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Particle tracks and the mechanism of decoherence in a model bubble chamber

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Abstract

We put forward a toy model for a "bubble chamber" and study its interaction with an incoming object particle. We discuss the notion of particle "tracks" inside the bubble chamber and analyze the mechanisms that provoke a loss of quantum mechanical coherence (decoherence). The model is solvable and provides interesting insights into some of the most salient features of the interaction between a microscopic particle and a macroscopic device. © 1998 Published by Elsevier Science B.V.

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1. Introduction

The analysis of the interaction between an elementary quantum mechanical system and a macroscopic object is a challenging and interesting problem: On one hand, it is obvious that there are situations in which a macroscopic object follows classical, rather than quantum mechanical laws. On the other hand, one should prove, and not postulate, that there are physical situations in which classical mechanics is a good approximation: In other words, one should attain a coherent description of macroscopic, "classical" objects *within* the quantum mechanical framework.

The quantum measurement problem [1,2] is deeply related to the above issue. In the present paper we shall investigate this topic by means of a toy model describing the interaction between an incoming particle Q and a "bubble chamber" B. Our work has two goals. First, we shall investigate in which sense it is possible to recover the notion of "track" of the Q particle inside B. Second, we shall endeavour to clarify which mechanisms provoke a loss of quantum mechanical coherence (decoherence) on our system.

The study of explicit models [3–8] has always turned out to be very useful in this context, for it has helped clarifying several delicate aspects of the measurement process and of the appearance of irreversibility in quantum mechanics. The starting point of this paper will be a famous model Hamiltonian, proposed by Hepp [3] about 20 years ago, that has played an important role in the analysis of quantum measurement processes because

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it is solvable and yields, in spite of its simplicity, nice physical insights [9–11]. This Hamiltonian describes the interaction between an ultrarelativistic particle Q and a one-dimensional array made up of N spins of magnitude $\frac{1}{2}$. Each spin represents a two-level (AgBr) molecule, and each molecule, by interacting with Q, can be dissociated (into Ag + Br atoms), so that the array shows trace of its interaction with Q and plays the role of a "detector". The abovementioned Hamiltonian is usually referred to as Coleman–Hepp (CH) or "AgBr" Hamiltonian.

The CH Hamiltonian was recently modified [12] in order to enable it to take into account the intrinsic energy of the detector and the energy-exchange processes between the Q particle and each molecule. These features bring to light interesting connections [12,13] with the Jaynes–Cummings [4] Hamiltonian, as well as with another model Hamiltonian, proposed by Cini a few years ago [5]. Additional contributions to the study of the CH Hamiltonian or its modified version were also given by other authors [14,15].

The modified CH model was also extended to arbitrary spin s [16], yielding interesting results. In particular, the case $s \gg 1$ turns out to be noteworthy, because as far as low-lying excited states are concerned, each spin may be replaced by a boson and one is left with an array of boson sources (or harmonic oscillators). This is accomplished by making use of the Holstein–Primakoff transformation [17].

The aim of this work is to analyze the large-spin limit of the modified CH Hamiltonian, put forward an interesting one-dimensional toy model for a bubble chamber and analyze the resulting Schrödinger equation. This paper is organized as follows: We introduce the model Hamiltonian for the bubble chamber in Section 2 and discuss the idea of particle "tracks" in Section 3. A typical double slit experiment and the loss of quantum mechanical coherence (decoherence) are analyzed in Section 4. Our model bubble chamber is extended to the continuous case in Section 5. Section 6 contains some concluding remarks.

2. Model for a bubble chamber

We start from the large-spin limit of the modified Coleman–Hepp Hamiltonian

$$H = H_0 + H', \qquad H_0 = H_Q + H_B, \tag{2.1}$$

where H_Q and H_B are the free Hamiltonians of the Q particle and of the array, respectively, and H' is the interaction Hamiltonian. (The subscript B stands for "bubble chamber", as the array will henceforth be referred to.) The explicit expressions are

$$H_Q = c\widehat{p}, \qquad H_B = \sum_{\ell=1}^{N} \hbar\omega a_\ell^\dagger a_\ell, \qquad H' = \sum_{\ell=1}^{N} u(\widehat{x} - x_\ell)\left[a_\ell \exp(\mathrm{i}\omega\widehat{x}/c) + \text{h.c.}\right], \tag{2.2}$$

where $\widehat{p}$ is the momentum of the (ultrarelativistic) Q particle, $\widehat{x}$ its position, a_ℓ and $a_\ell^\dagger$ boson annihilation and creation operators of the oscillator at the ℓth site, u is a real function, x_ℓ ($\ell = 1, \dots, N$) are the positions of the oscillators in the array ($x_{\ell+1} > x_\ell$), and we made use of the caret only for the position and momentum operators.

The model described above can be viewed as a caricature of a bubble chamber, in the following sense: At every site ℓ there is a harmonic oscillator that interacts with the Q particle. After the interaction is over, the state of the array has changed and the "bubble chamber" has recorded some signature of the passage of Q.

The Hamiltonian (2.2) can be viewed as a linear chain of N Jaynes–Cummings Hamiltonians, with the important difference that the latter contains terms of the type $\tau_\pm$, instead of $\exp(\pm\mathrm{i}\omega\widehat{x}/c)$, $\tau_\pm$ being the raising/lowering operators for a two-level system. In the case we are considering, the Q-particle has a continuous spectrum, and can exchange an arbitrary number of quanta of energy $\hbar\omega$.

An alternative physical picture of $u(x)$ was suggested in Ref. [16], where it was shown that in a suitably chosen rotating frame H' reduces to

$$\sum_\ell u(\hat{x} - x_\ell) q_\ell, \qquad q_\ell \equiv (a_\ell + a_\ell^\dagger)/\sqrt{2} \tag{2.3}$$

and $u(x)$ is nothing but the force exerted on the ℓth harmonic oscillator.

The ground state of the array is defined by

$$a_\ell |0\rangle_N = 0, \qquad \forall \ell = 1, 2, \ldots, N \tag{2.4}$$

and the Hilbert space is spanned by the vectors

$$|n_1, n_2, \ldots, n_N\rangle = \prod_{\ell=1}^{N} \frac{(a_\ell^\dagger)^{n_\ell}}{\sqrt{n_\ell!}} |0\rangle_N, \qquad n_\ell \in \mathbb{N}, \qquad \ell = 1, 2, \ldots, N. \tag{2.5}$$

The interaction Hamiltonian in the interaction picture can be computed exactly as

$$H'_I(t) = e^{iH_0 t/\hbar} H' e^{-iH_0 t/\hbar} = \sum_{\ell=1}^{N} u(\hat{x} + ct - x_\ell) [a_\ell \exp(i\omega\hat{x}/c) + \text{h.c.}]. \tag{2.6}$$

(In the interaction picture $\hat{x} \to \hat{x} + ct$, while $a_\ell \to a_\ell \exp(-i\omega t)$, so that $a_\ell \exp(i\omega\hat{x}/c)$ and its hermitian conjugate remain unchanged.) Since

$$[H'_I(t), H'_I(t')] = 0, \tag{2.7}$$

no ordering problems arise with the operators. The evolution operator in the interaction picture reads

$$\begin{aligned} U(t,t') &= \exp\left(-\frac{i}{\hbar} \int_{t'}^{t} H'_I(t'')\, dt''\right) \\ &= \exp\left(-\frac{i}{\hbar} \int_{t'}^{t} dt'' \sum_{\ell=1}^{N} u(\hat{x} + ct'' - x_\ell) [a_\ell \exp(i\omega\hat{x}/c) + \text{h.c.}]\right), \end{aligned} \tag{2.8}$$

and a straightforward calculation yields the S-matrix

$$\begin{aligned} S = \lim_{\substack{t\to\infty \\ t'\to-\infty}} U(t,t') &= \exp\left(-i\frac{u_0\Omega}{\hbar c} \sum_{\ell=1}^{N} [a_\ell \exp(i\omega\hat{x}/c) + \text{h.c.}]\right) \\ &= \exp(-i\lambda[b_N \exp(i\omega\hat{x}/c) + \text{h.c.}]), \end{aligned} \tag{2.9}$$

where $u_0\Omega \equiv \int dz\, u(z)$, and we defined

$$\lambda \equiv \frac{u_0\Omega}{\hbar c}\sqrt{N}, \tag{2.10}$$

$$b_N \equiv \frac{1}{\sqrt{N}} \sum_{\ell=1}^{N} a_\ell, \qquad [b_N, b_N^\dagger] = 1. \tag{2.11}$$

The operator $b_N^\dagger$ creates global boson-like excitations of the whole system B. Let the initial state of the total (Q + B) system be $|p, 0\rangle_N \equiv |p\rangle|0\rangle_N$, where $|p\rangle$ is a plane wave of Q and $|0\rangle_N$ the ground state of the bubble chamber. The structure (2.9) of the S-matrix and the commutator (2.11) yield the final state

$$S|p,0\rangle_N = e^{-\lambda^2/2} \sum_{j=0}^{\infty} \frac{(-i\lambda)^j}{\sqrt{j!}} |p_j, j\rangle_N \equiv |\lambda)_N, \tag{2.12}$$

where $|p_j, j\rangle_N \equiv |p_j\rangle|j\rangle_N$, $p_j \equiv p - j\hbar\omega/c$ and $|j\rangle_N$ is the "number state" of the bubble chamber, defined by

$$|j\rangle_N \equiv \frac{(b_N^\dagger)^j}{\sqrt{j!}}|0\rangle_N. \tag{2.13}$$

This result is exact. It is worth noting that the appearance of a coherent state [18] in the above formula (2.12) is a general characteristic of Jaynes–Cummings-like and CH-like [10–12] Hamiltonians. We can also easily compute the energy lost by the particle and acquired by the bubble chamber after the interaction is over ($t \gg x_N/c$). A straightforward calculation yields

$$\langle H_{\rm B}\rangle_{\rm F} = \lambda^2\hbar\omega, \tag{2.14}$$

where F denotes the final state (2.12). As was to be expected, the above equations allow us to interpret λ^2 as the average number of "global" boson excitations in the bubble chamber.

3. Particle "tracks"

Interestingly, it is possible to put forward the notion of "track" of the Q particle inside the bubble chamber. Such an interpretation holds only in the following *statistical* sense. Consider a number state of the bubble chamber, according to Eqs. (2.11) and (2.13)

$$\begin{aligned}|n\rangle_N &= \frac{(b_N^\dagger)^n}{\sqrt{n!}}|0\rangle_N = \frac{1}{\sqrt{N}}\sum_{\ell_1=1}^{N}\cdots\frac{1}{\sqrt{N}}\sum_{\ell_n=1}^{N}\frac{a_{\ell_1}^\dagger\cdots a_{\ell_n}^\dagger}{\sqrt{n!}}|0\rangle_N \\ &= N^{-n/2}\{|n,0,\ldots,0\rangle + \sqrt{n}\,[\,|n-1,1,0,\ldots,0\rangle + |0,n-1,1,0,\ldots,0\rangle + \ldots] + \ldots \\ &\quad + \sqrt{n!}\,[\,|1,\ldots,1,0,\ldots,0\rangle + |0,1,\ldots,1,0,\ldots,0\rangle + \ldots]\},\end{aligned} \tag{3.1}$$

where each ket has N arguments, in accordance with (2.5), and the vectors in the last term contain n 1's and $N-n$ 0's. If $n = O(N) \gg 1$, one can say that "the most probable contribution" to $|n\rangle_N$ comes from the last term in (3.1). This is, in turn, the most probable contribution to the coherent state $|\lambda\rangle_N$, if n is the closest integer to $\lambda^2 = (u_0\Omega/\hbar c)^2 N$.

For the sake of clarity, let us look explicitly at the case $N = 6$, $n = 4$. It is helpful to think of the ket itself as a linear bubble chamber. The most probable contribution to $|n\rangle_N$ is

$$\begin{aligned}&|1,1,1,1,0,0\rangle + |1,1,1,0,1,0\rangle + |1,1,1,0,0,1\rangle + |1,1,0,1,0,1\rangle + |1,1,0,0,1,1\rangle \\ &\quad + \ldots \qquad (6!\ \text{terms}),\end{aligned} \tag{3.2}$$

rather than, say,

$$|4,0,0,0,0,0\rangle + |0,4,0,0,0,0\rangle + \ldots \qquad (6\ \text{terms}),$$

or

$$|2,2,0,0,0,0\rangle + |2,0,2,0,0,0\rangle + \ldots \qquad (6!/(2!)^2\ \text{terms}). \tag{3.3}$$

It seems to us that the above description gives "the most classical looking quantum picture of a track" obtainable without the artificial postulate of reduction of quantum states. The result (3.1) says that a "track" is a superposition which is "irreducible" any further. It is a virtue of the model under consideration that it displays this salient feature of a track in a clear-cut way. In some sense, the above considerations may be said to be complementary to the classic work of Mott [19], who endeavored to "picture how it is that an outgoing

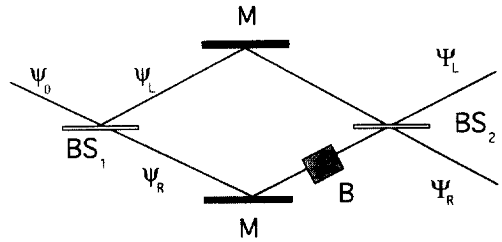

Fig. 1. A typical interference experiment. A Q-wave packet ψ_0 is coherently split by a beam splitter BS_1 into two branch waves ψ_L and ψ_R. Only ψ_R interacts with our bubble chamber B. The two waves Ψ_L and Ψ_R (which *include* the state of B, as well) are finally recombined at a second beam splitter BS_2. The M's are perfect mirrors. We denoted by R (L) *both* the lower (upper) route of the interferometer and the lower (upper) detection channel after recombination at BS_2.

spherical wave can produce a straight track" by "considering only two atoms" and showing that "the atoms cannot both be ionised unless they lie in a straight line with the radioactive nucleus". Here emphasis is on the linear shape (in space) of the track, while in our model emphasis is on the coherent character of the state describing the track, the spatial shape being out of the scope of the model.

4. Double slit experiment and decoherence

Let us analyze now a completely different aspect of the interaction between Q and B: We will focus our attention on the experiment sketched in Fig. 1. We take the incoming Q-state ψ_0 to be a Gaussian wave packet, that is initially well localized far away from the bubble chamber:

$$\langle x|\psi_0\rangle = \psi_0(x) = \left(\frac{1}{\pi\Delta^2}\right)^{1/4} e^{-x^2/2\Delta^2}, \tag{4.1}$$

$$\langle p|\psi_0\rangle = \tilde{\psi}_0(p) = \left(\frac{\Delta^2}{\pi\hbar^2}\right)^{1/4} e^{-p^2\Delta^2/\hbar^2}. \tag{4.2}$$

By interacting with the divider BS_1, ψ_0 is split into two branch waves ψ_L and ψ_R in the following way,

$$\psi_1 \equiv \begin{pmatrix} 0 \\ \psi_0 \end{pmatrix} \xrightarrow{BS_1} \begin{pmatrix} \hat{t}^\dagger & \hat{r} \\ -\hat{r}^\dagger & \hat{t} \end{pmatrix} \begin{pmatrix} 0 \\ \psi_0 \end{pmatrix} = \begin{pmatrix} r\psi_0' \\ t\psi_0 \end{pmatrix} \equiv \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \tag{4.3}$$

with $\hat{t}$ and $\hat{r}$ being operators such that

$$\hat{t}\psi_0 = t\psi_0, \qquad \hat{r}\psi_0 = r\psi_0', \qquad \hat{t}\psi_0' = t\psi_0', \qquad \hat{r}\psi_0' = r\psi_0, \tag{4.4}$$

where r and t are the reflection and transmission coefficients of BS_1 ($|r|^2 + |t|^2 = 1$). Here ψ_0' is a Gaussian wave packet (with the same characteristics of (4.1), (4.2), but in which the supports of the variables p and x are totally different from those of ψ_0 and do not overlap with them), propagating along the left route. Notice that the action of BS_1 is represented by a unitary transformation that completely preserves the quantum coherence, as it should. If we place no bubble chamber along the right route, the two branch waves are simply recombined by making use of the beam splitter BS_2, to yield

$$\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \xrightarrow{BS_2} \begin{pmatrix} \hat{t} & -\hat{r} \\ \hat{r}^\dagger & \hat{t}^\dagger \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} 0 \\ \psi_0 \end{pmatrix} \equiv \psi_1. \tag{4.5}$$

For simplicity, and without loss of generality, BS_2 has been taken to act as the "inverse" of BS_1, and the (trivial) effects of the mirrors M on the wave packets have not been considered. Under these conditions, and with no phase shifter in the interferometer, the particle will be detected in the R-channel, after recombination at BS_2. Notice also that the wave packets do not disperse, due to the form of the free Hamiltonian H_Q in (2.2).

Let us now "switch on" the bubble chamber and suppose that it is initially set in its ground state $|0\rangle_N$. The interaction Hamiltonian in (2.2) is conveniently rewritten as

$$H' = \frac{1}{2}(1-\tau_3)\sum_{\ell=1}^{N} u(\widehat{x}-x_\ell)[a_\ell \exp(i\omega\widehat{x}/c) + \text{h.c.}], \tag{4.6}$$

where τ_3 is the third Pauli matrix and the projector $\frac{1}{2}(1-\tau_3)$ ensures that only ψ_R interacts with B. A straightforward calculation yields the following final state for the whole $(Q+B)$ system,

$$|\Psi_I\rangle \equiv \begin{pmatrix} 0 \\ |\psi_0, 0\rangle_N \end{pmatrix} \quad\rightarrow\quad |\Psi_F\rangle \equiv \begin{pmatrix} tr|\psi_0', 0\rangle_N - rtS|\psi_0', 0\rangle_N \\ |r|^2|\psi_0, 0\rangle_N + |t|^2 S|\psi_0, 0\rangle_N \end{pmatrix} \equiv \begin{pmatrix} |\Psi_L\rangle \\ |\Psi_R\rangle \end{pmatrix}, \tag{4.7}$$

where $|\psi, 0\rangle_N \equiv |\psi\rangle \otimes |0\rangle_N$. The final density matrix of the total system reads

$$\Xi_F \equiv |\Psi_F\rangle\langle\Psi_F| = \begin{pmatrix} |\Psi_L\rangle\langle\Psi_L| & |\Psi_L\rangle\langle\Psi_R| \\ |\Psi_R\rangle\langle\Psi_L| & |\Psi_R\rangle\langle\Psi_R| \end{pmatrix}. \tag{4.8}$$

At this point, one usually traces over the states of the macroscopic object (B, in our case) and looks at the interference pattern (i.e. the coherent features) of Q. By contrast, we shall take a completely different attitude and trace over the Q-states: In this way, Q will act as a "probe" and will enable us to investigate the coherence properties of the macroscopic device B. The following calculation is done in the same spirit of Refs. [11,16]. The reduced density matrix of the bubble chamber is easily computed as

$$\rho_F \equiv \text{Tr}_Q\, \Xi_F = \begin{pmatrix} \int dp_L \langle p_L|\Psi_L\rangle\langle\Psi_L|p_L\rangle & 0 \\ 0 & \int dp_R \langle p_R|\Psi_R\rangle\langle\Psi_R|p_R\rangle \end{pmatrix} \equiv \begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{R} \end{pmatrix}, \tag{4.9}$$

where the integration domains of the two integrals are the supports of the two wave packets, respectively, and

$$\mathcal{R} = \int dp\, \langle p|[\,|r|^4|\psi_0, 0\rangle_N\, {}_N\langle 0, \psi_0| + |r|^2|t|^2 S|\psi_0, 0\rangle_N\, {}_N\langle 0, \psi_0| + \text{h.c.} + |t|^4 S|\psi_0, 0\rangle_N\, {}_N\langle 0, \psi_0|S^\dagger\,]|p\rangle, \tag{4.10}$$

$$\mathcal{L} = |r|^2|t|^2 \int dp\, \langle p|[\,|\psi_0', 0\rangle_N\, {}_N\langle 0, \psi_0'| - S|\psi_0', 0\rangle_N\, {}_N\langle 0, \psi_0'| - \text{h.c.} + S|\psi_0', 0\rangle_N\, {}_N\langle 0, \psi_0'|S^\dagger\,]|p\rangle. \tag{4.11}$$

(We used the same integration variable in the two equations above, because no confusion can arise.) Let us now compute the contributions of the terms in the r.h.s. of (4.10) and (4.11), by making use of the explicit expression of $|\Psi_{L,R}\rangle$. For the first term of (4.10) we simply obtain

$$\int dp\, \langle p|\psi_0, 0\rangle_N\, {}_N\langle 0, \psi_0|p\rangle = |0\rangle_N\, {}_N\langle 0|. \tag{4.12}$$

For the second term

$$\int \mathrm{d}p\, \langle p|S|\psi_0, 0\rangle_{N\,N}\langle 0, \psi_0|p\rangle = \int \mathrm{d}p\, \mathrm{d}p'\, \mathrm{d}p''\, \tilde{\psi}_0^*(p'')\, \tilde{\psi}_0(p')\langle p|S|p', 0\rangle_{N\,N}\langle 0, p''|p\rangle$$
$$= \mathrm{e}^{-\lambda^2/2} \sum_{j=0}^{\infty} \frac{(-\mathrm{i}\lambda)^j}{\sqrt{j!}} \int \mathrm{d}p\, \mathrm{d}p'\, \tilde{\psi}_0^*(p)\, \tilde{\psi}_0(p')\langle p|p' - j\hbar\omega/c\rangle \otimes |j\rangle_{N\,N}\langle 0|$$
$$= \mathrm{e}^{-\lambda^2/2} \sum_{j=0}^{\infty} \frac{(-\mathrm{i}\lambda)^j}{\sqrt{j!}} \int \mathrm{d}p'\, \tilde{\psi}_0^*(p' - j\hbar\omega/c)\, \tilde{\psi}_0(p')|j\rangle_{N\,N}\langle 0|$$
$$= \mathrm{e}^{-\lambda^2/2} \sum_{j=0}^{\infty} \frac{(-\mathrm{i}\lambda)^j}{\sqrt{j!}} \mathrm{e}^{-(j\omega\Delta/2c)^2} |j\rangle_{N\,N}\langle 0|. \tag{4.13}$$

The third term is nothing but the Hermitian conjugate of (4.13). The last term in (4.10) is easily evaluated by making use of a procedure similar to the previous one and reads

$$\int \mathrm{d}p\, \langle p|S|\psi_0, 0\rangle_{N\,N}\langle 0, \psi_0|S^\dagger|p\rangle = \int \mathrm{d}p\, \mathrm{d}p'\, \mathrm{d}p''\, \tilde{\psi}_0^*(p'')\, \tilde{\psi}_0(p')\langle p|S|p', 0\rangle_{N\,N}\langle 0, p''|S^\dagger|p\rangle$$
$$= \mathrm{e}^{-\lambda^2} \sum_{j,k=0}^{\infty} \frac{(-\mathrm{i})^{j-k}\lambda^{j+k}}{\sqrt{j!}\sqrt{k!}} \mathrm{e}^{-[(j-k)\omega\Delta/2c]^2} |j\rangle_{N\,N}\langle k|. \tag{4.14}$$

Analogously for the terms of (4.11). By substituting these expressions in (4.10) and (4.11) we finally obtain

$$\mathcal{R} = |r|^4 |0\rangle_{N\,N}\langle 0| + |r|^2|t|^2 \mathrm{e}^{-\lambda^2/2} \sum_{j=0}^{\infty} \frac{(-\mathrm{i}\lambda)^j}{\sqrt{j!}} \mathrm{e}^{-(j\omega\Delta/2c)^2} |j\rangle_{N\,N}\langle 0| + \text{h.c.}$$
$$+ |t|^4 \mathrm{e}^{-\lambda^2} \sum_{j,k=0}^{\infty} \frac{(-\mathrm{i})^{j-k}\lambda^{j+k}}{\sqrt{j!}\sqrt{k!}} \mathrm{e}^{-[(j-k)\omega\Delta/2c]^2} |j\rangle_{N\,N}\langle k|, \tag{4.15}$$

$$\mathcal{L} = |r|^2|t|^2 \Bigg(|0\rangle_{N\,N}\langle 0| - \mathrm{e}^{-\lambda^2/2} \sum_{j=0}^{\infty} \frac{(-\mathrm{i}\lambda)^j}{\sqrt{j!}} \mathrm{e}^{-(j\omega\Delta/2c)^2} |j\rangle_{N\,N}\langle 0| - \text{h.c.}$$
$$+ \mathrm{e}^{-\lambda^2} \sum_{j,k=0}^{\infty} \frac{(-\mathrm{i})^{j-k}\lambda^{j+k}}{\sqrt{j!}\sqrt{k!}} \mathrm{e}^{-[(j-k)\omega\Delta/2c]^2} |j\rangle_{N\,N}\langle k| \Bigg). \tag{4.16}$$

These results are exact and yield the reduced density matrix of B in (4.9). Notice the dependence of the off-diagonal terms on a factor that rapidly (exponentially) decreases as a function of $|j-k|$: The off-diagonal terms (representing the residual quantum mechanical coherence) rapidly vanish as their "distance" from the diagonal ($j = k$) increases.

It is interesting and instructive to compare this result with (4.5). The presence of the apparatus B profoundly modifies the coherence properties of the whole system, so that the L-component does not vanish anymore (or, in other words, there is a nonvanishing probability $p_{\mathrm{L}} = \mathrm{Tr_B}\, \mathcal{L} = 2|r|^2|t|^2[1 - \exp(-\lambda^2/2)] \neq 0$ of observing the Q particle in the L-channel after recombination at $\mathrm{BS_2}$).

Let us now focus our attention on the parameter $\omega\Delta/2c$ and consider the case

$$\eta \equiv \frac{\omega\Delta}{2c} \gg 1, \tag{4.17}$$

that can be viewed as a condition of "slow passage", or "broad packet", or "large energy exchange". In this case, Eqs. (4.15) and (4.16) reduce to

$$\mathcal{R} \xrightarrow{\eta\gg 1} (|r|^2 + |t|^2 \mathrm{e}^{-\lambda^2/2})^2 |0\rangle_{N\,N}\langle 0| + |t|^4 \sum_{j=1}^{\infty} \mathrm{e}^{-\lambda^2} \frac{\lambda^{2j}}{j!} |j\rangle_{N\,N}\langle j|, \tag{4.18}$$

$$\mathcal{L} \xrightarrow{\eta\gg 1} |r|^2|t|^2(1-\mathrm{e}^{-\lambda^2/2})^2|0\rangle_{N\,N}\langle 0| + |r|^2|t|^2\sum_{j=1}^{\infty}\mathrm{e}^{-\lambda^2}\frac{\lambda^{2j}}{j!}|j\rangle_{N\,N}\langle j|. \tag{4.19}$$

In this limit the reduced density matrix is completely diagonal: The quantum mechanical coherence is lost.

It is also possible to define a *degree of coherence* $C(t)$ of the total $(\mathrm{Q}+\mathrm{B})$ system as

$$C(t) \equiv \langle \Psi_0(t)|\Psi(t)\rangle = \langle \Psi_\mathrm{I}|U_\mathrm{int}(t)|\Psi_\mathrm{I}\rangle, \tag{4.20}$$
$$|\Psi(t)\rangle = \mathrm{e}^{-\mathrm{i}Ht/\hbar}|\Psi_\mathrm{I}\rangle, \qquad |\Psi_0(t)\rangle = \mathrm{e}^{-\mathrm{i}H_0t/\hbar}|\Psi_\mathrm{I}\rangle,$$

where $\Psi_0(t)$, $\Psi(t)$ are the evolved states of the total system under the action of the free and the total Hamiltonian, respectively, Ψ_I is the initial state in (4.7) and $U_\mathrm{int} \equiv \exp(\mathrm{i}H_0t)\exp(-\mathrm{i}Ht)$ is the evolution operator in the interaction picture. In the $t\to\infty$ limit, (4.20) becomes

$$C(\infty) \equiv \langle \Psi_\mathrm{I}|S|\Psi_\mathrm{I}\rangle, \tag{4.21}$$

where S is the S-matrix and $C(\infty)$ represents the degree of coherence of the system at the end of the interaction. $C(\infty)$ can be easily computed by noting that

$$S|\Psi_\mathrm{I}\rangle = \sum_j \int \mathrm{d}p\, S|p,j\rangle_{N\,N}\langle j,p|\Psi_\mathrm{I}\rangle = \int \mathrm{d}p\, S|p,0\rangle_N\tilde{\psi}_0(p), \tag{4.22}$$

since ${}_N\langle j,p|\Psi_\mathrm{I}\rangle = \delta_{0j}\tilde{\psi}_0(p)$. Hence, by use of (2.12) and (4.22) we find

$$C(\infty) = \langle \Psi_\mathrm{I}|S|\Psi_\mathrm{I}\rangle = \mathrm{e}^{-\lambda^2/2}\int \mathrm{d}p\,\tilde{\psi}_0^*(p)\,\tilde{\psi}_0(p) = \mathrm{e}^{-\lambda^2/2} = \exp\left(-\frac{\langle H_\mathrm{B}\rangle_\mathrm{F}}{2\hbar\omega}\right), \tag{4.23}$$

where we took into account the result (2.14). The degree of coherence $C(\infty)$ turns out to be a real number, such that $0 < C(\infty) < 1$. If $C(\infty) = 1$, the system $\mathrm{Q}+\mathrm{B}$ is perfectly coherent. If, on the other hand, $C(\infty) = 0$, the system $\mathrm{Q}+\mathrm{B}$ has lost all its coherence. (As a matter of fact, one can show that, in the model described in this paper, $C(\infty)$ is nothing but the visibility of the interference pattern of the Q particle [12] and therefore must be a real number.)

One realizes that the system decoheres more as λ ($\propto u_0\Omega$), the strength of the interaction between Q and the constituents of B, increases. Alternatively, one can say that the loss of coherence becomes larger as the energy exchanged between the Q particle and the bubble chamber increases. This is physically appealing: The limit of very large $\langle H_\mathrm{B}\rangle_\mathrm{F}$ signifies that the bubble chamber has stored a significant amount of energy, by detecting the route taken by the Q particle. In this sense, $C(\infty)$ plays also the role of a parameter that controls how "effective" a measurement of the Q particle trajectory is.

The quantity C is also closely related to the "decoherence parameter", originally introduced in the context of the "many Hilbert spaces" approach to quantum measurements [20], which allows one to give a quantitative estimate of the loss of quantum coherence (decoherence) undergone by a quantum system as a consequence of its interaction with a macroscopic object (such as our model bubble chamber). For some attempts at defining the decoherence parameter in the context of the AgBr Hamiltonian see Ref. [11]. A related but alternative definition of "coherency" is given in Ref. [16].

5. The continuous case

One might wish to enquire about a continuous model of a bubble chamber composed of a fluid, for example. To this end, define (we use the same symbols as in (2.2) for simplicity)

$$H_B = \hbar\omega \int_0^L \mathrm{d}y\, a(y)^\dagger a(y), \qquad H' = \frac{1}{\sqrt{L}} \int_0^L \mathrm{d}y\, v(\widehat{x} - y)[a(y)\exp(\mathrm{i}\omega\widehat{x}/c) + \mathrm{h.c.}],$$

$$[a(x), a(y)^\dagger] = \delta(x - y), \tag{5.1}$$

where L is the total length of the "bubble chamber". (It turns out to be more convenient to introduce explicitly the normalization factor $L^{-1/2}$ in H', so that v has dimension of energy, like u in Eq. (2.2).) The ground state is defined by

$$a(y)|0\rangle = 0, \qquad \forall y \in \mathbb{R} \tag{5.2}$$

and the Hilbert space is spanned by the vectors

$$|y_1, y_2, \ldots, y_j\rangle = \frac{1}{\sqrt{n!}} a^\dagger(y_1) a^\dagger(y_2) \cdots a^\dagger(y_j)|0\rangle. \tag{5.3}$$

The calculations of the previous section can be carried through in a very similar fashion: The interaction Hamiltonian, the evolution operator (in the interaction picture) and the S-matrix are exactly computed as

$$H_I'(t) = \frac{1}{\sqrt{L}} \int_0^L \mathrm{d}y\, v(\widehat{x} + ct - y)[a(y)\exp(\mathrm{i}\omega\widehat{x}/c) + \mathrm{h.c.}], \tag{5.4}$$

$$U(t, t') = \exp\left(-\frac{\mathrm{i}}{\hbar}\frac{1}{\sqrt{L}} \int_{t'}^{t} \mathrm{d}t'' \int_0^L \mathrm{d}y\, v(\widehat{x} + ct'' - y)[a(y)\exp(\mathrm{i}\omega\widehat{x}/c) + \mathrm{h.c.}]\right), \tag{5.5}$$

$$S = \exp\left(-\mathrm{i}\frac{v_0\Omega}{\hbar c}\frac{1}{\sqrt{L}} \int_0^L \mathrm{d}y\, [a(y)\exp(\mathrm{i}\omega\widehat{x}/c) + \mathrm{h.c.}]\right) = \exp(-\mathrm{i}\mu[b\exp(\mathrm{i}\omega\widehat{x}/c) + \mathrm{h.c.}]), \tag{5.6}$$

where $v_0\Omega = \int \mathrm{d}z\, v(z)$,

$$\mu \equiv \frac{v_0\Omega}{\hbar c} \tag{5.7}$$

plays the same role as λ in Eq. (2.10), and

$$b \equiv \frac{1}{\sqrt{L}} \int_0^L \mathrm{d}y\, a(y), \qquad [b, b^\dagger] = 1, \tag{5.8}$$

is the continuous limit of b_N, in Eq. (2.11). If the initial state is $|p, 0\rangle$, like in the previous section, one gets

$$S|p, 0\rangle = \mathrm{e}^{-\mu^2/2} \sum_{j=0}^{\infty} \frac{(-\mathrm{i}\mu)^j}{\sqrt{j!}} |p_j, j\rangle \equiv |\mu), \tag{5.9}$$

where $|p_j, j\rangle \equiv |p_j\rangle|j\rangle$, and $|j\rangle$, the "number state" of the bubble chamber, is defined by

$$|j\rangle \equiv \frac{(b^\dagger)^j}{\sqrt{j!}}|0\rangle = \prod_{\ell=1}^{j} \frac{1}{\sqrt{L}} \int_0^L \mathrm{d}y_\ell\, |y_1, y_2, \ldots, y_j\rangle. \tag{5.10}$$

Unfortunately, it is not easy to define a "track" of the Q particle, like in Section 3, unless one (rather artificially) divides L into N segments of length δL. If one does so, one is effectively back to the discrete model of Section 3. This is quite a natural circumstance in view of the fact that a real bubble chamber consists of discrete atoms rather than an idealised continuous fluid. All the calculations and results of Section 4 are still valid in this case, provided one formally identifies the vectors $|\ldots\rangle_N$ in Section 2 with $|\ldots\rangle$ in the present section and sets $\mu = \lambda$.

6. Summary and concluding remarks

We studied the interaction between an object particle Q and a model bubble chamber B and showed that it is possible to put forward the notion of "track", in a statistical sense. We also computed the density matrix of the total system, obtained the reduced density matrix of B and discussed its quantum coherence properties. This yielded a quantitative estimate of the loss of quantum mechanical coherence (decoherence) undergone by the macrosystem B as a consequence of its interaction with Q. Finally, we defined a quantitative degree of coherence C for the total system.

Our analysis only involves Schrödinger's equation: We have neither added external ingredients, nor proposed modifications of quantum mechanics. On the other hand, we made use of the technique of partial tracing. A critical discussion of this technique and of the projection postulate [2] is outside the scope of the present paper.

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