sqrt{\mu^2+\nu^2}}\right)^2+\frac{\mu^2X^2}{\mu^2+\nu^2}\right]}\mathrm{d}p\\ &=\frac{\alpha}{\sqrt{\pi(\mu^2+\nu^2)}}e^{-\alpha^2\frac{X^2}{\mu^2+\nu^2}},\end{aligned} \qquad (10)$$

which is a Gaussian with respect to $X$, but has a nontrivial dependence on $\mu$ and $\nu$. On the other hand, by making use of Equation (3), one gets the Radon transform

$$f^{\sharp}(\vartheta,X)=f^{M^2}(\cos\vartheta,\,\sin\vartheta,X)=\frac{\alpha}{\sqrt{\pi}}e^{-\alpha^2X^2}, \qquad (11)$$

which is simply a Gaussian, independent of the angle $\vartheta$, due to symmetry.

Let us tackle the inversion problem. We start from the inverse $M^2$ transform, which is easily solved in a few lines:

$$\begin{aligned}f(q,p)&=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^3}f^{M^2}(\mu,\nu,X)e^{i(X-\mu q-\nu p)}\mathrm{d}X\,\mathrm{d}\mu\,\mathrm{d}\nu\\ &=\frac{\alpha}{(2\pi)^2\sqrt{\pi(\mu^2+\nu^2)}}\\ &\quad\times\int_{\mathbb{R}^3}e^{-\frac{\alpha^2}{\mu^2+\nu^2}\left(X^2-iX\frac{\mu^2+\nu^2}{\alpha^2}\right)}e^{-i(\mu q+\nu p)}\mathrm{d}X\,\mathrm{d}\mu\,\mathrm{d}\nu\\ &=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}e^{-\frac{\mu^2+\nu^2}{4\alpha^2}}e^{-i(\mu q+\nu p)}\mathrm{d}\mu\,\mathrm{d}\nu\\ &=\frac{\alpha}{\pi}e^{-(q^2+p^2)\alpha^2}.\end{aligned} \qquad (12)$$

Let us now endeavour to invert the Radon transform. It would be tempting to leave this as an exercise for the reader, but we will sketch the main steps of the derivation. From Equation (6) we get

$$\begin{aligned}F'_{(q,p)}(r)=-\frac{\alpha^3}{\pi\sqrt{\pi}}\int_0^{2\pi}&(q\cos\vartheta+p\sin\vartheta+r)\\ &\times e^{-\alpha^2(q\cos\vartheta+p\sin\vartheta+r)^2}\mathrm{d}\vartheta\end{aligned} \qquad (13)$$

and thus $f(q,p) = \lim_{\varepsilon\downarrow 0} f_\varepsilon(q,p)$, where

$$f_\varepsilon(q,p) = \frac{\alpha^3}{\pi^2\sqrt{\pi}}\int_\varepsilon^{+\infty}\mathrm{d}r\int_0^{2\pi}\mathrm{d}\vartheta\,\frac{(q\cos\vartheta + p\sin\vartheta + r)}{r} \times e^{-\alpha^2(q\cos\vartheta + p\sin\vartheta + r)^2}. \tag{14}$$

Already in the very simple case of a Gaussian function with a Gaussian Radon transform the above inversion formula is not easy to manage. First, introduce a step function, $\theta(r) = 1$ if $r > 0$, and $\theta(r) = 0$ otherwise, and change the period of the angle integration

$$f_\varepsilon(q,p) = \frac{\alpha^3}{\pi^2\sqrt{\pi}}\int_{\mathbb{R}}\mathrm{d}r\int_{-\pi}^{\pi}\mathrm{d}\vartheta\,\frac{\theta(r-\varepsilon)}{r} \times (r - q\cos\vartheta + -p\sin\vartheta)e^{-\alpha^2(r - q\cos\vartheta - p\sin\vartheta)^2}. \tag{15}$$

Then change variables $z = r - q\cos\vartheta - p\sin\vartheta$ and $t = \tan(\vartheta/2)$

$$f_\varepsilon(q,p) = \frac{2\alpha^3}{\pi^2\sqrt{\pi}}\int_{\mathbb{R}^2}\theta\left(\frac{t^2(z-q-\varepsilon) + 2pt + (z+q-\varepsilon)}{1+t^2}\right) \times \frac{ze^{-\alpha^2z^2}}{t^2(z-q) + 2pt + (z+q)}\mathrm{d}t\,\mathrm{d}z. \tag{16}$$

Now look at the region where the argument of the theta function is positive. One gets two roots: $t_{1,2}(z-\varepsilon)$, with

$$t_{1,2}(z) = \frac{-p \pm \sqrt{p^2+q^2-z^2}}{z-q}, \tag{17}$$

whose discriminant is negative for $z \notin [-\sqrt{p^2+q^2}+\varepsilon, \sqrt{p^2+q^2}+\varepsilon]$. Therefore,

$$f_\varepsilon(q,p) = I_1(\varepsilon; q,p) + I_2(\varepsilon; q,p), \tag{18}$$

where

$$I_1(\varepsilon,q,p) = \frac{2\alpha^3}{\pi^2\sqrt{\pi}}\int_{\sqrt{p^2+q^2}+\varepsilon}^{+\infty}\mathrm{d}z\,ze^{-\alpha^2z^2} \times \int_{-\infty}^{+\infty}\mathrm{d}t\,\frac{1}{t^2(z-q) + 2pt + (z+q)} \tag{19}$$

and

$$I_2(\varepsilon,q,p) = \frac{2\alpha^3}{\pi^2\sqrt{\pi}}\int_{-\sqrt{p^2+q^2}+\varepsilon}^{\sqrt{p^2+q^2}+\varepsilon}\mathrm{d}z\int_{-\infty}^{+\infty}\mathrm{d}t \times \theta\left(\frac{t^2(z-q-\varepsilon) + 2pt + (z+q-\varepsilon)}{1+t^2}\right) \times \frac{ze^{-\alpha^2z^2}}{t^2(z-q) + 2pt + (z+q)}. \tag{20}$$

Let us evaluate $I_1(\varepsilon, q, p)$. The integration over $t$ yields $\pi/\sqrt{z^2-q^2-p^2}$ and thus

$$I_1(\varepsilon,q,p) = \frac{2\alpha^3}{\pi\sqrt{\pi}}\int_{\sqrt{p^2+q^2}+\varepsilon}^{+\infty}\frac{ze^{-\alpha^2z^2}}{\sqrt{z^2-q^2-p^2}}\mathrm{d}z. \tag{21}$$

An integration by parts gives

$$\begin{aligned} I_1(\varepsilon,q,p) &= \frac{4\alpha^5}{\pi\sqrt{\pi}}\int_{\sqrt{p^2+q^2}+\varepsilon}^{+\infty} z\sqrt{z^2-q^2-p^2}\,e^{-\alpha^2z^2}\mathrm{d}z + O(\sqrt{\varepsilon}) \\ &= \frac{4\alpha^5}{\pi\sqrt{\pi}}\int_{(2\varepsilon\sqrt{p^2+q^2}+\varepsilon^2)^{1/2}}^{+\infty} y^2e^{-\alpha^2(y^2+q^2+p^2)}\mathrm{d}y + O(\sqrt{\varepsilon}) \\ &= \frac{2\alpha^5}{\pi\sqrt{\pi}}e^{-\alpha^2(q^2+p^2)}\int_{-\infty}^{+\infty}y^2e^{-\alpha^2y^2}\mathrm{d}y + O(\sqrt{\varepsilon}), \end{aligned} \tag{22}$$

where $y^2 = z^2 - q^2 - p^2$. Since the Gaussian integral equals $\sqrt{\pi}/2\alpha^3$, we finally get

$$\lim_{\varepsilon\downarrow 0} I_1(\varepsilon,q,p) = f(q,p). \tag{23}$$

Therefore, it remains to prove that $I_2(\varepsilon,q,p)$ vanishes for $\varepsilon \to 0$. We will leave this as a very instructive exercise and just give three hints. First, one can see that the theta function in $I_2(\varepsilon,q,p)$ is different from zero when $z - \varepsilon > q$ and

$$t \in \mathbb{R} \setminus [t_1(z-\varepsilon), t_2(z-\varepsilon)], \tag{24}$$

or when $z - \varepsilon < q$ and

$$t \in [t_2(z-\varepsilon), t_1(z-\varepsilon)]. \tag{25}$$

Therefore, the integral (Equation (20)) splits into the sum of three integrals. Secondly, by noticing that for any $z$,

$$\int_{-\infty}^{+\infty}\mathrm{d}t\,\frac{1}{(t-t_1(z))(t-t_2(z))} = 0, \tag{26}$$

one can trade the integration over the range in Equation (24), with an integration over $[t_1(z-\varepsilon), t_2(z-\varepsilon)]$ and gather together the three integrals. Thirdly, the integration over $t$ yields the logarithm of a rational function that can be shown to vanish for $\varepsilon \to 0$.

Notice that the Radon transform can also be inverted by using the following alternative formula due to Helgason [19], which is suitable for generalizations to symmetric homogeneous spaces

$$f(q,p) = \frac{1}{4\pi}(-\Delta)^{1/2}\int_0^{2\pi} f^\sharp(\vartheta, q\cos\vartheta + p\sin\vartheta)\mathrm{d}\vartheta. \tag{27}$$

Here the fractional Laplacian

$$(-\Delta)^{1/2} = \left(-\frac{\partial^2}{\partial q^2} - \frac{\partial^2}{\partial p^2}\right)^{1/2} \tag{28}$$

is defined by a Fourier transform

$$(-\Delta)^{1/2}g(q,p) = \int_{\mathbb{R}^2}(k_1+k_2)^{1/2}\hat{g}(k_1,k_2)e^{i(qk_1+pk_2)} \times \frac{\mathrm{d}k_1\mathrm{d}k_2}{2\pi}, \tag{29}$$

where

$$\hat{g}(k_1,k_2) = \int_{\mathbb{R}^2} g(q,p)e^{-i(qk_1+pk_2)}\frac{\mathrm{d}q\,\mathrm{d}p}{2\pi}. \tag{30}$$

In our case we would have to compute

$$\frac{\alpha}{4\pi\sqrt{\pi}}\left(-\frac{\partial^2}{\partial q^2}-\frac{\partial^2}{\partial p^2}\right)^{1/2}\int_0^{2\pi} e^{-\alpha^2(q\cos\vartheta+p\sin\vartheta)^2}\mathrm{d}\vartheta, \tag{31}$$

a task even more difficult than the previous one.

## 4. Extension to *n* dimensions and a few comments

The definitions and conclusions of the previous sections can be easily extended to $n$ dimensions. The Radon transform of a function $f$ of the $n$-dimensional vector $x\in\mathbb{R}^n$ reads

$$f^\sharp(\omega,X) = \int_{\mathbb{R}^n} f(x)\delta(X-\langle\omega,x\rangle)\mathrm{d}x, \tag{32}$$

where $\omega\in\mathbb{S}^{n-1}$, the $(n-1)$-dimensional sphere, $\langle\cdot,\cdot\rangle$ denotes scalar product and $X\in\mathbb{R}$. The $M^2$ transform is

$$f^{M^2}(\mu,X) = \int_{\mathbb{R}^n} f(x)\delta(X-\langle\mu,x\rangle)\mathrm{d}x, \tag{33}$$

where $\mu\in\mathbb{R}^n$ and $X\in\mathbb{R}$. Obviously, from $f^{M^2}(\mu,X)$ one can immediately recover $f^\sharp(\omega,X)$ by setting $\mu=\omega\in\mathbb{S}^{n-1}$:

$$f^\sharp(\omega,X) = f^{M^2}(\omega,X). \tag{34}$$

However, notice that, although $f^\sharp$ is the restriction of $f^{M^2}$ on the unit sphere $\mathbb{S}^{n-1}$, there is actually a bijection between the two transforms. Therefore, they carry exactly the same information. Indeed, since the Dirac distribution is positive homogeneous of degree $-1$, i.e. $\delta(\alpha x)=|\alpha|^{-1}\delta(x)$, for every $\alpha\neq 0$, one gets from Equation (33)

$$f^{M^2}(\mu,X) = \frac{1}{|\mu|}f^{M^2}\left(\frac{\mu}{|\mu|},\frac{X}{|\mu|}\right), \tag{35}$$

for $\mu\neq 0$. In words, the tomogram $f^{M^2}(\mu,X)$ at a generic point $\mu\in\mathbb{R}^n$ is completely determined by the tomogram at $\mu/|\mu|\in\mathbb{S}^{n-1}$. But the latter is nothing but the Radon transform, by Equation (34). Therefore, we get the bijection

$$f^\sharp(\omega,X) = f^{M^2}(\omega,X), \tag{36}$$

$$f^{M^2}(\mu,X) = \frac{1}{|\mu|}f^\sharp\left(\frac{\mu}{|\mu|},\frac{X}{|\mu|}\right) \quad (\mu\neq 0). \tag{37}$$

Notice also that at the origin $\mu=0$ the $M^2$ transform

$$f^{M^2}(0,X) = \delta(X)\int_{\mathbb{R}^n} f(x)\mathrm{d}x, \tag{38}$$

depends only on the total mass.

The inversion formulae for the transforms (32) and (33), read

$$f(x) = \frac{1}{2^n\pi^{n-1}}(-\Delta)^{(n-1)/2}\int_{\mathbb{S}^{n-1}} f^\sharp(\omega,\langle\omega,x\rangle)\mathrm{d}\omega \tag{39}$$

and

$$f(x) = \int_{\mathbb{R}^{n+1}} f^{M^2}(\mu,X)e^{i(X-\langle\mu,x\rangle)}\frac{\mathrm{d}X\,\mathrm{d}\mu}{(2\pi)^n}, \tag{40}$$

respectively. While formula (40), which is nothing but a Fourier transform, is quite easy to handle, the inversion formula (39) is in general very hard to tackle, especially for even $n$, due to the presence of a fractional Laplacian. Therefore, from a practical point of view, our message is the following: in order to invert the Radon transform (32), dilate it by Equation (37) into the $M^2$ transform and then use the Fourier inversion formula (40). This simple trick enables one to avoid long and tedious calculations.

One final comment is in order. The difficulties that one encounters in inverting the Radon transform in the conventional framework are due to the following 'geometrical' picture. Reducing the space (or the number of variables) *de facto* introduces 'interactions'. This makes inversions more cumbersome. By contrast, the smart introduction of an additional fictitious variable may lead to an unfolding of the interacting problem into a non-interacting one, making inversions elementary.


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