

Measurement-Induced Quantum Diffusion

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The dynamics of a “kicked” quantum system undergoing repeated measurements of momentum is investigated. A diffusive behavior is obtained even when the dynamics of the classical counterpart is not chaotic. The diffusion coefficient is explicitly computed for a large class of Hamiltonians and compared to the classical case.

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The classical and quantum dynamics of bound Hamiltonian systems under the action of periodic “kicks” are in general very different. Classical systems can follow very complicated trajectories in phase space, while the evolution of the wave function in the quantum case is more regular. This phenomenon, discovered two decades ago [1,2], as well as the features of the quantum mechanical suppression of classical chaos and the semiclassical approximation ($\hbar \rightarrow 0$) are now well understood [3,4].

The “kicked” rotator (standard map) has played an important role in the study of the different features of the classical and quantum case. This model is very useful not only because it elucidates several conceptual differences between these two cases, but also for illustrative purposes. One of the most distinctive features of an underlying chaotic behavior is the diffusive character of the dynamics of the classical action variable in phase space. In the quantum case, this diffusion is always suppressed after a sufficiently long time. On the other hand, it has been shown [5] that, in the case of the kicked rotator, a diffusive behavior is obtained even in the quantum case, if a measurement is performed after every kick. The purpose of this Letter is to investigate this situation in more detail, focusing our attention on the role played by the measurement process in the quantum dynamics. We will see that quantum measurements provoke diffusion in a very large class of kicked systems, even when the corresponding classical dynamics is regular.

We consider the following Hamiltonian (in action-angle variables):

$$H = H_0(p) + \lambda V(x)\delta_T(t), \quad (1)$$

where

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT), \quad (2)$$

with T being the period of the perturbation. The interaction $V(x)$ is defined for $x \in [-\pi, \pi]$, with periodic boundary conditions. This Hamiltonian gives rise to the radial twisting map. This is a wide class of maps, including as a particular case the standard map [6], which describes the local behavior of a perturbed integrable map near resonance. The free Hamiltonian H_0 has a dis-

crete spectrum and a countable complete set of eigenstates $\{|m\rangle\}$,

$$\langle x|m\rangle = \frac{1}{\sqrt{2\pi}} \exp(imx), \quad m = 0, \pm 1, \pm 2, \dots \quad (3)$$

We shall consider the evolution engendered by (1) interspersed with quantum measurements, in the following sense: the system evolves under the action of the free Hamiltonian for $(N-1)T + \tau < t < NT$ ($0 < \tau < T$), undergoes a kick at $t = NT$, evolves again freely, and then undergoes a “measurement” of p at $t = NT + \tau$. The evolution of the density matrix between measurements is

$$\rho_{NT+\tau} = U_{\text{free}}(\tau)U_{\text{kick}}U_{\text{free}}(T-\tau)\rho_{(N-1)T+\tau} \\ \times U_{\text{free}}^\dagger(T-\tau)U_{\text{kick}}^\dagger U_{\text{free}}^\dagger(\tau), \quad (4)$$

$$U_{\text{kick}} = \exp(-i\lambda V/\hbar), \quad U_{\text{free}}(t) = \exp(-iH_0 t/\hbar). \quad (5)$$

At each measurement, the wave function is “projected” onto the n th eigenstate of p with probability

$$P_n(NT + \tau) = \text{Tr}(|n\rangle\langle n|\rho_{NT+\tau}) \quad (6)$$

and the off-diagonal terms of the density matrix disappear. The occupation probabilities $P_n(t)$ change discontinuously at times NT and their evolution is governed by the master equation

$$P_n(N) = \sum_m W_{nm}P_m(N-1), \quad (7)$$

where we defined, with a little abuse of notation,

$$P_n(N) \equiv P_n(NT + \tau) \quad (8)$$

and where

$$W_{nm} \equiv |\langle n|U_{\text{free}}(\tau)U_{\text{kick}}U_{\text{free}}(T-\tau)|m\rangle|^2 \\ = |\langle n|U_{\text{kick}}|m\rangle|^2 \quad (9)$$

are the transition probabilities. Although the map (7) depends on λ, V, H_0 in a complicated way, very general conclusions can be drawn about the average value of a generic regular function of momentum $g(p)$. Let

$$\langle g(p) \rangle_t \equiv \text{Tr}[g(p)\rho(t)] = \sum_n g(p_n)P_n(t), \quad (10)$$

where $p|n\rangle = p_n|n\rangle$ ($p_n = n\hbar$), and consider

$$\langle g(p) \rangle_N = \sum_n g(p_n) P_n(N) = \sum_{n,m} g(p_n) W_{nm} P_m(N-1), \quad (11)$$

where $\langle g(p) \rangle_N \equiv \langle g(p) \rangle_{NT+\tau}$ is the average value of g after N kicks. Substitute W_{nm} from (9) to obtain

$$\begin{aligned} \langle g(p) \rangle_N &= \sum_{n,m} g(p_n) \langle m|U_{\text{kick}}^\dagger|n\rangle \langle n|U_{\text{kick}}|m\rangle P_m(N-1) \\ &= \sum_m \langle m|U_{\text{kick}}^\dagger g(p) U_{\text{kick}}|m\rangle P_m(N-1), \end{aligned} \quad (12)$$

where we used $g(p)|n\rangle = g(p_n)|n\rangle$. We are mostly interested in the evolution of the quantities p and p^2 . By the Baker-Hausdorff lemma

$$\begin{aligned} U_{\text{kick}}^\dagger g(p) U_{\text{kick}} &= g(p) + i \frac{\lambda}{\hbar} [V, g(p)] \\ &\quad + \frac{1}{2!} \left(\frac{i\lambda}{\hbar} \right)^2 [V, [V, g(p)]] + \dots, \end{aligned} \quad (13)$$

we obtain the exact expressions [7]

$$U_{\text{kick}}^\dagger p U_{\text{kick}} = p + i \frac{\lambda}{\hbar} [V, p], \quad (14)$$

$$U_{\text{kick}}^\dagger p^2 U_{\text{kick}} = p^2 + i \frac{\lambda}{\hbar} [V, p^2] + \lambda^2 (V')^2, \quad (15)$$

where prime denotes derivative. Substituting into (12), we get

$$\begin{aligned} \langle p \rangle_N &= \sum_m \langle m| \left(p + i \frac{\lambda}{\hbar} [V, p] \right) |m\rangle P_m(N-1) \\ &= \langle p \rangle_{N-1}, \end{aligned} \quad (16)$$

$$\begin{aligned} \langle p^2 \rangle_N &= \sum_m \langle m| \left(p^2 + i \frac{\lambda}{\hbar} [V, p^2] \right. \\ &\quad \left. + \lambda^2 (V')^2 \right) |m\rangle P_m(N-1) \\ &= \langle p^2 \rangle_{N-1} + \lambda^2 \langle (V')^2 \rangle. \end{aligned} \quad (17)$$

Iterating, we obtain

$$\langle p \rangle_N = \langle p \rangle_0, \quad (18)$$

$$\langle p^2 \rangle_N = \langle p^2 \rangle_0 + \lambda^2 \langle f^2 \rangle_N, \quad (19)$$

where $f = -V'(x)$ is the force and

$$\begin{aligned} \langle f^2 \rangle &= \text{Tr}(f^2 \rho_{NT+\tau}) = \sum_n \langle n|f^2|n\rangle P_n(N) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} dx f^2(x) \end{aligned} \quad (20)$$

is a constant that does not depend on N because $\langle n|f^2|n\rangle$ is independent of the state $|n\rangle$ [see (3)] and $\sum P_n = 1$. In particular, the kinetic energy $K = p^2/2m$ grows at a constant rate: $\langle K \rangle_N = \langle K \rangle_0 + \lambda^2 \langle f^2 \rangle N/2m$. By using (18) and (19) we obtain the friction coefficient

$$F = \frac{\langle p \rangle_N - \langle p \rangle_0}{NT} = 0 \quad (21)$$

and the diffusion coefficient

$$D = \frac{\langle \Delta p^2 \rangle_N - \langle \Delta p^2 \rangle_0}{NT} = \frac{\lambda^2 \langle f^2 \rangle}{T}, \quad (22)$$

where $\langle \Delta p^2 \rangle_N = \langle p^2 \rangle_N - \langle p \rangle_N^2$. The above results are *exact*: their derivation involves no approximation. This shows that this class of Hamiltonian systems, if measured after every kick, has a constant diffusion rate in momentum with no friction, for *any* perturbation $V = V(x)$.

In particular, in the seminal kicked-rotator model, one gets ($H_0 = p^2/2I$ and $V = \cos x$)

$$D = \frac{\lambda^2}{2T} : \quad (23)$$

this is nothing but the diffusion constant obtained in the classical case [1,5]. Notice that one obtains the quasilinear diffusion constant without higher-order correction [6].

The above results may seem somewhat puzzling, essentially because one finds that in the quantum case, when repeated measurements of momentum (action variable) are performed on the system, a chaotic behavior is obtained for *every* value of λ and for *any* potential $V(x)$. On the other hand, in the classical case, diffusion occurs only for some $V(x)$, when λ exceeds some critical value λ_{crit} . (For instance, the kicked rotator displays diffusion for $\lambda \geq \lambda_{\text{crit}} \approx 0.972$ [1,6].) It appears, therefore, that quantum measurements not only yield a chaotic behavior in a quantum context, they even produce chaos when the classical motion is regular. In order to bring to light the causes of this peculiar situation, it is necessary to look at the classical case. The classical map for the Hamiltonian (1) reads

$$\begin{aligned} x_N &= x_{N-1} + H'_0(p_{N-1})T, \\ p_N &= p_{N-1} - \lambda V'(x_N). \end{aligned} \quad (24)$$

A quantum measurement of p yields an exact determination of momentum p and, as a consequence, makes position x completely undetermined (uncertainty principle). This situation has no classical analog: it is inherently quantal. However, the classical "map" that best mimics this physical picture is obtained by assuming that position x_N at time τ after each kick (i.e., when the quantum counterpart undergoes a measurement) behaves like a random variable ξ_N uniformly distributed over $[-\pi, \pi]$,

$$\begin{aligned} x_N &= \xi_N, \\ p_N &= p_{N-1} - \lambda V'(\xi_N). \end{aligned} \quad (25)$$

Introducing the ensemble average $\langle\langle \dots \rangle\rangle$ over the stochastic process (i.e., over the set of independent random variables $\{\xi_k\}_{k \leq N}$), it is straightforward to obtain

$$\begin{aligned} \langle\langle p_N \rangle\rangle &= \langle\langle p_{N-1} \rangle\rangle - \lambda \langle V'(\xi_N) \rangle, \\ \langle\langle \Delta p_N^2 \rangle\rangle &= \langle\langle \Delta p_{N-1}^2 \rangle\rangle + \lambda^2 [\langle V'(\xi_N)^2 \rangle - \langle V'(\xi_N) \rangle^2], \end{aligned} \quad (26)$$

where $\Delta p_N^2 = p_N^2 - \langle\langle p_N \rangle\rangle^2$ and

$$\langle g(\xi) \rangle \equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\xi) d\xi \quad (27)$$

is the average over the single random variable ξ [this coincides with the quantum average: see, for instance, the last term of (20)]. In deriving (26), the average of $V'(\xi_N)p_{N-1}$ was factorized because p_{N-1} depends only on $\{\xi_k\}_{k \leq N-1}$. The average of $V'(\xi_N)$ in (26) vanishes due to the periodic boundary conditions on V , so that

$$\langle \langle \Delta p_N^2 \rangle \rangle = \langle \langle \Delta p_{N-1}^2 \rangle \rangle + \lambda^2 \langle f^2 \rangle \quad (28)$$

and the momentum diffuses at the rate (22), as in the quantum case with measurements. What we obtain in this case is a diffusion taking place in the whole phase space, without effects due to the presence of adiabatic islands.

It is interesting to frame our conclusion in a proper context, by comparing the different cases analyzed: (A) a classical system, under the action of a suitable kicked perturbation, displays a diffusive behavior if the coupling constant exceeds a certain threshold (KAM theorem); (B) on the other hand, in its quantum counterpart, this diffusion is always suppressed. (C) The introduction of measurements between kicks encompasses this limitation, yielding diffusion in the quantum case. More so, diffusion takes place for any potential and all values of the coupling constant (namely, even when the classical motion is regular). (D) The same behavior is displayed by a “randomized classical map,” in the sense explained above. These conclusions are sketched in Table I. As we have seen, the effect of measurements is basically equivalent to a complete randomization of the classical angle variable x , at least for the calculation of the diffusion coefficient in the chaotic regime. There are two points which deserve clarification. Indeed, one might think the following: (i) The randomized classical map (25) and the quantum map with measurements (7), (18)–(22) are identical; (ii) the diffusive features in a quantum context are to be ascribed to the projection process (6) (hence to a nonunitary dynamics). Both expectations would be incorrect. As for (i), there are

corrections in \hbar : it is indeed straightforward to show that the two maps have equal moments up to third order, while the fourth moment displays a difference of order $O(\hbar^2)$,

$$\langle p^4 \rangle_N - \langle p^4 \rangle_{N-1} = \langle \langle p_N^4 \rangle \rangle - \langle \langle p_{N-1}^4 \rangle \rangle + \lambda^2 \hbar^2 \langle (f')^2 \rangle. \quad (29)$$

As for (ii), it suffices to observe that the very same results can be obtained by making use only of a purely unitary evolution (albeit, as we will see, of a larger system). To this end, we must give a model for measurement, by looking more closely at the physics of such a process. When a quantum measurement is performed, the relevant information is recorded in an apparatus. For example, the measured system scatters one or more photons (phonons) and each p eigenstate gets entangled with the photon (phonon) wave function. A process of this sort can be schematized by associating an additional degree of freedom (a “spin” is the simplest possible case) with every momentum eigenstate, at time τ after every kick. This is easily accomplished by adding the following “decomposition” [8] Hamiltonian to (1):

$$H_{\text{dec}} = \frac{\pi}{2} \sum_{n,k} |n\rangle \langle n| \otimes \sigma^{(n,k)} \delta(t - kT - \tau), \quad (30)$$

where $|n\rangle$ is an eigenstate of p and $\sigma^{(n,k)} \mathbf{V}(n,k)$ is the first Pauli matrix, whose action is given by

$$\sigma^{(n,k)} |\pm\rangle_{(n,k)} = |\mp\rangle_{(n,k)}, \quad (31)$$

where $|+\rangle_{(n,k)}, |-\rangle_{(n,k)}$ denote spin up, spin down, respectively, in “channel” (n,k) . Let us prepare the total (rotator + spins) system in the initial ($t = 0^+$) state

$$|\Psi_{\text{in}}\rangle = \sum_m c_m |m\rangle \bigotimes_{k,n} |-\rangle_{(n,k)} \quad (32)$$

(all spins down). For the sake of simplicity, we shall concentrate our attention on the first two kicks. In the same notation as in (8), the evolution of the state $|\Psi(N)\rangle \equiv |\Psi(NT + \tau^+)\rangle$ reads

$$|\Psi(0)\rangle = -i \sum_m c'_m |m\rangle \otimes |+\rangle_{(m,0)} \bigotimes_{k \geq 1, n} |-\rangle_{(n,k)}, \quad (33)$$

$$|\Psi(1)\rangle = (-i)^2 \sum_{\ell, m} |\ell\rangle \otimes |+\rangle_{(\ell,1)} \otimes A_{\ell m} c'_m |+\rangle_{(m,0)} \bigotimes_{k \geq 2, n} |-\rangle_{(n,k)}, \quad (34)$$

$$|\Psi(2)\rangle = (-i)^3 \sum_{j, \ell, m} |j\rangle \otimes |+\rangle_{(j,2)} \otimes A_{j\ell} |+\rangle_{(\ell,1)} \otimes A_{\ell m} c'_m |+\rangle_{(m,0)} \bigotimes_{k \geq 3, n} |-\rangle_{(n,k)}, \quad (35)$$

where $c'_m = c_m \exp[-iH_0(p_m)\tau]$ and

$$A_{\ell m} \equiv \langle \ell | U_{\text{free}}(\tau) U_{\text{kick}} U_{\text{free}}(T - \tau) | m \rangle \quad (36)$$

is the transition amplitude ($W_{\ell m} = |A_{\ell m}|^2$). We see that at time τ after the k th kick, the n th eigenstate of the system becomes associated with spin up in channel (n,k) .

By using (34) and (35) one readily shows that the occupation probabilities evolve according to

$$\begin{aligned} P_n(2) &\equiv \langle \Psi(2) | (|n\rangle \langle n| \otimes \mathbf{1}_{\text{spins}}) | \Psi(2) \rangle \\ &= \sum_m W_{nm} P_m(1). \end{aligned} \quad (37)$$

TABLE I. Classical vs quantum diffusion.

A	classical	diffusion for $\lambda > \lambda_{\text{crit}}$
B	quantum	no diffusion
C	quantum + measurements	diffusion $\forall \lambda$
D	classical + random	diffusion $\forall \lambda$

The generalization to N kicks is straightforward and it is very easy to obtain the same master equation (7). The observables of the quantum particle evolve therefore as in (11): in particular, the average value of the quantum observable $\bar{p} = p \otimes \mathbf{1}_{\text{spins}}$ displays diffusion with coefficients (21) and (22). This shows that the *unitary* dynamics engendered by (1) and (30) yields the same quantal diffusive behavior that is obtained by making use of “projections” (6). We notice indeed that, although the combined system (rotator + spins) evolves unitarily, if one chooses to “look” only at the rotator, by tracing away the spin degrees of freedom, the resulting dynamics (37) is nonunitary. This is in line with the projection postulate [9], used in the first part of this Letter [up to Eq. (29)].

Our analysis can be easily generalized to radial twisting maps in higher dimensions. It would be interesting to extend it to a slightly different class of Hamiltonians, such as those used in [10] to analyze the effect of an oscillating perturbation on an atomic system.

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