XV Workshop on Statistical Mechanics and Nonperturbative Field Theory

Bari

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UNIVERSAL RESULTS FOR TWO-DIMENSIONAL PERCOLATION

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SISSA

Percolation and conformal field theory

- Exact results of conformal field theory (CFT) can be used to determine universal critical properties in 2D

- Complete results for magnetic critical points from "minimal models" of CFT. Correlators satisfy differential equations

- Percolative critical points are not minimal models. However, differential equations are known for *boundary* connectivities $\longrightarrow$ Cardy's crossing probabilities, relation with SLE

Recently we obtained results for *bulk* connectivities (critical and off-critical)

Percolative universality classes

- Random percolation : a site is occupied with probability p

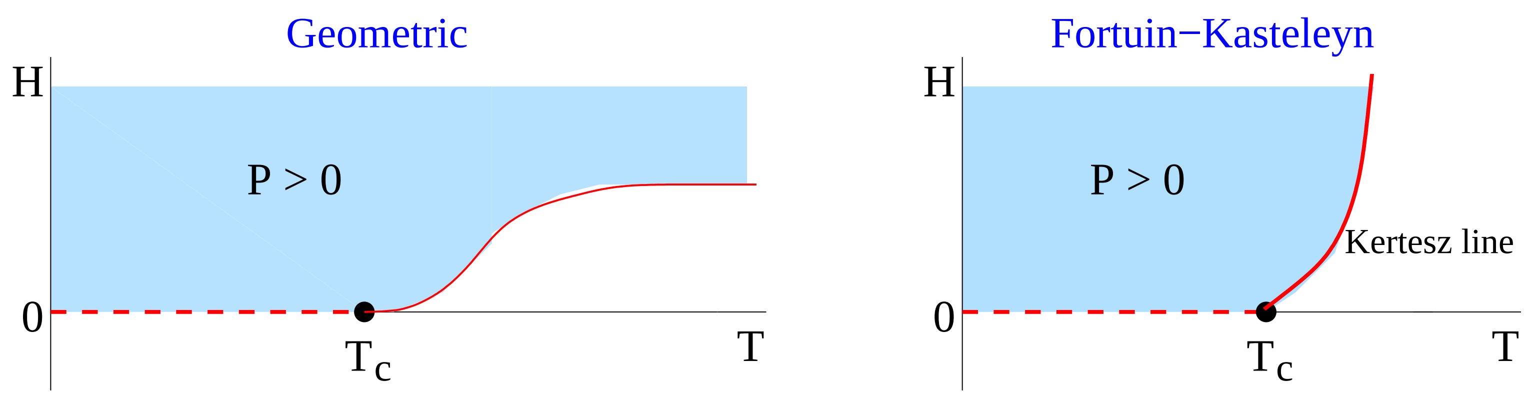
- Correlated percolation : occupation is determined by interaction

Ising: $$-\mathcal{H} = \frac{1}{T}\sum_{\langle ij\rangle}\sigma_i\sigma_j + H\sum_i \sigma_i\,, \qquad \sigma_i = \pm 1\,, \qquad\qquad Z = \sum_{\{\sigma_i\}} e^{-\mathcal{H}}$$

nearest neighbors with spin + are in the same cluster with probability:

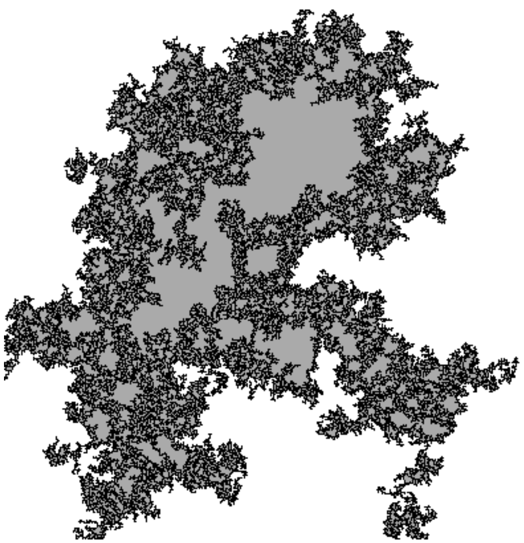
1, geometric clusters

$1 - e^{-1/T}$, Fortuin-Kasteleyn (or Coniglio-Klein) clusters (or droplets)

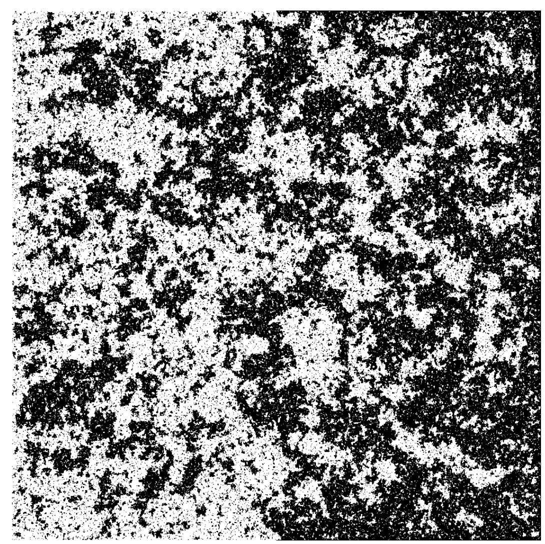


$(T, H) = (T_c, 0)$ fixed point for both geometric and FK clusters cluster size $\sim$ (linear extension)^D $\qquad$ D fractal dimension

$$D = \begin{cases} 91/48 = 1.89.. & \text{random (den Nijs '83; Nienhuis '84)} \\ 187/96 = 1.94.. & \text{Ising geometric (Stella, Vanderzande '89)} \\ 15/8 = 1.87.. & \text{Ising FK (Coniglio, Klein '80)} \end{cases}$$



random



Ising geom

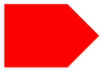
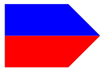
Percolation as the $(q \to 1)$-state Potts model

$$\mathcal{H}_{Potts} = -J \sum_{\langle x,y \rangle} \delta_{s(x),s(y)}\,, \qquad s(x) = 1, \ldots, q \qquad\qquad S_q \text{ invariance}$$

$$\begin{aligned} Z_{Potts} &= \sum_{\{s(x)\}} e^{-\mathcal{H}_{Potts}} = \quad \text{(Kasteleyn, Fortuin, '69)} \\ &= \sum_{\text{bond configs}} p^{\#\,bonds}(1-p)^{\#\,absent\,bonds}\, q^{\#\,clusters}\,, \qquad p = 1 - e^{-J} \end{aligned}$$

$P = \partial_q(\text{Potts magnetization})|_{q=1}$

cluster connectivities $\longleftrightarrow$ Potts spin correlators $\overset{2D}{\longleftrightarrow}$ kink field correlators

kink fields $\mu(x)$ create domain walls in the Potts broken phase: μ  = 

(GD, Viti, '11)

Critical three-point connectivity (GD, Viti, '10)

$P_n(x_1,\dots,x_n) \;\equiv\;$ probability $x_1,\dots,x_n$ in the same cluster

$P_2 \propto r_{12}^{-2X}, \qquad P_3 \propto (r_{12}r_{13}r_{23})^{-X} \qquad r_{ij} \equiv |x_i - x_j|, \qquad X = 2 - D$

$$R = \frac{P_3(x_1,x_2,x_3)}{\sqrt{P_2(x_1,x_2)P_2(x_1,x_3)P_2(x_2,x_3)}}$$ universal constant at a fixed point

$$\begin{cases} P_2(x_1,x_2) = \lim_{q\to 1} \langle \phi(x_1)\phi(x_2)\rangle \\ \\ P_3(x_1,x_2,x_3) = \lim_{q\to 1} \sqrt{2}\,\langle \phi(x_1)\phi(x_2)\phi(x_3)\rangle \end{cases}$$

$\phi = \frac{\mu+\bar\mu}{\sqrt{2}}, \quad \mu =$ Potts kink field

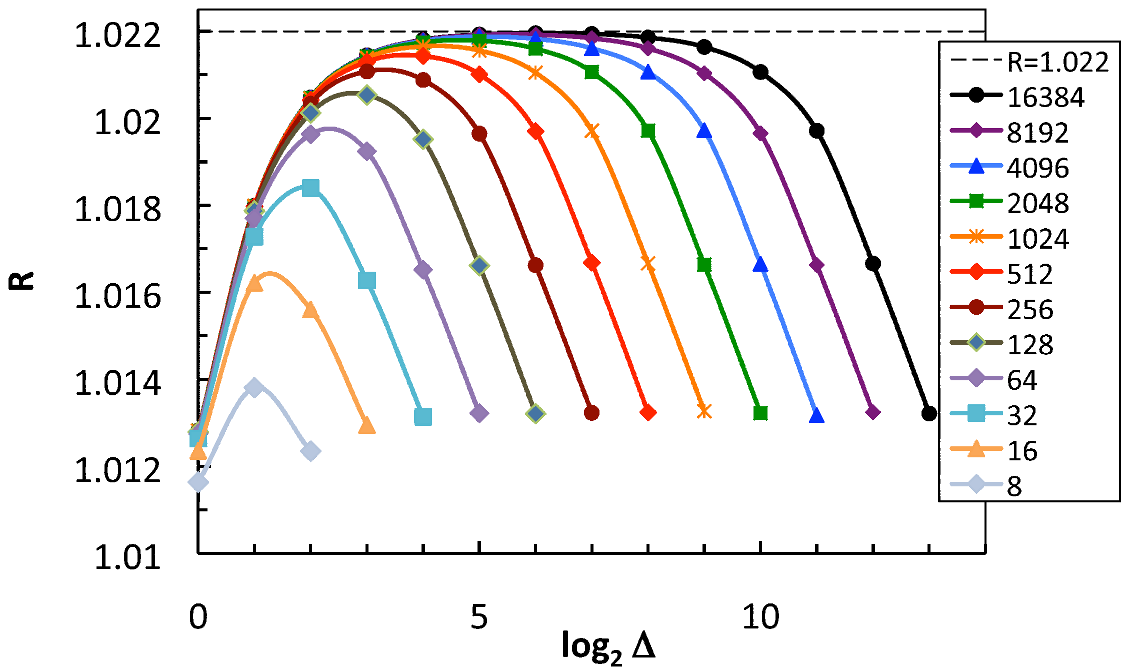
$$R = \sqrt{2} \lim_{c\to c_{perc}} \lim_{X_\phi \to X} C_{\phi\phi\phi}(c, X_\phi)$$

$C_{\phi\phi\phi}$ from "analytic continuation" of minimal models (Zamolodchikov, '05)

R	fixed point
1.0220..	random
1.3767..	Ising geometric
1.0524..	Ising FK

Monte Carlo determination for random percolation (Ziff, Simmons, Kleban, '10)

Equilateral triangle of side Δ on $L \times L$ lattice with periodic b.c. at p_c



Off-critical universality

Critical behavior as $p \to p_c^{\pm}$:

order parameter	$P \simeq B\,(p-p_c)^{\beta}$
mean cluster size	$S \simeq \Gamma^{\pm}\,\lvert p-p_c\rvert^{-\gamma}$
correlation length	$\xi \simeq f^{\pm}\,\lvert p-p_c\rvert^{-\nu}$
mean cluster number	$\dfrac{\langle N_c\rangle}{N} \simeq A^{\pm}\,\lvert p-p_c\rvert^{2-\alpha}$

One can construct universal critical amplitude combinations, e.g.

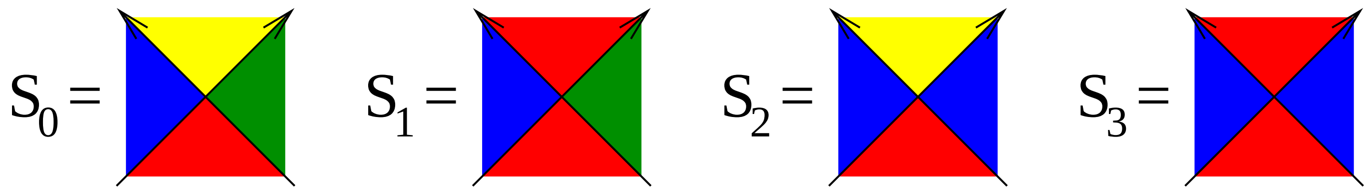
$$\frac{\Gamma_-}{\Gamma_+} = \frac{\int d^2x\,P_2(x,0)|_{p\to p_c^-}}{\int d^2x\,P_2(x,0)|_{p\to p_c^+}}$$

The scaling Potts model is integrable

- **Exact S-matrix** (Chim, A.Zamolodchikov, '92)

q degenerate vacua at $J > J_c$; elementary excitations are domain walls

Scattering amplitudes:



$$S_0(\theta) = \frac{\sinh\lambda\theta \, \sinh\lambda(\theta - i\pi)}{\sinh\lambda\left(\theta - \frac{2\pi i}{3}\right) \, \sinh\lambda\left(\theta - \frac{i\pi}{3}\right)} \, \Pi\left(\frac{\lambda\theta}{i\pi}\right) \qquad S_1(\theta) = \frac{\sin\frac{2\pi\lambda}{3} \, \sinh\lambda(\theta - i\pi)}{\sin\frac{\pi\lambda}{3} \, \sinh\lambda\left(\theta - \frac{2i\pi}{3}\right)} \, \Pi\left(\frac{\lambda\theta}{i\pi}\right)$$

$$S_2(\theta) = \frac{\sin\frac{2\pi\lambda}{3} \, \sinh\lambda\theta}{\sin\frac{\pi\lambda}{3} \, \sinh\lambda\left(\theta - \frac{i\pi}{3}\right)} \, \Pi\left(\frac{\lambda\theta}{i\pi}\right) \qquad S_3(\theta) = \frac{\sin\lambda\pi}{\sin\frac{\pi\lambda}{3}} \, \Pi\left(\frac{\lambda\theta}{i\pi}\right)$$

$$\sqrt{q} = 2\sin\frac{\pi\lambda}{3} \qquad \Pi\left(\frac{\lambda\theta}{i\pi}\right) = \frac{\sinh\lambda\left(\theta + i\frac{\pi}{3}\right)}{\sinh\lambda(\theta - i\pi)} \, \exp\left(\int_0^\infty \frac{dx}{x} \, \frac{\sinh\frac{x}{2}\left(1 - \frac{1}{\lambda}\right) - \sinh\frac{x}{2}\left(\frac{1}{\lambda} - \frac{5}{3}\right)}{\sinh\frac{x}{2\lambda}\cosh\frac{x}{2}} \, \sinh\frac{x\theta}{i\pi}\right)$$

θ parameterizes the center of mass energy

• **Spectral series for Potts correlators** (GD, Cardy, '98)

Example: susceptibility amplitude ratio, two-particle approximation :

q	Field Theory	Lattice
2	37.699	37.6936..[a]
3	13.848	13.83(8)[b,c]
4	4.01	3.9(1)[d]

[a] Wu, McCoy, Tracy, Barouch, '76

[b] Enting, Guttmann, '03

[c] Shchur, Berche, Butera, '08

[d] Shchur, Janke, '10 (Baxter-Wu model)

Universal amplitude ratios for random percolation (GD, Viti, Cardy, '10)

$$P_2(x_1,x_2)|_{p\to p_c^-} = \frac{F_\mu^2}{\pi}\,K_0(r_{12}/\xi) + O(e^{-2r_{12}/\xi})$$

$$P_2(x_1,x_2)|_{p\to p_c^+} = \frac{F_\sigma}{\pi^2}\int_0^\infty d\theta\,|f(2\theta)\Omega(2\theta)|^2\,K_0\left(2\frac{r_{12}}{\xi}\cosh\theta\right) + O(e^{-3r_{12}/\xi})$$

$$f(\theta) = -i\sinh\frac{\theta}{2}\,\exp\left\{-2\int_0^\infty \frac{dx}{x}\,\frac{\sinh\frac{x}{3}\cosh\frac{x}{6}}{\sinh^2 x\cosh\frac{x}{2}}\,\sin^2\frac{(i\pi-\theta)x}{2\pi}\right\}$$

$$\Omega(\theta) = \mathcal{C}\int_{-\infty}^{+\infty}\frac{dx}{2\pi}\,W\left(-x-\frac{\theta}{2}+i\pi\right)W\left(-x+\frac{\theta}{2}+i\pi\right)e^{-x/6}$$

$$\mathcal{C} = \exp\left[\int_0^\infty \frac{dt}{t}\,\frac{4\sinh^2\frac{t}{4}\sinh\frac{t}{2}}{\sinh^2 t}\right], \qquad W(\theta) = \frac{1}{\cosh\theta}\,\exp\left[\int_0^\infty \frac{dt}{t}\,\frac{2\sinh\frac{t}{2}}{\sinh^2 t}\sin^2\left(\frac{t}{2\pi}(i\pi-\theta)\right)\right]$$

$$F_\mu^2 = F_\sigma\,\lim_{\theta\to\infty}|f(\theta)\Omega(\theta)|$$

Amplitude Ratio	Field Theory	Lattice
A^-/A^+	1	1^a
f^-/f^+	2	-
f^-_{2nd}/f^-	1.001	-
f^-_{2nd}/f^+_{2nd}	3.73	4.0 ± 0.5^c
$4B^2(f^-_{2nd})^2/\Gamma^-$	2.22	2.23 ± 0.10^d
$(-80/27\,A^-)^{1/2}f^-_{2nd}$	0.926	$\approx 0.93^{a+b}$
Γ^-/Γ^+	160.2	162.5 ± 2^e

[a] Domb, Pearce, '76

[b] Aharony, Stauffer, '97

[c] Corsten, Jan, Jerrard, '89

[d] Daboul, Aharony, Stauffer, '00

[e] Jensen, Ziff, '06

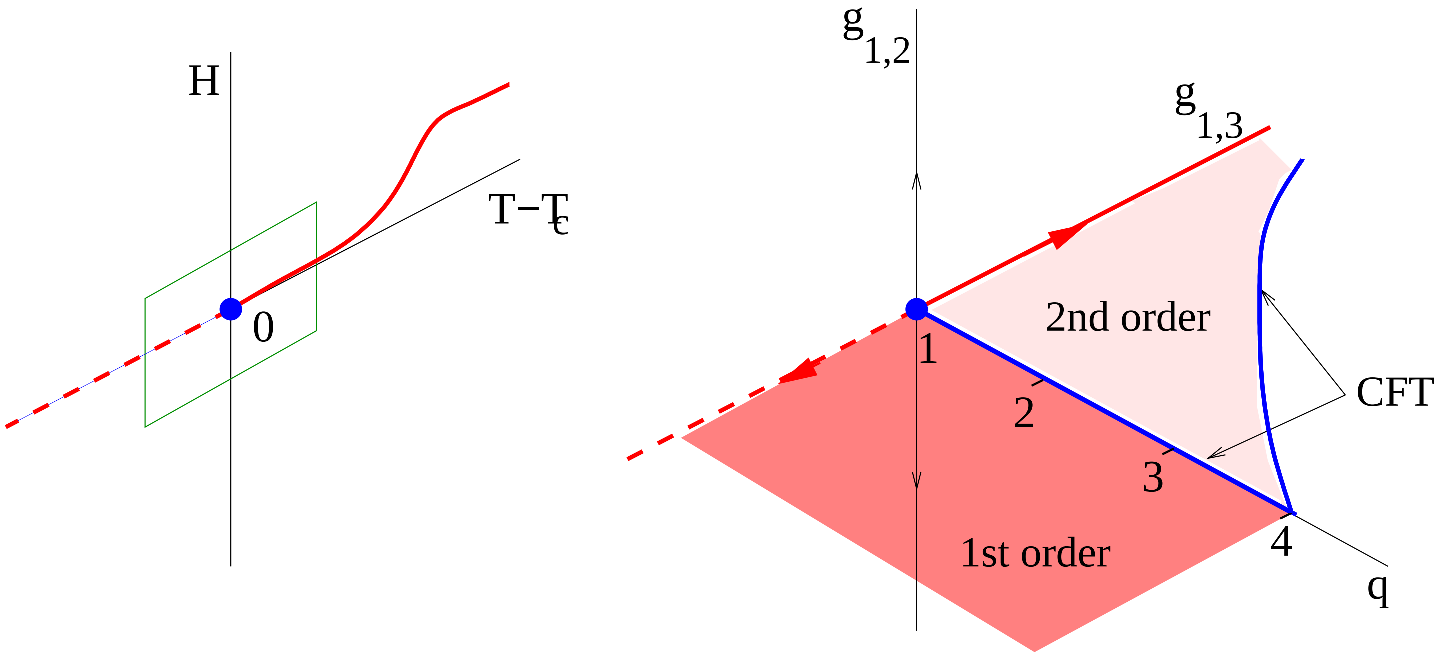
30 years of efforts by the numerical community, Γ^-/Γ^+ most elusive

Numerical estimates for Γ^-/Γ^+ in random percolation (from Ziff, '11)

year	author	system, method	Γ^-/Γ^+
1976	Sykes, Gaunt, Glen	lattice, series (12-20 order)	1.3-2.0
1976	Stauffer	lattice, series analysis	≈ 100
1978	Nakanishi, Stanley	lattice, MC	25(10)
1978	Wolff, Stauffer	lattice, series, fit to gaussian	180(36)
1979	Hoshen et al	lattice, MC	196(40)
1980	Nakanishi, Stanley	lattice, MC (reanalyze)	219(25)
1981	Gawlinsky, Stanley	overlapping disks, MC	50(26)
1985	Rushton, Family, Herrmann	additive polymerization, MC	140(45)
1987	Meir	lattice, series	210(10)
1987	Kim, Herrmann, Landau	continuum model, MC	14(10)
1987	Nakanishi	AB percolation, MC	139(24)
1988	Balberg	widthless sticks, MC	≈ 3
1988	Ottavi	approx. theory (gaussian fit)	193.9
1989	Corsten, Jan, Jerrard	lattice, MC	75(+40, -25)
1990	S. B. Lee, Torquato	penetrable conc. shell	1050(32)
1990	S. B. Lee	disks, MC	192(20)
1991	Hund	random contour model, MC	≈ 200
1993	Zhang, De'Bell	Penrose quasi-lattice, series	310(60)
1995	Conway, Guttmann	lattice, series (26-33 order)	$45(+20, -10)$
1996	S. B. Lee	penetrable conc. shell, disks	175(50)
1997	S. B. Lee, Jeon	kinetic gelation, MC	170(20)
2006	Jensen, Ziff	lattice, MC, series	162.5(2.0)

Field theory of Ising clusters (GD, '09)

Geometric clusters :



1,2 and 1,3 CFT perturbations are integrable

Particles on 2nd order surface :

- $q = 1 + \epsilon$: ϵ massless, 1 massive with lifetime $\sim 1/\epsilon$ $\Longrightarrow$ $\xi_{perc} = \infty$
- $q = 1$: 0 massless, one stable massive $\Longrightarrow$ $\xi_{magn} < \infty$

Universal amplitude ratios for Ising clusters (GD, Viti, '10)

Amplitude Ratio	geometric clusters	FK clusters
Γ_a/Γ_b^+	not defined	40.3
$f_{2nd,a}/f_a$	''	0.99959..
f_a/f_b^+	''	2
$f_a/\widehat{f}_a$	''	1
$A_{k,a}/A_{k,b}^+; k=0,-1$	''	1
Γ_b^+/Γ_b^-	-	1
f_b^+/f_b^-	$1/2$	1
$f_{2nd,b}^-/f_b^-$	0.6799	0.61
$f_{2nd,b}^+/f_{2nd,b}^-$	-	1
$f_b^+/\widehat{f}_b^\pm$	$1/2$	1
U_b	24.72	15.2
$A_{k,b}^+/A_{k,b}^-; k=0,-1$	1	1
$A_{0,b}^\pm/A_{-1,b}^\pm$	$-\gamma-\ln\pi=-1.7219..$	$-\gamma-\ln\pi=-1.7219..$
r_b	$\frac{3\sqrt{3}(\gamma+\ln\pi)}{64\pi^2}=0.014165..$	$-\frac{\gamma+\ln\pi}{12\pi^2}=-0.014539..$
f_c^+/f_c^-	$1/2$	-
$f_{2nd,c}^-/f_c^-$	1.002	-
$f_c^+/\widehat{f}_c^\pm$	$\sin\frac{\pi}{5}=0.58778..$	-
$A_{k,c}^+/A_{k,c}^-; k=0,1$	1	-
$A_{0,c}^\pm/A_{1,c}^\pm$	$-0.42883..$	-
r_c	$-3.7624..\times 10^{-3}$	-

$\gamma = 0.5772..$ Euler-Mascheroni constant