Open Systems & Information Dynamics
Vol. 19, No. 1 (2012) 1250002 (10 pages)
DOI:10.1142/S1230161212500023

The Observables of a Dissipative Quantum System

Dariusz Chruściński[1], Paolo Facchi[2,3], Giuseppe Marmo[4]
and Saverio Pascazio[5,3]

[1] *Institute of Physics, Nicolaus Copernicus University, Grudziądzka 5/7, 87–100 Toruń, Poland*

[2] *Dipartimento di Matematica and MECENAS, Università di Bari, I–70125 Bari, Italy*

[3] *INFN, Sezione di Bari, I–70126 Bari, Italy*

[4] *Dipartimento di Scienze Fisiche and MECENAS, Università di Napoli "Federico II", I–80126 Napoli, Italy, and INFN, Sezione di Napoli, I–80126 Napoli, Italy*

[5] *Dipartimento di Fisica and MECENAS, Università di Bari, I-70126 Bari, Italy*

(Received: July 29, 2011)

Abstract. A time-dependent product is introduced between the observables of a dissipative quantum system, that accounts for the effects of dissipation on observables and commutators. In the $t \to \infty$ limit this yields a contracted algebra. This product can be transported back by duality on the space of states. The general ideas are corroborated by a few explicit examples.

1. Introduction

One of the most distinctive traits of quantum mechanics is the non-commutativity of some of its observables. If a commutator vanishes, the associated observables can be simultaneously measured and can be considered "classical" with respect to each other. The system is classical when *all* its observables commute. The transition from quantum to classical is a fascinating subject of investigation and interesting approaches have been proposed in order to emphasize the role of observables in this context and give a consistent definition of classicality [1, 2, 3].

A dissipative quantum system loses some of its genuine quantum features (such as the ability to interfere) and eventually displays a "classical" behaviour [4, 5]. In this letter we suggest a mechanism that yields classicality (in the afore-mentioned sense) starting from dissipative dynamics and the physics of open quantum systems. Besides being of interest in themselves, these subjects have profound conceptual consequences and lead to applications, for example in quantum enhanced applications and quantum technologies [6]. It is therefore of interest to understand what happens to the observables of a dissipative quantum system and in which sense measurements yield less information at the end of a dissipative process. The approach we shall propose is general, but for the sake of simplicity we shall limit our discussion to the master equation. Generalizations and further discussion will be postponed to a forthcoming publication.

The description of quantum systems makes use of states ρ and an algebra $\mathcal{A}$ of observables A. One can describe the dynamical evolution in terms of the former or the latter, the two pictures being equivalent, according to Dirac's prescription [7]

$$\mathrm{Tr}(\rho_t A_0) \;=\; \mathrm{Tr}(\rho_0 A_t)\,. \tag{1.1}$$

We shall work in the Markovian approximation, when the dynamics is governed by the master equation

$$\dot{\rho}_t \;=\; L\rho_t\,, \tag{1.2}$$

where ρ_t is the density matrix of the quantum system, the subscript t denotes the evolved quantity at time t and L is the time-independent generator of a dynamical semigroup. Equation (1.2) can be formally solved

$$\rho_t \;=\; e^{tL}\rho_0 \;=\; \Lambda_t(\rho_0)\,, \qquad t \geq 0 \tag{1.3}$$

and it is well known that under certain conditions on L [8] the dynamics Λ_t is completely positive and trace preserving [9, 5].

Equation (1.1) leads to the (adjoint) evolution equation for observables (Heisenberg picture)

$$\dot{A}_t \;=\; L^\sharp A_t \iff A_t \;=\; \Lambda_t^\sharp(A_0)\,. \tag{1.4}$$

In this article we address the following question: what can be meaningfully observed in a dissipative quantum system, in particular when it has reached its equilibrium state? Our strategy will be to interpret the effects of the adjoint evolution $\Lambda^\sharp$ on the product on the algebra $\mathcal{A}$ of observables, with basis $\{A_j\}$:

$$A_i A_j \;=\; \alpha_{ij}^k A_k\,. \tag{1.5}$$

In turn, this product defines the commutators (Lie product) through the structure constants C:

$$[A_i, A_j] \;=\; C_{ij}^k A_k\,, \tag{1.6}$$

where $C_{ij}^k = \alpha_{ij}^k - \alpha_{ji}^k$. We shall see that in general, the aforementioned question will lead to a *contraction* of the algebra of observables [10, 11].

2. First Example and General Ideas

Let us start from a simple but interesting case study. Let

$$L\rho \;=\; -\frac{\gamma}{2}(\rho - \sigma_3\rho\sigma_3)\,, \tag{2.1}$$

where σ_α, $\alpha = 1, 2, 3$, are the Pauli matrices, $\sigma_0 = \mathbb{1}$, and $\gamma > 0$. This describes the dissipative dynamics of a qubit undergoing phase damping. The asymptotic solution is

$$\rho_0 = \frac{1}{2}(\mathbf{1} + \boldsymbol{x} \cdot \boldsymbol{\sigma}) \stackrel{t\to\infty}{\longrightarrow} \Lambda_\infty(\rho_0) = \rho_\infty = \frac{1}{2}(\mathbf{1} + x_3\sigma_3)\,, \tag{2.2}$$

$\boldsymbol{x}$ being a vector in the unit 3-dimensional ball $|\boldsymbol{x}| \leq 1$. It is very simple to see that (2.1) yields

$$\Lambda_t^\sharp(\sigma_{0,3}) = \Lambda_\infty^\sharp(\sigma_{0,3}) = \sigma_{0,3}\,, \tag{2.3}$$
$$\Lambda_t^\sharp(\sigma_{1,2}) = e^{-\gamma t}\sigma_{1,2} \longrightarrow \Lambda_\infty^\sharp(\sigma_{1,2}) = 0\,. \tag{2.4}$$

These equations must be understood in the weak sense, according to (1.1): for example, the expectation value of $\sigma_{1,2}$ in the asymptotic state (2.2) vanishes. This result offers a remarkable interpretation: as time goes by, it becomes increasingly difficult to measure the coherence between the two states of the qubit. In the $t \to \infty$ limit, coherence is lost and the only nontrivial observables are populations. This interpretation, although suggestive, must face a serious problem: can one consistently define a novel product among observables, in such a way that

$$\mathcal{A}_\infty = \Lambda_\infty^\sharp(\mathcal{A}) = \lim_{t\to\infty} \Lambda_t^\sharp(\mathcal{A}) \tag{2.5}$$

be a well-defined algebra? The following result [14] helps answering this question.

Let $\mathcal{A}$ be a complex topological algebra, i.e., a topological vector space over $\mathbb{C}$ with a continuous bilinear operation

$$(X, Y) \in \mathcal{A} \times \mathcal{A} \mapsto X \cdot Y \in \mathcal{A} \tag{2.6}$$

and $U_\lambda : \mathcal{A} \to \mathcal{A}$ a family of linear morphisms that continuously depends on a real parameter λ. If U_λ are invertible in a neighborhood of the origin $\lambda \in I \setminus \{0\}$, then we can consider the continuous family of products

$$X \cdot_\lambda Y = U_\lambda^{-1}(U_\lambda(X) \cdot U_\lambda(Y))\,, \tag{2.7}$$

for $\lambda \in I \setminus \{0\}$. All these products are isomorphic by definition, since $U_\lambda(X \cdot_\lambda Y) = U_\lambda(X) \cdot U_\lambda(Y)$ and if U_0 is invertible, then clearly

$$\lim_{\lambda\to 0} X \cdot_\lambda Y = U_0^{-1}(U_0(X) \cdot U_0(Y))\,. \tag{2.8}$$

However, the $\lim_{\lambda\to 0} X \cdot_\lambda Y$ may exist for all $X, Y \in \mathcal{A}$ even if U_0 is not invertible and the right-hand side of (2.8) does not make sense. We say then that $\lim_{\lambda\to 0} X \cdot_\lambda Y$ is a *contraction* of the product $X \cdot Y$. The existence and the form of the contracted product heavily depends on the family U_λ [11].

We therefore identify $\lambda = 1/t$, $U_\lambda = \Lambda_t^\sharp$ and adopt the prescription

$$A \cdot_t B \;\equiv\; (\Lambda_t^\sharp)^{-1}(\Lambda_t^\sharp(A) \cdot \Lambda_t^\sharp(B)), \quad \forall\, A, B \in \mathcal{A}\,. \tag{2.9}$$

Clearly, in general, $\Lambda_\infty (= U_0)$ is not invertible, but the limiting product "$\cdot_\infty$" can make sense. Having defined a product, we can now define the commutators according to the rule

$$[A_i, A_j]_t \;\equiv\; (\Lambda_t^\sharp)^{-1}[\Lambda_t^\sharp(A_i), \Lambda_t^\sharp(A_j)] \;\equiv\; C_{ij}^k(t) A_k\,, \tag{2.10}$$

where $[A, B] = A \cdot B - B \cdot A$. In the $t \to \infty$ limit (2.10) yields a contraction of the original algebra (1.6).[1] We can now "transport" the product back on the states by duality, by defining[2]

$$\langle \rho_1 | \rho_2 \rangle_t \;\equiv\; \mathrm{Tr}[(\Lambda_t)^{-1}(\Lambda_t(\rho_1) \cdot \Lambda_t(\rho_2))] \;=\; \mathrm{Tr}[(\Lambda_t(\rho_1) \cdot \Lambda_t(\rho_2))]\,, \tag{2.11}$$

where we used the fact that Λ_t preserves the trace. Notice that the above product embodies the customary notion of "likeliness" between two quantum states. As time goes by, dissipation tends to erase information, and states become more difficult to distinguish. Equation (2.11) also expresses transition probabilities between states, some of which may vanish. The scalar product on states (2.11) shows that it is not just the commutator (Lie) product (2.10) which is relevant, but also the symmetrized (Jordan) product, for $\langle \rho_1 | \rho_2 \rangle_t = \langle \rho_2 | \rho_1 \rangle_t$. Finally, observe that when we consider in particular pure states $\rho = |\psi\rangle\langle\psi|$, we find that the contraction procedure changes the Hermitian product on the Hilbert space.

For instance, in the simple model (2.1)–(2.4), the contracted algebra is the Lie algebra of the Euclidean group $E(2)$ of isometries of the plane:

$$[\sigma_1, \sigma_2]_t \;=\; 2ie^{-2\gamma t}\sigma_3 \;\longrightarrow\; [\sigma_1, \sigma_2]_\infty \;=\; 0\,, \tag{2.12}$$

$$[\sigma_2, \sigma_3]_t \;=\; 2i\sigma_1 \;\longrightarrow\; [\sigma_2, \sigma_3]_\infty \;=\; 2i\sigma_1\,, \tag{2.13}$$

$$[\sigma_1, \sigma_3]_t \;=\; -2i\sigma_2 \;\longrightarrow\; [\sigma_1, \sigma_3]_\infty \;=\; -2i\sigma_2\,. \tag{2.14}$$

The transition from su(2) to e(2) is that taking from the spin of massive particles to the helicity of massless particles [12], by looking at a small portion of the sphere (tangent plane). For a recent application with atomic ensembles, see [13]. Since the asymptotic state is given by (2.2), one observes that a

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[1] One can also take $U_\lambda = (\Lambda_t^\sharp)^{-1}$ and define $A \cdot B \equiv (\Lambda_t^\sharp)^{-1}(\Lambda_t^\sharp(A) \cdot_t \Lambda_t^\sharp(B))$, that preserves the product and the commutators for any invertible evolution. This definition is also mathematically consistent, but does not yield the same physical interpretation as (2.9).

[2] This definition enables one to discuss purities in a straightforward way. Alternatively, one can define the product on $\sqrt{\rho_j} \in \mathcal{A}$. This and additional possibilities will be analyzed in a forthcoming article.

measurement of $\sigma_{1,2}$ yields zero, while a measurement of σ_3 (populations) yields a nontrivial result. Also, given two states $\rho_j = \frac{1}{2}(\mathbf{1}+\boldsymbol{x}^{(j)}\cdot\boldsymbol{\sigma})$, $j=1,2$,

$$\langle\rho_1|\rho_2\rangle_{t=0} = \frac{1}{2}(1+\boldsymbol{x}^{(1)}\cdot\boldsymbol{x}^{(2)}) \longrightarrow \langle\rho_1|\rho_2\rangle_{t=\infty} = \frac{1}{2}(1+x_3^{(1)}x_3^{(2)}), \quad (2.15)$$

so that only populations matter in the end.

A few additional observations will hopefully clarify some subtle points. The contraction of the angular momentum operator algebra, say su(2), going to the e(2) algebra, when interpreted in terms of representation theory, shows that in order to deal with Hermitian operators we must go from possibly finite-dimensional representations to infinite-dimensional ones. In particular, when we contract from su(2) to e(2), if we want our operators to be Hermitian in the limit, we have to work with the entire Hilbert space of square integrable functions on the sphere S^2 and not just finite-dimensional irreducible representations.

In general, the contraction procedure will create invariant Abelian subalgebras [$(\mathbf{1},\sigma_1,\sigma_2)$ in (2.12)–(2.14)], which may be interpreted as the creation of some "classical content" in the system, due to dissipation. These aspects need further analysis to exhibit all its potentiality in dealing with the quantum-classical transition.

If one adds to (2.1) a unitary evolution $-i[H,\rho]$, with Hamiltonian $H=\Omega\sigma_3$, nothing changes. However, a Hamiltonian $H=\Omega\sigma_1$ yields a more involved dynamics [15] and makes $\Lambda^\sharp_\infty(\sigma_3)$ vanish as well: in this case the contracted algebra is Abelian and even measurement of populations becomes trivial. The interpretation is straightforward: the Hamiltonian provokes Rabi oscillations between the two levels, the asymptotic state is $\rho_\infty=\mathbf{1}/2$ [rather than (2.2)] and the final state is totally mixed. Having tested our general scheme on a simple but significant example, we can now look at more complicated situations.

3. Second Example

Let

$$L\rho = -\frac{\gamma}{2}\left(\{a^\dagger a,\rho\} - 2a\rho a^\dagger\right) \quad (3.1)$$

describe a harmonic oscillator undergoing energy damping. Here, $\{A,B\}=AB+BA$. It is easy to check that

$$\Lambda^\sharp_t(a) = e^{-\gamma t/2}a\,, \qquad \Lambda^\sharp_t(a^\dagger) = e^{-\gamma t/2}a^\dagger,$$
$$\Lambda^\sharp_t(N) = e^{-\gamma t}N\,, \qquad N=a^\dagger a\,, \quad (3.2)$$

so that the oscillator algebra is contracted to an Abelian algebra, with $[a,a^\dagger]_\infty=[a,N]_\infty=0$ (remember that the above equations are understood in the weak sense). The physical picture is straightforward: dissipation drives the system to its ground state and in the limit not only the relative coherence, but even the populations of the excited states vanish. The introduction of a Hamiltonian $H = \omega a^\dagger a$ does not change the global picture.

4. Third Example

Let

$$L\rho \;=\; -\frac{\gamma}{2}\left(\{(a^\dagger a)^2, \rho\} - 2a^\dagger a\rho a^\dagger a\right) \tag{4.1}$$

describes a harmonic oscillator undergoing phase damping. A generic density matrix

$$\rho_0 \;=\; \sum c_{mn}|m\rangle\langle n| \;\stackrel{t\to\infty}{\longrightarrow}\; \sum |c_n|^2|n\rangle\langle n| \tag{4.2}$$

becomes diagonal in the energy representation, so that the product (2.11) contains only information on the population differences.

Since $L^\sharp = L$ and $\Lambda^\sharp = \Lambda$, one finds

$$\Lambda_t^\sharp(a) \;=\; e^{-\gamma t/2}a\,, \qquad \Lambda_t^\sharp(a^\dagger) \;=\; e^{-\gamma t/2}a^\dagger\,, \qquad \Lambda_t^\sharp(N) \;=\; N\,, \tag{4.3}$$

so that, unlike in the second example, N is left unaltered. The contraction of the oscillator algebra yields the Lie algebra of the Poincaré group in 1+1 dimensions ISO(1,1):

$$[a, a^\dagger]_\infty = 0\,, \qquad [a, N]_\infty = a\,, \qquad [a^\dagger, N]_\infty = -a^\dagger\,. \tag{4.4}$$

The physical picture is straightforward: in the presence of phase damping the system is driven to an incoherent mixture (in the energy basis). However, in the asymptotic limit it is still possible to measure nonvanishing populations of the different states: see (4.2). The introduction of a Hamiltonian $H = \omega a^\dagger a$ does not change anything.

5. Fourth Example

Let

$$L\rho \;=\; -\gamma(\{x^2, \rho\} - 2x\rho x) \;=\; -\gamma[x, [x, \rho]]\,, \tag{5.1}$$

describes a massive particle undergoing decoherence:

$$L|x\rangle\langle y| \;=\; -\gamma(x-y)^2|x\rangle\langle y|\,. \tag{5.2}$$

Also in this case, the generator (5.1) is self-dual, $L = L^\sharp$.

By considering formally x and p as bounded operators, one gets

$$\Lambda_t^\sharp(p) \;=\; p\,, \qquad \Lambda_t^\sharp(x) \;=\; x\,, \tag{5.3}$$

for all t, so that the CCR are preserved. However one gets, for $n \geq 2$,

$$L(p^n) \;=\; \gamma n(n-1)p^{n-2}\,, \tag{5.4}$$

so higher order commutation relations change.

These findings can be corroborated by working with the (bounded) unitary groups generated by x and p, that is the Weyl operators

$$U(\alpha) \;=\; e^{i\alpha x}\,, \qquad V(\beta) \;=\; e^{i\beta p}\,, \qquad \alpha, \beta \in \mathbb{R}\,. \tag{5.5}$$

They satisfy

$$U(\alpha)V(\beta) \;=\; e^{-i\alpha\beta}V(\beta)U(\alpha)\,. \tag{5.6}$$

One has $[x, U(\alpha)] = 0$ and $[x, V(\beta)] = -\beta V(\beta)$, yielding

$$L\,U(\alpha) \;=\; 0\,, \qquad L\,V(\beta) \;=\; -\gamma\beta^2\,V(\beta)\,, \tag{5.7}$$

and hence

$$\Lambda_t^\sharp U(\alpha) \;=\; U(\alpha)\,, \qquad \Lambda_t^\sharp V(\beta) \;=\; e^{-\gamma\beta^2 t}V(\beta)\,. \tag{5.8}$$

Notice, that for any $\beta \neq 0$ $\Lambda_t^\sharp V(\beta)$ is no longer unitary for $t > 0$, and asymptotically vanishes. However, for any t one has

$$\begin{aligned}
U(\alpha)\cdot_t V(\beta) &= \Lambda_t^{\#-1}(\Lambda_t^{\#}U(\alpha)\cdot\Lambda_t^{\#}V(\beta)) \\
&= \Lambda_t^{\#-1}(e^{-\gamma\beta^2 t}U(\alpha)\cdot V(\beta)) \\
&= \Lambda_t^{\#-1}(e^{-\gamma b^2 t}e^{-i\alpha\beta}V(\beta)\cdot U(\alpha)) \\
&= e^{-i\alpha\beta}\Lambda_t^{\#-1}(\Lambda_t^{\#}V(\beta)\cdot\Lambda_t^{\#}U(\alpha)) \\
&= e^{-i\alpha\beta}V(\beta)\cdot_t U(\alpha)\,,
\end{aligned} \tag{5.9}$$

that is, the commutation relations of the Weyl system are preserved. However, the Weyl system itself is not preserved, since $\Lambda_t^\sharp V(\beta)$ is not unitary. This example clarifies that, while the contraction does not affect the basic Lie algebra, it changes the *whole* associative algebra, and thus the higher-order commutators. Finally, notice that the presence of a free Hamiltonian changes the picture considerably [16] and will not be considered here.

6. Fifth Example

Finite dimensional version of the fourth example. Consider a d-level system and let

$$X \;=\; \sum_{m=1}^{d} m|m\rangle\langle m| \tag{6.1}$$

be the discrete position operator on a circle. Consider the analogous of (5.1)

$$L\rho \;=\; -\gamma[X, [X, \rho]]\,. \tag{6.2}$$

Let us introduce Schwinger's unitary operators [17]

$$U = \sum_{m=1}^{d} \lambda^m |m\rangle\langle m| , \qquad V = \sum_{k=1}^{d} \lambda^{-k} |\widetilde{k}\rangle\langle\widetilde{k}| , \tag{6.3}$$

where $\lambda = e^{2\pi i/d}$, and the momentum eigenbasis $\{|\widetilde{k}\rangle\}$, defined by a discrete Fourier transform,

$$|\widetilde{k}\rangle = \frac{1}{\sqrt{d}} \sum_{m=1}^{d} \lambda^{-km} |m\rangle . \tag{6.4}$$

Schwinger's system, which is the finite dimensional version of Weyl's, satisfies

$$U^k V^l = \lambda^{kl} V^l U^k , \tag{6.5}$$

for $k, l = 1, \ldots, d$. One easily finds [compare with (5.8)]

$$\Lambda_t^\sharp U^k = U^k , \qquad \Lambda_t^\sharp V^l = e^{-\gamma l^2 t} V^l , \tag{6.6}$$

so that V^l asymptotically vanishes. Again, $\Lambda_t^\sharp V^l$ is no longer unitary for $t > 0$. As a consequence, like in the previous example, we get

$$U^k \cdot_t V^l = \lambda^{kl} V^l \cdot_t U^k , \tag{6.7}$$

and the commutation relations are preserved. However, Schwinger's system is not preserved, since $\Lambda_t^\sharp V^l$ is no longer unitary. From (6.2) one has the discrete version of (5.2)

$$L|m\rangle\langle n| = -\gamma(m-n)^2 |m\rangle\langle n| , \tag{6.8}$$

so that each observable becomes asymptotically diagonal in the position eigenbasis $|m\rangle$. It is clear that the introduction of a unitary evolution with Hamiltonian $H = \sum_m h_m |m\rangle\langle m|$ does not change the global picture.

7. Sixth Example

Finally, let us consider the following model of pure decoherence of a d-level system. Define d unitary operators

$$U_k = \sum_{l=0}^{d-1} \lambda^{-kl} P_l , \quad k = 0, \ldots, d-1 , \tag{7.1}$$

where $P_l = |l\rangle\langle l|$ and $\lambda = e^{2\pi i/d}$. Note that $U_0 = \mathbf{1}_d$, and $\mathrm{Tr}\, U_k = 0$ for $k \geq 1$. Now, for $\gamma_1, \ldots, \gamma_{d-1} \geq 0$ let us define the following generator

$$L\rho = -\frac{1}{d} \sum_{k=1}^{d-1} \gamma_k \left(\rho - U_k \rho U_k^\dagger \right) . \tag{7.2}$$

It is clear that for $d = 2$ one has $U_1 = \sigma_3$ and hence (7.2) reproduces (2.1) as a particular case. Using (7.1) one easily derives the dynamical map

$$\Lambda_t \rho = \sum_{m,n=0}^{d-1} c_{mn}(t) P_m \rho P_n \,, \tag{7.3}$$

where the decoherence matrix $c_{mn}(t)$ reads

$$\begin{aligned} c_{mn}(t) &= e^{-(i\omega_{mn}+\gamma_{mn})t} \,, \\ \gamma_{mn} &= \frac{1}{d}\sum_{k=1}^{d-1} \gamma_k \,\mathrm{Re}\left(1 - \lambda^{-k(m-n)}\right) , \\ \omega_{mn} &= -\mathrm{Im}\left(\sum_{k=1}^{d-1} \gamma_k \lambda^{-k(m-n)}\right) . \end{aligned} \tag{7.4}$$

Note that $\gamma_{mn} = \gamma_{nm}$, with $\gamma_{mm} = 0$, and $\omega_{mn} = -\omega_{nm}$, which implies $\omega_{mm} = 0$. In particular, if all $\gamma_j = \gamma$, then

$$\gamma_{mn} = \gamma \quad m \neq n \,, \quad \omega_{mn} = 0 \,, \tag{7.5}$$

and one finds

$$\Lambda_t^\sharp |m\rangle\langle n| = c_{nm}(t)\, |m\rangle\langle n| \,. \tag{7.6}$$

Hence, due to $\gamma_{mn} > 0$, only the diagonal elements P_m survive asymptotically. If one adds to (7.2) the Hamiltonian $H = \sum_k h_k P_k$, the asymptotic picture does not change. Finally, one finds the following formula for the product $A \cdot_t B$

$$|m\rangle\langle n| \cdot_t |k\rangle\langle l| = \frac{c_{nm}(t) c_{lk}(t)}{c_{lm}(t)} |m\rangle\langle n| \cdot |k\rangle\langle l| \,. \tag{7.7}$$

In particular, if all decoherence rates are equal $\gamma_j = \gamma$,

$$|m\rangle\langle n| \cdot_t |k\rangle\langle l| = e^{-\gamma[1+\delta_{ml}-\delta_{mn}-\delta_{kl}]t}\, \delta_{nk}\, |m\rangle\langle l| \,. \tag{7.8}$$

This formula is useful both for states and operators.

8. Conclusions

Starting from the adjoint evolution of a dissipative quantum system, we have defined a product that yields a contracted algebra of observables. Other definitions, fully consistent from a mathematical point of view, are clearly possible, but do not yield an equally appealing physical interpretation. In some sense, the ansatz (2.9) "ascribes" to the product $\cdot_t$ the dissipative features of the evolution and the increasing difficulty in measuring those observables that are more affected by decoherence and dissipation.

In the present framework, ample room is left for noncommutative (quantum) observables, that do not belong to the center of the contracted algebra. These are associated with the kernel of $L^\sharp$. These observables are not affected by dissipation and preserve their quantum features. One can find many examples, e.g. in models like those discussed in the sixth example (when some $\gamma_{mn} = 0$).

We confined our analysis to Markovian systems, described by the master equation (1.2). However, our main conclusions remain valid when the evolution is described by a map (quantum channel). This unearths additional possibilities that will be discussed in a forthcoming paper.

Acknowledgment

The authors thank SVYASA University, Bangalore (India) for their warm hospitality during the final part of this work.

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