Analysis of strongly coupled quantum field theories: A perturbation approach

Marco Frasca

marcofrasca@mclink.it

SM&FT 2011, Bari, 21-23 September 2011

Plan of the talk

- Classical field theory
  - Scalar field theory
  - Yang-Mills theory
  - Mapping theorem: Formulation
  - Mapping theorem: Yang-Mills-Green function
  - Mapping theorem: A comparison
- Quantum field theory
  - Scalar field theory
  - Next-to-leading order
  - Running coupling
  - Yang-Mills theory
- Conclusions

Classical field theory: Scalar field

- A classical field theory for a massless scalar field is given by

$$\Box\phi + \lambda\phi^3 = j$$

Classical field theory: Scalar field

- A classical field theory for a massless scalar field is given by

$$\Box\phi + \lambda\phi^3 = j$$

- The homogeneous equation can be solved exactly by

$$\phi = \mu\left(\frac{2}{\lambda}\right)^{\frac{1}{4}} \mathrm{sn}(p\cdot x + \theta, i)$$

being sn an elliptic Jacobi function and μ and θ two constant. This solution holds provided the following dispersion relation holds

$$p^2 = \mu^2\sqrt{\frac{\lambda}{2}}$$

so this solution represents a free massive solution notwithstanding we started from a massless theory.

Classical field theory: Scalar field

- A classical field theory for a massless scalar field is given by

$$\Box\phi+\lambda\phi^3=j$$

- The homogeneous equation can be solved exactly by

$$\phi=\mu\left(\frac{2}{\lambda}\right)^{\frac{1}{4}}\mathrm{sn}(p\cdot x+\theta,i)$$

  being sn an elliptic Jacobi function and μ and θ two constant. This solution holds provided the following dispersion relation holds

$$p^2=\mu^2\sqrt{\frac{\lambda}{2}}$$

  so this solution represents a free massive solution notwithstanding we started from a massless theory.

- Mass arises from the nonlinearities when λ is taken to be finite rather than going to zero.

Classical field theory: Scalar field

- When there is a current we ask for a solution in the limit $\lambda \to \infty$ as our aim is to understand a strong coupling limit. So, we check a solution

$$\phi = \kappa \int d^4x' G(x - x')j(x') + \delta\phi$$

being $\delta\phi$ all higher order corrections.

Classical field theory: Scalar field

- When there is a current we ask for a solution in the limit $\lambda \to \infty$ as our aim is to understand a strong coupling limit. So, we check a solution

$$\phi = \kappa \int d^4x' G(x-x') j(x') + \delta\phi$$

being $\delta\phi$ all higher order corrections.

- One can prove that this is indeed so provided

$$\delta\phi = \kappa^2 \lambda \int d^4x' d^4x'' G(x-x')[G(x'-x'')]^3 j(x') + O(j(x)^3)$$

with the identification $\kappa = \mu$, the same of the exact solution, and $\Box G(x-x') + \lambda [G(x-x')]^3 = \mu^{-1} \delta^4(x-x')$.

Classical field theory: Scalar field

- When there is a current we ask for a solution in the limit $\lambda \to \infty$ as our aim is to understand a strong coupling limit. So, we check a solution

$$\phi = \kappa \int d^4x' G(x - x') j(x') + \delta\phi$$

being $\delta\phi$ all higher order corrections.

- One can prove that this is indeed so provided

$$\delta\phi = \kappa^2 \lambda \int d^4x' d^4x'' G(x - x') [G(x' - x'')]^3 j(x') + O(j(x)^3)$$

with the identification $\kappa = \mu$, the same of the exact solution, and $\Box G(x - x') + \lambda [G(x - x')]^3 = \mu^{-1} \delta^4(x - x')$.

- This implies that the corresponding quantum field theory, in a very strong coupling limit, takes a Gaussian form and is trivial (triviality of the scalar field theory in the infrared limit).

Classical field theory: Scalar field

- When there is a current we ask for a solution in the limit $\lambda \to \infty$ as our aim is to understand a strong coupling limit. So, we check a solution

$$\phi = \kappa \int d^4x' G(x-x')j(x') + \delta\phi$$

  being $\delta\phi$ all higher order corrections.
- One can prove that this is indeed so provided

$$\delta\phi = \kappa^2\lambda \int d^4x' d^4x'' G(x-x')[G(x'-x'')]^3 j(x') + O(j(x)^3)$$

  with the identification $\kappa = \mu$, the same of the exact solution, and $\Box G(x-x') + \lambda[G(x-x')]^3 = \mu^{-1}\delta^4(x-x')$.
- This implies that the corresponding quantum field theory, in a very strong coupling limit, takes a Gaussian form and is trivial (triviality of the scalar field theory in the infrared limit).
- All we need now is to find the exact form of the propagator $G(x-x')$ and we have completely solved the classical theory for the scalar field in a strong coupling limit.

Classical field theory: Scalar field

- In order to solve the equation

$$\Box G(x-x') + \lambda[G(x-x')]^3 = \mu^{-1}\delta^4(x-x')$$

we can start from the $d=1+0$ case $\partial_t^2 G_0(t-t') + \lambda[G_0(t-t')]^3 = \mu^2\delta(t-t')$.

Classical field theory: Scalar field

- In order to solve the equation

$$\Box G(x-x') + \lambda[G(x-x')]^3 = \mu^{-1}\delta^4(x-x')$$

we can start from the $d = 1+0$ case $\partial_t^2 G_0(t-t') + \lambda[G_0(t-t')]^3 = \mu^2\delta(t-t')$.

- It is straightforwardly obtained the Fourier transformed solution

$$G_0(\omega) = \sum_{n=0}^{\infty}(2n+1)\frac{\pi^2}{K^2(i)}\frac{(-1)^n e^{-(n+\frac{1}{2})\pi}}{1+e^{-(2n+1)\pi}}\frac{1}{\omega^2 - m_n^2 + i\epsilon}$$

being $m_n = (2n+1)\frac{\pi}{2K(i)}\left(\frac{\lambda}{2}\right)^{\frac{1}{4}}\mu$ and $K(i) \approx 1.3111028777$ an elliptic integral.

Classical field theory: Scalar field

- In order to solve the equation

$$\Box G(x-x') + \lambda[G(x-x')]^3 = \mu^{-1}\delta^4(x-x')$$

we can start from the $d = 1+0$ case $\partial_t^2 G_0(t-t') + \lambda[G_0(t-t')]^3 = \mu^2\delta(t-t')$.

- It is straightforwardly obtained the Fourier transformed solution

$$G_0(\omega) = \sum_{n=0}^{\infty}(2n+1)\frac{\pi^2}{K^2(i)}\frac{(-1)^n e^{-(n+\frac{1}{2})\pi}}{1+e^{-(2n+1)\pi}}\frac{1}{\omega^2 - m_n^2 + i\epsilon}$$

being $m_n = (2n+1)\frac{\pi}{2K(i)}\left(\frac{\lambda}{2}\right)^{\frac{1}{4}}\mu$ and $K(i) \approx 1.3111028777$ an elliptic integral.

- We are able to recover the full covariant propagator by boosting from the rest reference frame obtaining finally

$$G(p) = \sum_{n=0}^{\infty}(2n+1)\frac{\pi^2}{K^2(i)}\frac{(-1)^n e^{-(n+\frac{1}{2})\pi}}{1+e^{-(2n+1)\pi}}\frac{1}{p^2 - m_n^2 + i\epsilon}.$$

This shows that our solution given above indeed represents a strong coupling expansion being meaningful for $\lambda \to \infty$.

Classical field theory: Yang-Mills field

- A classical field theory for the Yang-Mills field is given by

$$\partial^\mu\partial_\mu A_\nu^a-\left(1-\frac{1}{\alpha}\right)\partial_\nu(\partial^\mu A_\mu^a)+gf^{abc}A^{b\mu}(\partial_\mu A_\nu^c-\partial_\nu A_\mu^c)+gf^{abc}\partial^\mu(A_\mu^b A_\nu^c)+g^2f^{abc}f^{cde}A^{b\mu}A_\mu^d A_\nu^e=-j_\nu^a$$

Classical field theory: Yang-Mills field

- A classical field theory for the Yang-Mills field is given by

$$\partial^\mu\partial_\mu A^a_\nu-\left(1-\frac{1}{\alpha}\right)\partial_\nu(\partial^\mu A^a_\mu)+gf^{abc}A^{b\mu}(\partial_\mu A^c_\nu-\partial_\nu A^c_\mu)+gf^{abc}\partial^\mu(A^b_\mu A^c_\nu)+g^2f^{abc}f^{cde}A^{b\mu}A^d_\mu A^e_\nu=-j^a_\nu$$

- For the homogeneous equation, we want to study it in the formal limit $g\to\infty$. We note that a class of exact solutions exists if we take the potential A^a_μ just depending on time, after a proper selection of the components [see Smilga (2001)]. These solutions are the same of the scalar field when spatial coordinates are set to zero (rest frame).

Classical field theory: Yang-Mills field

- A classical field theory for the Yang-Mills field is given by

$$\partial^\mu\partial_\mu A^a_\nu-\left(1-\frac{1}{\alpha}\right)\partial_\nu(\partial^\mu A^a_\mu)+gf^{abc}A^{b\mu}(\partial_\mu A^c_\nu-\partial_\nu A^c_\mu)+gf^{abc}\partial^\mu(A^b_\mu A^c_\nu)+g^2f^{abc}f^{cde}A^{b\mu}A^d_\mu A^e_\nu=-j^a_\nu$$

- For the homogeneous equation, we want to study it in the formal limit $g \to \infty$. We note that a class of exact solutions exists if we take the potential A^a_μ just depending on time, after a proper selection of the components [see Smilga (2001)]. These solutions are the same of the scalar field when spatial coordinates are set to zero (rest frame).
- Differently from the scalar field, we cannot just boost away these solutions to get a general solution to Yang-Mills equations due to gauge symmetry. Anyhow, one can prove that the mapping persists but is just approximate in the limit of a very large coupling.

Classical field theory: Yang-Mills field

- A classical field theory for the Yang-Mills field is given by

$$\partial^\mu\partial_\mu A^a_\nu-\left(1-\frac{1}{\alpha}\right)\partial_\nu(\partial^\mu A^a_\mu)+gf^{abc}A^{b\mu}(\partial_\mu A^c_\nu-\partial_\nu A^c_\mu)+gf^{abc}\partial^\mu(A^b_\mu A^c_\nu)+g^2f^{abc}f^{cde}A^{b\mu}A^d_\mu A^e_\nu=-j^a_\nu$$

- For the homogeneous equation, we want to study it in the formal limit $g\rightarrow\infty$. We note that a class of exact solutions exists if we take the potential A^a_μ just depending on time, after a proper selection of the components [see Smilga (2001)]. These solutions are the same of the scalar field when spatial coordinates are set to zero (rest frame).
- Differently from the scalar field, we cannot just boost away these solutions to get a general solution to Yang-Mills equations due to gauge symmetry. Anyhow, one can prove that the mapping persists but is just approximate in the limit of a very large coupling.
- This mapping would imply that we will have at our disposal a starting solution to build a quantum field theory for a strongly coupled Yang-Mills field. This solution has a mass gap already at a classical level!

Classical field theory: Yang-Mills field

- Exactly as in the case of the scalar field we assume the following solution to our field equations

$$A^a_\mu = \kappa \int d^4x' D^{ab}_{\mu\nu}(x - x') j^{b\nu}(x') + \delta A^a_\mu$$

Classical field theory: Yang-Mills field

- Exactly as in the case of the scalar field we assume the following solution to our field equations

$$A^a_\mu = \kappa \int d^4x' D^{ab}_{\mu\nu}(x-x') j^{b\nu}(x') + \delta A^a_\mu$$

- Also in this case, apart from a possible correction, this boils down to an expansion in powers of the currents as already guessed in the ’80 [R. T. Cahill and C. D. Roberts (1985)].

Classical field theory: Yang-Mills field

- Exactly as in the case of the scalar field we assume the following solution to our field equations

$$A^a_\mu = \kappa \int d^4x' D^{ab}_{\mu\nu}(x-x') j^{b\nu}(x') + \delta A^a_\mu$$

- Also in this case, apart from a possible correction, this boils down to an expansion in powers of the currents as already guessed in the '80 [R. T. Cahill and C. D. Roberts (1985)].
- This implies that the corresponding quantum theory, in a very strong coupling limit, takes a Gaussian form and is trivial.

Classical field theory: Yang-Mills field

- Exactly as in the case of the scalar field we assume the following solution to our field equations

$$A^a_\mu = \kappa \int d^4x' D^{ab}_{\mu\nu}(x - x') j^{b\nu}(x') + \delta A^a_\mu$$

- Also in this case, apart from a possible correction, this boils down to an expansion in powers of the currents as already guessed in the '80 [R. T. Cahill and C. D. Roberts (1985)].
- This implies that the corresponding quantum theory, in a very strong coupling limit, takes a Gaussian form and is trivial.
- The crucial point, as already pointed out in the eighties [T. Goldman and R. W. Haymaker (1981), T. Cahill and C. D. Roberts (1985)], is the exact determination of the gluon propagator in the low-energy limit. Then, a lot of physics will be at our hands!

Classical field theory: Yang-Mills field

- Exactly as in the case of the scalar field we assume the following solution to our field equations

$$A^a_\mu = \kappa \int d^4x' D^{ab}_{\mu\nu}(x-x') j^{b\nu}(x') + \delta A^a_\mu$$

- Also in this case, apart from a possible correction, this boils down to an expansion in powers of the currents as already guessed in the ’80 [R. T. Cahill and C. D. Roberts (1985)].
- This implies that the corresponding quantum theory, in a very strong coupling limit, takes a Gaussian form and is trivial.
- The crucial point, as already pointed out in the eighties [T. Goldman and R. W. Haymaker (1981), T. Cahill and C. D. Roberts (1985)], is the exact determination of the gluon propagator in the low-energy limit. Then, a lot of physics will be at our hands!
- The mapping theorem helps to solve this problem definitely.

Mapping theorem: Formulation

- Exact determination of the gluon propagator can be largely simplified if we are able to map Yang-Mills theory on a theory with known results. With this aim in mind the following theorem has been proved:

Mapping theorem: Formulation

- Exact determination of the gluon propagator can be largely simplified if we are able to map Yang-Mills theory on a theory with known results. With this aim in mind the following theorem has been proved:
- **MAPPING THEOREM:** *An extremum of the action*

$$S=\int d^4x\left[\frac{1}{2}(\partial\phi)^2-\frac{\lambda}{4}\phi^4\right]$$

is also an extremum of the SU(N) Yang-Mills Lagrangian when one properly chooses A_μ^a with some components being zero and all others being equal, and $\lambda=Ng^2$, being g the coupling constant of the Yang-Mills field, when only time dependence is retained. In the most general case the following mapping holds

$$A_\mu^a(x)=\eta_\mu^a\phi(x)+O(1/\sqrt{N}g),$$

being η_μ^a some constants properly chosen, that becomes exact for the Lorenz gauge.

Mapping theorem: Formulation

- Exact determination of the gluon propagator can be largely simplified if we are able to map Yang-Mills theory on a theory with known results. With this aim in mind the following theorem has been proved:
- **MAPPING THEOREM:** *An extremum of the action*

$$S = \int d^4x \left[\frac{1}{2}(\partial\phi)^2 - \frac{\lambda}{4}\phi^4\right]$$

is also an extremum of the SU(N) Yang-Mills Lagrangian when one properly chooses A_μ^a *with some components being zero and all others being equal, and* $\lambda = Ng^2$*, being* g *the coupling constant of the Yang-Mills field, when only time dependence is retained. In the most general case the following mapping holds*

$$A_\mu^a(x) = \eta_\mu^a\phi(x) + O(1/\sqrt{N}g),$$

being η_μ^a *some constants properly chosen, that becomes exact for the Lorenz gauge.*

- This theorem was proved in the following papers: M. Frasca, Phys. Lett. **B670**, 73-77 (2008) [0709.2042]; Mod. Phys. Lett. A **24**, 2425-2432 (2009) [0903.2357] after considering a criticism by Terry Tao. Tao agreed with the latest proof.

Mapping theorem: Yang-Mills-Green function

- The mapping theorem permits us to write down immediately the propagator for the Yang-Mills equations in the Landau gauge for SU(N):

$$\Delta_{\mu\nu}^{ab}(p)=\delta_{ab}\left(\eta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\sum_{n=0}^{\infty}\frac{B_n}{p^2-m_n^2+i\epsilon}+O\left(\frac{1}{\sqrt{N}g}\right)$$

being

$$B_n=(2n+1)\frac{\pi^2}{K^2(i)}\frac{(-1)^{n+1}e^{-(n+\frac{1}{2})\pi}}{1+e^{-(2n+1)\pi}}$$

and

$$m_n=(2n+1)\frac{\pi}{2K(i)}\left(\frac{Ng^2}{2}\right)^{\frac{1}{4}}\Lambda$$

Mapping theorem: Yang-Mills-Green function

- The mapping theorem permits us to write down immediately the propagator for the Yang-Mills equations in the Landau gauge for SU(N):

$$\Delta_{\mu\nu}^{ab}(p)=\delta_{ab}\left(\eta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\sum_{n=0}^{\infty}\frac{B_n}{p^2-m_n^2+i\epsilon}+O\left(\frac{1}{\sqrt{N}g}\right)$$

being

$$B_n=(2n+1)\frac{\pi^2}{K^2(i)}\frac{(-1)^{n+1}e^{-(n+\frac{1}{2})\pi}}{1+e^{-(2n+1)\pi}}$$

and

$$m_n=(2n+1)\frac{\pi}{2K(i)}\left(\frac{Ng^2}{2}\right)^{\frac{1}{4}}\Lambda$$

- The constant Λ must be the same constant that appears in the ultraviolet limit by dimensional transmutation, here arises as an integration constant [M. Frasca, arXiv:1007.4479v2 [hep-ph]].

Mapping theorem: Yang-Mills-Green function

- The mapping theorem permits us to write down immediately the propagator for the Yang-Mills equations in the Landau gauge for SU(N):

$$\Delta^{ab}_{\mu\nu}(p)=\delta_{ab}\left(\eta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\sum_{n=0}^{\infty}\frac{B_n}{p^2-m_n^2+i\epsilon}+O\left(\frac{1}{\sqrt{N}g}\right)$$

being

$$B_n=(2n+1)\frac{\pi^2}{K^2(i)}\frac{(-1)^{n+1}e^{-(n+\frac{1}{2})\pi}}{1+e^{-(2n+1)\pi}}$$

and

$$m_n=(2n+1)\frac{\pi}{2K(i)}\left(\frac{Ng^2}{2}\right)^{\frac{1}{4}}\Lambda$$

- The constant Λ must be the same constant that appears in the ultraviolet limit by dimensional transmutation, here arises as an integration constant [M. Frasca, arXiv:1007.4479v2 [hep-ph]].
- This is the propagator of a massive field theory but the mass poles arise dynamically from the non-linearities in the equations of motion. At this stage we are working classically yet.

Mapping theorem: Yang-Mills-Green function

- The mapping theorem permits us to write down immediately the propagator for the Yang-Mills equations in the Landau gauge for SU(N):

$$\Delta_{\mu\nu}^{ab}(p)=\delta_{ab}\left(\eta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\sum_{n=0}^{\infty}\frac{B_n}{p^2-m_n^2+i\epsilon}+O\left(\frac{1}{\sqrt{N}g}\right)$$

  being

$$B_n=(2n+1)\frac{\pi^2}{K^2(i)}\frac{(-1)^{n+1}e^{-(n+\frac{1}{2})\pi}}{1+e^{-(2n+1)\pi}}$$

  and

$$m_n=(2n+1)\frac{\pi}{2K(i)}\left(\frac{Ng^2}{2}\right)^{\frac{1}{4}}\Lambda$$

- The constant Λ must be the same constant that appears in the ultraviolet limit by dimensional transmutation, here arises as an integration constant [M. Frasca, arXiv:1007.4479v2 [hep-ph]].
- This is the propagator of a massive field theory but the mass poles arise dynamically from the non-linearities in the equations of motion. At this stage we are working classically yet.
- All this classical analysis could be easier to work out on the lattice than the corresponding quantum field theory and would already be an important step beyond.

Mapping theorem: A comparison

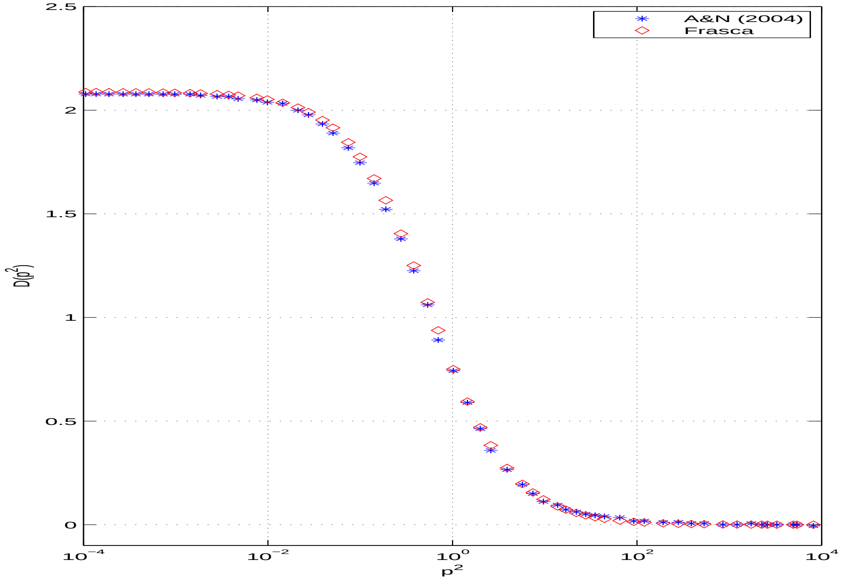
- Could the propagator we obtained for the classical Yang-Mills field be the right one for the quantum field theory in the infrared limit?

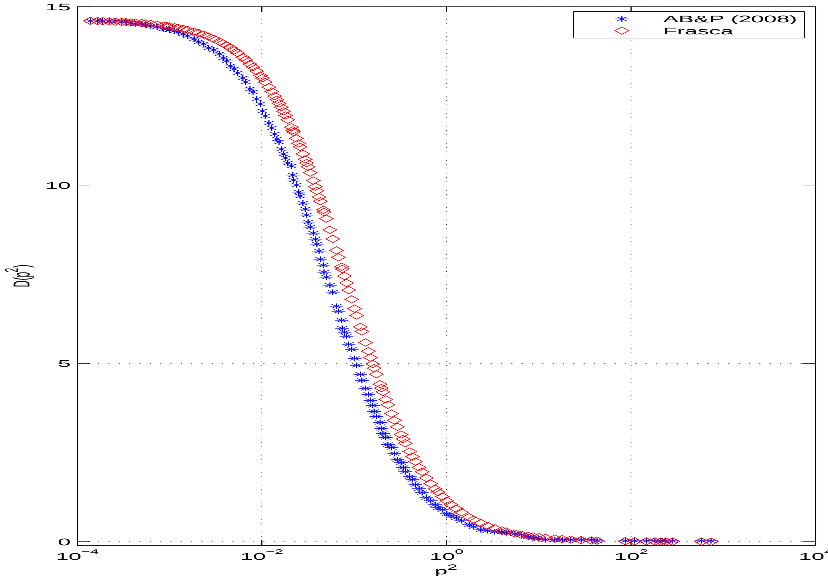
Mapping theorem: A comparison

- Could the propagator we obtained for the classical Yang-Mills field be the right one for the quantum field theory in the infrared limit?
- In order to answer this question, we compared it with a solution obtained numerically from Dyson-Schwinger equations [A. C. Aguilar and A. A. Natale (2004) but see also A. C. Aguilar, D. Binosi, J. Papavassiliou, arXiv:0802.1870v3 [hep-ph]].

Mapping theorem: A comparison

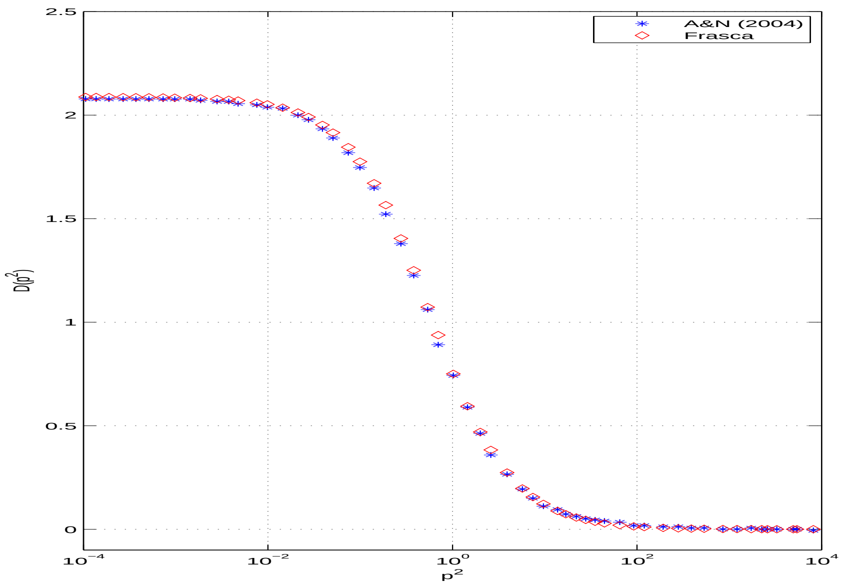
- Could the propagator we obtained for the classical Yang-Mills field be the right one for the quantum field theory in the infrared limit?
- In order to answer this question, we compared it with a solution obtained numerically from Dyson-Schwinger equations [A. C. Aguilar and A. A. Natale (2004) but see also A. C. Aguilar, D. Binosi, J. Papavassiliou, arXiv:0802.1870v3 [hep-ph]].
- We obtained the following figure for a gluon mass of 746 (A&N, 2004) and 282 (AB&P, 2008) MeV (only fitting parameter):

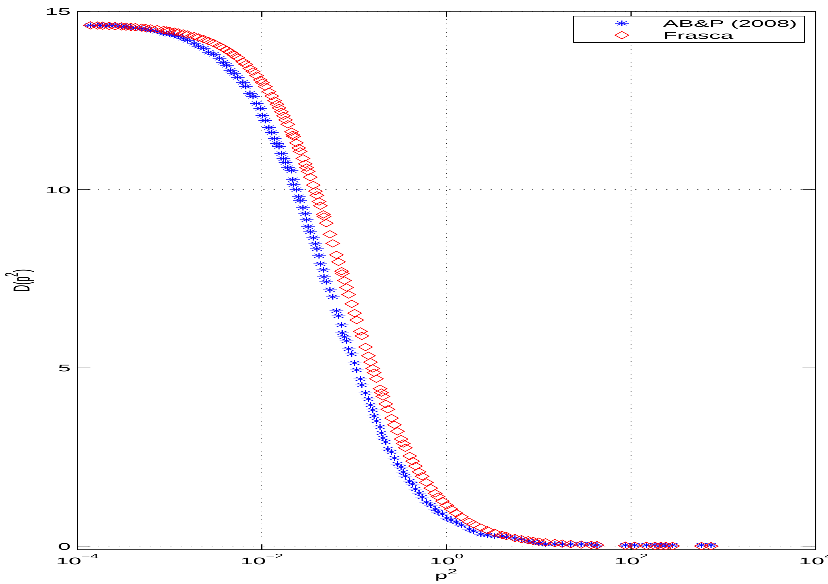




Mapping theorem: A comparison

- Could the propagator we obtained for the classical Yang-Mills field be the right one for the quantum field theory in the infrared limit?
- In order to answer this question, we compared it with a solution obtained numerically from Dyson-Schwinger equations [A. C. Aguilar and A. A. Natale (2004) but see also A. C. Aguilar, D. Binosi, J. Papavassiliou, arXiv:0802.1870v3 [hep-ph]].
- We obtained the following figure for a gluon mass of 746 (A&N, 2004) and 282 (AB&P, 2008) MeV (only fitting parameter):





- The agreement is strikingly good but is worsening in the intermediate range of energies. This should be expected by our approximation.

Quantum field theory: Scalar field (1)

- We can formulate a quantum field theory for the scalar field starting from the generating functional

$$Z[j] = N \int [d\phi] \exp \left[i \int d^4x \left(\frac{1}{2}(\partial\phi)^2 - \frac{\lambda}{4}\phi^4 + j\phi \right) \right].$$

Quantum field theory: Scalar field (1)

- We can formulate a quantum field theory for the scalar field starting from the generating functional

$$Z[j]=N\int[d\phi]\exp\left[i\int d^4x\left(\frac{1}{2}(\partial\phi)^2-\frac{\lambda}{4}\phi^4+j\phi\right)\right].$$

- We can rescale the space-time variable as $x\rightarrow\sqrt{\lambda}x$ and rewrite the functional as

$$Z[j]=N\int[d\phi]\exp\left[i\frac{1}{\lambda}\int d^4x\left(\frac{1}{2}(\partial\phi)^2-\frac{1}{4}\phi^4+\frac{1}{\lambda}j\phi\right)\right].$$

  Then we can seek for a solution series as $\phi=\sum_{n=0}^{\infty}\lambda^{-n}\phi_n$ and rescale the current $j\rightarrow j/\lambda$ being this arbitrary.

Quantum field theory: Scalar field (1)

- We can formulate a quantum field theory for the scalar field starting from the generating functional

$$Z[j]=N\int[d\phi]\exp\left[i\int d^4x\left(\frac{1}{2}(\partial\phi)^2-\frac{\lambda}{4}\phi^4+j\phi\right)\right].$$

- We can rescale the space-time variable as $x\rightarrow\sqrt{\lambda}x$ and rewrite the functional as

$$Z[j]=N\int[d\phi]\exp\left[i\frac{1}{\lambda}\int d^4x\left(\frac{1}{2}(\partial\phi)^2-\frac{1}{4}\phi^4+\frac{1}{\lambda}j\phi\right)\right].$$

  Then we can seek for a solution series as $\phi=\sum_{n=0}^{\infty}\lambda^{-n}\phi_n$ and rescale the current $j\rightarrow j/\lambda$ being this arbitrary.

- It is not difficult to see that the leading order correction can be computed solving the classical equation

$$\Box\phi_0+\phi_0^3=j$$

  that we already know how to manage. This is completely consistent with our preceding formulation [M. Frasca (2006)] but now all is fully covariant. We are just using our ability to solve the classical theory.

Quantum field theory: Scalar field (2)

- Using the approximation holding at strong coupling

$$\phi_0 = \mu \int d^4x G(x - x')j(x') + \ldots$$

it is not difficult to write the generating functional at the leading order in a Gaussian form

$$Z_0[j] = N \exp\left[\frac{i}{2} \int d^4x' d^4x'' j(x')G(x' - x'')j(x'')\right].$$

Quantum field theory: Scalar field (2)

- Using the approximation holding at strong coupling

$$\phi_0 = \mu \int d^4 x G(x - x') j(x') + \ldots$$

it is not difficult to write the generating functional at the leading order in a Gaussian form

$$Z_0[j] = N \exp\left[\frac{i}{2}\int d^4x' d^4x'' j(x') G(x' - x'') j(x'')\right].$$

- This conclusion is really important: It says that the scalar field theory in d=3+1 is <u>trivial</u> in the infrared limit!

Quantum field theory: Scalar field (2)

- Using the approximation holding at strong coupling

$$\phi_0 = \mu \int d^4x G(x-x')j(x') + \dots$$

it is not difficult to write the generating functional at the leading order in a Gaussian form

$$Z_0[j] = N \exp\left[\frac{i}{2}\int d^4x' d^4x'' j(x')G(x'-x'')j(x'')\right].$$

- This conclusion is really important: It says that the scalar field theory in d=3+1 is trivial in the infrared limit!
- This functional describes a set of free particles with a mass spectrum

$$m_n = (2n+1)\frac{\pi}{2K(i)}\left(\frac{\lambda}{2}\right)^{\frac{1}{4}}\mu$$

that are the poles of the propagator, the one of the classical theory.

Quantum field theory: Scalar field (3)

- Accounting for next-to-leading order corrections one has:

$$Z[j] \approx Z_0[j] \int [d\phi_1] e^{i\frac{1}{\lambda}\int d^4x\left\{\frac{1}{2}(\partial\phi_1)^2 - \frac{3}{2}\mu^2\lambda\left[\int d^4x'\Delta(x-x')j(x')\right]^2\phi_1^2\right\}}$$

being

$$\Delta(x-x') = G(x-x') + \mu^2\lambda \int d^4x'' G(x-x'')[G(x''-x')]^3.$$

Quantum field theory: Scalar field (3)

- Accounting for next-to-leading order corrections one has:

$$Z[j] \approx Z_0[j] \int [d\phi_1] e^{i\frac{1}{\lambda}\int d^4x \left\{\frac{1}{2}(\partial\phi_1)^2 - \frac{3}{2}\mu^2\lambda\left[\int d^4x' \Delta(x-x')j(x')\right]^2\phi_1^2\right\}}$$

being

$$\Delta(x-x') = G(x-x') + \mu^2\lambda \int d^4x'' G(x-x'')[G(x''-x')]^3.$$

- In the small momenta limit one has for the propagator

$$\Delta(p) \approx G(p)\left[1 - \frac{0.086}{4(2\pi)^4\lambda^{\frac{1}{2}}} - \frac{1}{4(2\pi)^4\lambda}\left(0.337 - 0.086\frac{p^2}{\mu^2}\right) + O\left(\frac{1}{\lambda^{\frac{3}{2}}}\right)\right].$$

Quantum field theory: Scalar field (3)

- Accounting for next-to-leading order corrections one has:

$$Z[j] \approx Z_0[j]\int [d\phi_1] e^{i\frac{1}{\lambda}\int d^4x\left\{\frac{1}{2}(\partial\phi_1)^2 - \frac{3}{2}\mu^2\lambda\left[\int d^4x'\Delta(x-x')j(x')\right]^2\phi_1^2\right\}}$$

  being

$$\Delta(x-x') = G(x-x') + \mu^2\lambda\int d^4x'' G(x-x'')[G(x''-x')]^3.$$

- In the small momenta limit one has for the propagator

$$\Delta(p) \approx G(p)\left[1 - \frac{0.086}{4(2\pi)^4\lambda^{\frac{1}{2}}} - \frac{1}{4(2\pi)^4\lambda}\left(0.337 - 0.086\frac{p^2}{\mu^2}\right) + O\left(\frac{1}{\lambda^{\frac{3}{2}}}\right)\right].$$

- This gives a renormalization constant of the field as

$$Z_\phi = \sqrt{1 - \frac{0.086}{4(2\pi)^4\lambda^{\frac{1}{2}}} + O\left(\frac{1}{\lambda}\right)} \approx 1 - \frac{0.086}{8(2\pi)^4\lambda^{\frac{1}{2}}} + O\left(\frac{1}{\lambda}\right).$$

Quantum field theory: Scalar field (4)

- The theory presents contributions from a massless propagator. From the generating functional with NLO correction one has

$$\frac{1}{i^2 Z}\frac{\delta^2 Z}{\delta j(x_2)\delta j(x_1)}\bigg|_{j=0} = \tilde{\Delta}(x_2 - x_1) = \Delta(x_2 - x_1) - 3\mu^2\lambda\Delta(x_2 - x_1)\Delta_0(0)$$

being

$$\Delta_0(x) = \frac{1}{\mu^4}\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot x}\frac{1}{p^2 - i\epsilon}.$$

Quantum field theory: Scalar field (4)

- The theory presents contributions from a massless propagator. From the generating functional with NLO correction one has

$$\frac{1}{i^2 Z}\frac{\delta^2 Z}{\delta j(x_2)\delta j(x_1)}\bigg|_{j=0} = \tilde{\Delta}(x_2 - x_1) = \Delta(x_2 - x_1) - 3\mu^2\lambda\Delta(x_2 - x_1)\Delta_0(0)$$

being

$$\Delta_0(x) = \frac{1}{\mu^4}\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot x}\frac{1}{p^2 - i\epsilon}.$$

- This integral needs to be regularized but we can accomplish this through the ultraviolet cut-off μ. So, we put

$$\Delta_0(0) = \frac{1}{32\pi^2\mu^2}.$$

Quantum field theory: Scalar field (4)

- The theory presents contributions from a massless propagator. From the generating functional with NLO correction one has

$$\frac{1}{i^2 Z}\frac{\delta^2 Z}{\delta j(x_2)\delta j(x_1)}\Big|_{j=0} = \tilde{\Delta}(x_2 - x_1) = \Delta(x_2 - x_1) - 3\mu^2\lambda\Delta(x_2 - x_1)\Delta_0(0)$$

being

$$\Delta_0(x) = \frac{1}{\mu^4}\int \frac{d^4p}{(2\pi)^4} e^{ip\cdot x}\frac{1}{p^2 - i\epsilon}.$$

- This integral needs to be regularized but we can accomplish this through the ultraviolet cut-off μ. So, we put

$$\Delta_0(0) = \frac{1}{32\pi^2\mu^2}.$$

- This contribution can be computed exactly with a functional Taylor expansion of the generating functional at all orders.

Quantum field theory: Scalar field (4)

- The theory presents contributions from a massless propagator. From the generating functional with NLO correction one has

$$\frac{1}{i^2 Z}\frac{\delta^2 Z}{\delta j(x_2)\delta j(x_1)}\Big|_{j=0} = \tilde{\Delta}(x_2 - x_1) = \Delta(x_2 - x_1) - 3\mu^2\lambda\Delta(x_2 - x_1)\Delta_0(0)$$

being

$$\Delta_0(x) = \frac{1}{\mu^4}\int\frac{d^4p}{(2\pi)^4}e^{ip\cdot x}\frac{1}{p^2 - i\epsilon}.$$

- This integral needs to be regularized but we can accomplish this through the ultraviolet cut-off μ. So, we put

$$\Delta_0(0) = \frac{1}{32\pi^2\mu^2}.$$

- This contribution can be computed exactly with a functional Taylor expansion of the generating functional at all orders.
- This NLO contribution arises by a massless propagator. This is a zero mode due to translational invariance and just gives an overall multiplicative constant to the Gaussian generating functional.

Running coupling (1)

- We can determine the beta function in the infrared limit from the propagator. This beta function must be the same both for the scalar theory and the Yang-Mills field for this limit due to the mapping theorem.

Running coupling (1)

- We can determine the beta function in the infrared limit from the propagator. This beta function must be the same both for the scalar theory and the Yang-Mills field for this limit due to the mapping theorem.
- It is very easy to check that the following Callan-Symanzik equation holds in this case:

$$\mu\frac{\partial G(p)}{\partial\mu}-4\lambda\frac{\partial G(p)}{\partial\lambda}-\gamma G(p)=0$$

  and we can identify $\beta(\lambda)=4\lambda$ and $\gamma=0$. Using mapping theorem we can state $\beta(g)=4Ng^2$ for a Yang-Mills theory.

Running coupling (1)

- We can determine the beta function in the infrared limit from the propagator. This beta function must be the same both for the scalar theory and the Yang-Mills field for this limit due to the mapping theorem.
- It is very easy to check that the following Callan-Symanzik equation holds in this case:

$$\mu\frac{\partial G(p)}{\partial\mu}-4\lambda\frac{\partial G(p)}{\partial\lambda}-\gamma G(p)=0$$

  and we can identify $\beta(\lambda)=4\lambda$ and $\gamma=0$. Using mapping theorem we can state $\beta(g)=4Ng^2$ for a Yang-Mills theory.
- This implies that the running coupling in the infrared limit for the Yang-Mills theory increases like p^4.

Running coupling (1)

- We can determine the beta function in the infrared limit from the propagator. This beta function must be the same both for the scalar theory and the Yang-Mills field for this limit due to the mapping theorem.
- It is very easy to check that the following Callan-Symanzik equation holds in this case:

$$\mu\frac{\partial G(p)}{\partial\mu} - 4\lambda\frac{\partial G(p)}{\partial\lambda} - \gamma G(p) = 0$$

  and we can identify $\beta(\lambda) = 4\lambda$ and $\gamma = 0$. Using mapping theorem we can state $\beta(g) = 4Ng^2$ for a Yang-Mills theory.
- This implies that the running coupling in the infrared limit for the Yang-Mills theory increases like p^4.
- Such a conclusion would support a view of Yang-Mills vacuum as an instanton liquid [T. Schäfer and E. V. Shuryak (1998), P. Boucaud et al. (2003)].

Running coupling (1)

- We can determine the beta function in the infrared limit from the propagator. This beta function must be the same both for the scalar theory and the Yang-Mills field for this limit due to the mapping theorem.
- It is very easy to check that the following Callan-Symanzik equation holds in this case:

$$\mu\frac{\partial G(p)}{\partial\mu} - 4\lambda\frac{\partial G(p)}{\partial\lambda} - \gamma G(p) = 0$$

  and we can identify $\beta(\lambda) = 4\lambda$ and $\gamma = 0$. Using mapping theorem we can state $\beta(g) = 4Ng^2$ for a Yang-Mills theory.
- This implies that the running coupling in the infrared limit for the Yang-Mills theory increases like p^4.
- Such a conclusion would support a view of Yang-Mills vacuum as an instanton liquid [T. Schäfer and E. V. Shuryak (1998), P. Boucaud et al. (2003)].
- Recent analysis on scalar field theory supports such a conclusion [I. Suslov arXiv:0911.1149v1 [hep-th], D. Podolsky, arXiv:1003.3670v1 [hep-th]].

Running coupling (2)

- We can easily get higher order corrections to the β function by noting that, in the limit $\boldsymbol{p} \to 0$,

$$G(0) \approx -\frac{1.11}{\mu^2\lambda^{\frac{1}{2}}}\left[1 - \frac{0.086}{4(2\pi)^4\lambda^{\frac{1}{2}}} - \frac{0.337}{4(2\pi)^4\lambda} + O\left(\frac{1}{\lambda^{\frac{3}{2}}}\right)\right].$$

and we recognize at the leading order the Nambu-Jona-Lasinio limit [Frasca (2009)].

Running coupling (2)

- We can easily get higher order corrections to the β function by noting that, in the limit $\boldsymbol{p} \to 0$,

$$G(0) \approx -\frac{1.11}{\mu^2\lambda^{\frac{1}{2}}}\left[1 - \frac{0.086}{4(2\pi)^4\lambda^{\frac{1}{2}}} - \frac{0.337}{4(2\pi)^4\lambda} + O\left(\frac{1}{\lambda^{\frac{3}{2}}}\right)\right].$$

  and we recognize at the leading order the Nambu-Jona-Lasinio limit [Frasca (2009)].

- The corresponding Callan-Symanzik equation gives to next-to-leading order:

$$\beta(\lambda) = 4\lambda - \frac{0.344}{(2\pi)^4}\sqrt{\lambda} + \ldots.$$

Running coupling (2)

- We can easily get higher order corrections to the β function by noting that, in the limit $\boldsymbol{p} \to 0$,

$$G(0) \approx -\frac{1.11}{\mu^2\lambda^{\frac{1}{2}}}\left[1 - \frac{0.086}{4(2\pi)^4\lambda^{\frac{1}{2}}} - \frac{0.337}{4(2\pi)^4\lambda} + O\left(\frac{1}{\lambda^{\frac{3}{2}}}\right)\right].$$

and we recognize at the leading order the Nambu-Jona-Lasinio limit [Frasca (2009)].

- The corresponding Callan-Symanzik equation gives to next-to-leading order:

$$\beta(\lambda) = 4\lambda - \frac{0.344}{(2\pi)^4}\sqrt{\lambda} + \ldots.$$

- Similarly, we get an anomalous dimension

$$\gamma = \frac{0.344}{(2\pi)^4\sqrt{\lambda}}.$$

and we prove in this way that $\beta(\lambda)/\lambda$ has an expansion in $\lambda^{-1/2}$ in agreement with Suslov (2011).

Quantum field theory: Yang-Mills field (1)

- We now use the mapping theorem fixing the form of the propagator in the infrared, e.g. in the Landau gauge, as

$$D_{\mu\nu}^{ab}(p)=\delta_{ab}\left(\eta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\sum_{n=0}^{\infty}\frac{B_n}{p^2-m_n^2+i\epsilon}+O\left(\frac{1}{\sqrt{N}g}\right)$$

but this can be recomputed in any gauge by the classical equations with the mapping theorem. We note the presence of the gap.

Quantum field theory: Yang-Mills field (1)

- We now use the mapping theorem fixing the form of the propagator in the infrared, e.g. in the Landau gauge, as

$$D_{\mu\nu}^{ab}(p)=\delta_{ab}\left(\eta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\sum_{n=0}^{\infty}\frac{B_n}{p^2-m_n^2+i\epsilon}+O\left(\frac{1}{\sqrt{N}g}\right)$$

  but this can be recomputed in any gauge by the classical equations with the mapping theorem. We note the presence of the gap.

- The next step is to use the approximation that holds in a strong coupling limit

$$A_\mu^a=\Lambda\int d^4x' D_{\mu\nu}^{ab}(x-x')j^{b\nu}(x')+O\left(\frac{1}{\sqrt{N}g}\right)+O(j^3)$$

Quantum field theory: Yang-Mills field (1)

- We now use the mapping theorem fixing the form of the propagator in the infrared, e.g. in the Landau gauge, as

$$D^{ab}_{\mu\nu}(p)=\delta_{ab}\left(\eta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\sum_{n=0}^\infty \frac{B_n}{p^2-m_n^2+i\epsilon}+O\left(\frac{1}{\sqrt{N}g}\right)$$

  but this can be recomputed in any gauge by the classical equations with the mapping theorem. We note the presence of the gap.

- The next step is to use the approximation that holds in a strong coupling limit

$$A^a_\mu=\Lambda\int d^4x' D^{ab}_{\mu\nu}(x-x')j^{b\nu}(x')+O\left(\frac{1}{\sqrt{N}g}\right)+O(j^3)$$

- and we note that, in this approximation, the ghost field just decouples and becomes free and one finally has at the leading order

$$Z_0[j]=N\exp\left[\frac{i}{2}\int d^4x'd^4x''j^{a\mu}(x')D^{ab}_{\mu\nu}(x'-x'')j^{b\nu}(x'')\right].$$

  This functional describes free massive glueballs that are the proper states in the infrared limit. Yang-Mills theory is trivial in the limit of the coupling going to infinity and we expect the running coupling to go to zero lowering energies.

Quantum field theory: Yang-Mills field (2)

- Let us consider the following two-point function

$$D^{ab}_{\mu\nu}(t-t',0) = \langle T A^a_\mu(t,0) A^b_\nu(t',0)\rangle$$

Quantum field theory: Yang-Mills field (2)

- Let us consider the following two-point function

$$D^{ab}_{\mu\nu}(t-t',0) = \langle T A^a_\mu(t,0) A^b_\nu(t',0)\rangle$$

- From the mapping theorem and the corresponding propagator for the scalar theory it is not difficult to see that

$$D^{ab}_{\mu\nu}(t-t',0) = \eta^a_\mu \eta^b_\nu \sum_{n=0}^{\infty} B_n e^{-im_n(t-t')}$$

being η^a_μ some constants.

Quantum field theory: Yang-Mills field (2)

- Let us consider the following two-point function

$$D^{ab}_{\mu\nu}(t-t',0) = \langle T A^a_\mu(t,0) A^b_\nu(t',0)\rangle$$

- From the mapping theorem and the corresponding propagator for the scalar theory it is not difficult to see that

$$D^{ab}_{\mu\nu}(t-t',0) = \eta^a_\mu \eta^b_\nu \sum_{n=0}^{\infty} B_n e^{-im_n(t-t')}$$

  being η^a_μ some constants.

- So, the spectrum of the theory is uncovered to be

$$m_n = (2n+1)\frac{\pi}{2K(i)}\sqrt{\sigma}$$

  being $\sqrt{\sigma} = \left(Ng^2/2\right)^{\frac{1}{4}}\Lambda$ the string tension to be fixed experimentally.

Quantum field theory: Yang-Mills field (2)

- Let us consider the following two-point function

$$D^{ab}_{\mu\nu}(t-t',0)=\langle TA^a_\mu(t,0)A^b_\nu(t',0)\rangle$$

- From the mapping theorem and the corresponding propagator for the scalar theory it is not difficult to see that

$$D^{ab}_{\mu\nu}(t-t',0)=\eta^a_\mu\eta^b_\nu\sum_{n=0}^{\infty}B_ne^{-im_n(t-t')}$$

being η^a_μ some constants.

- So, the spectrum of the theory is uncovered to be

$$m_n=(2n+1)\frac{\pi}{2K(i)}\sqrt{\sigma}$$

being $\sqrt{\sigma}=\left(Ng^2/2\right)^{\frac{1}{4}}\Lambda$ the string tension to be fixed experimentally.

- We see that, in the infrared limit, Yang-Mills theory displays a spectrum of free massive particles with a superimposed spectrum of a harmonic oscillator (they are structure-like).

Conclusions

- We provided a strong coupling expansion both for classical and quantum field theory of a massless quartic scalar field.

Conclusions

- We provided a strong coupling expansion both for classical and quantum field theory of a massless quartic scalar field.
- Using a mapping theorem, results obtained for the scalar field are straightforwardly applied to the Yang-Mills field in the limit of an increasingly large coupling.

Conclusions

- We provided a strong coupling expansion both for classical and quantum field theory of a massless quartic scalar field.
- Using a mapping theorem, results obtained for the scalar field are straightforwardly applied to the Yang-Mills field in the limit of an increasingly large coupling.
- Agreement with numerical solutions to the Dyson-Schwinger equations is strikingly good. Worsening is seen in the intermediate region where it should be expected.

Conclusions

- We provided a strong coupling expansion both for classical and quantum field theory of a massless quartic scalar field.
- Using a mapping theorem, results obtained for the scalar field are straightforwardly applied to the Yang-Mills field in the limit of an increasingly large coupling.
- Agreement with numerical solutions to the Dyson-Schwinger equations is strikingly good. Worsening is seen in the intermediate region where it should be expected.
- Running coupling is seen to go to zero as a power of decreasing momenta. Pure Yang-Mills theory is proved trivial in the infrared even if QCD is surely not.

Conclusions

- We provided a strong coupling expansion both for classical and quantum field theory of a massless quartic scalar field.
- Using a mapping theorem, results obtained for the scalar field are straightforwardly applied to the Yang-Mills field in the limit of an increasingly large coupling.
- Agreement with numerical solutions to the Dyson-Schwinger equations is strikingly good. Worsening is seen in the intermediate region where it should be expected.
- Running coupling is seen to go to zero as a power of decreasing momenta. Pure Yang-Mills theory is proved trivial in the infrared even if QCD is surely not.
- Higher order corrections were also provided obtaining an expansion in $1/\lambda^{\frac{1}{2}}$.

Conclusions

- We provided a strong coupling expansion both for classical and quantum field theory of a massless quartic scalar field.
- Using a mapping theorem, results obtained for the scalar field are straightforwardly applied to the Yang-Mills field in the limit of an increasingly large coupling.
- Agreement with numerical solutions to the Dyson-Schwinger equations is strikingly good. Worsening is seen in the intermediate region where it should be expected.
- Running coupling is seen to go to zero as a power of decreasing momenta. Pure Yang-Mills theory is proved trivial in the infrared even if QCD is surely not.
- Higher order corrections were also provided obtaining an expansion in $1/\lambda^{\frac{1}{2}}$.
- The main conclusion is that computations for strongly coupled quantum field theory can be done much in the same way as for a weak coupling.