A Reflexive toy-model for financial market

An alternative route towards intermittency.

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SM&FT 2011 - The XV Workshop on Statistical
Mechanics and nonpertubative Field Theory
Bari, 21-23 Settembre 2011

Brief Summary

1. We will mathematically define the reflexive systems.

2. We will apply this definition to the financial market.

3. We will propose an explicit reflexive model and we will describe its intermittent behavior

4. We will compare the model behavior with some stylized properties of actual financial market.

5. We will discuss the reason of the intermittent behavior and we describe a possible alternative “route towards intermittency”.

Reflexive models: a mathematical definition

The cognitive function $y = f(x)$ $\rightarrow$ the observer tries to model the behavior of the system and, depending on its ideas, chooses a cognitive function $f(x)$ of the system variable x. The observer is then driven in his/her choices by the value of the variable y.

The manipulative function $\Phi(y)$ $\rightarrow$ describes the effect of the observer opinion and choices (depending on the value of y) on the behavior of the system.

Completeley reflexive model:

$$\begin{cases} y(t) = f(x(t)) \\ x(t+1) = \Phi(y(t)) \end{cases} \tag{1}$$

Mixed model with intrinsic dynamics (not depending on the observer):

$$\begin{cases} y(t) = f(x(t)) \\ x(t+1) = g(x(t)) + \Phi(y(t), x(t)). \end{cases} \tag{2}$$

If $\Phi(y(t), x(t)) = 0$ this system reduces to a simple dynamical system (also stochastic) driven by the function $g(x)$, without reflexive effect.

Application to financial market

Cognitive function $y = f(x) \longrightarrow$ all statistical analysis techniques that the agents use to predict the future behavior of the market.

Manipulative function $\Phi(y, x) \longrightarrow$ the effect of agents opinion on the behavior of the price.

The main idea is not new (Soros, Kalecki and Minsky) but it is typically mathematically treated in the context of multi-agent models. In particular minority game models show a practically vanishing manipulative function.

We assume that the minority game is not the correct paradigm of financial market: during trends and bubbles, for example, almost all agents earn something and this is in contrast with minority game

$$\Downarrow$$

In some sense minority game do not treat the case where money supply and/or average values of investment goods increase.

An explicit model: the linear reflexive model (LRM)*

What is the best linear model fitting the last N observations $r(t-\tau)$, with $0 \leq \tau \leq N-1$?

$$\tilde{r}(t+1) = f(t)r(t), \tag{3}$$

where $r(t)$ is the logarithmic return of price, i.e. $r(t) \equiv \log\left[S(t)/S(t-1)\right]$,

Minimizing the error of the model χ^2

$$\chi^2 = \sum_{\tau=0}^{N-1}\left[\tilde{r}(t-\tau) - r(t-\tau)\right]^2 = \sum_{\tau=0}^{N-1}\left[f(t)r(t-\tau-1) - r(t-\tau)\right]^2, \tag{4}$$

we obtain:

$$f(t) = \frac{\sum\limits_{\tau=0}^{N-1} r(t-\tau)r(t-\tau-1)}{\sum\limits_{\tau=0}^{N-1} r(t-\tau-1)^2}. \tag{5}$$

Thus the best choice for $f(t)$ is similar to the normalized correlation of the previous observations.

*L. Palatella, A reflexive toy model for financial market, Physica A **389**, 315-322 (2010)

Still on the model

If we want $f(t)$ to exactly fulfill the condition $-1 \leq f(t) \leq +1$ we choose:

$$f(t) = \frac{\sum_{\tau=0}^{N-2} r(t-\tau) r(t-\tau-1)}{\left(\sum_{\tau=0}^{N-2} r(t-\tau)^2\right)^{1/2} \left(\sum_{\tau=1}^{N-1} r(t-\tau)^2\right)^{1/2}} = \frac{\mathbf{r}(t) \cdot \mathbf{r}(t-1)}{\|\mathbf{r}(t)\| \|\mathbf{r}(t-1)\|} \tag{6}$$

We make the simplifying hypothesis that N is identical for all the market operators.

We build up a model where the next logarithmic return price is a weighted average of the previous return and of a random component

$$r(t+1) = \underbrace{r_0(1 - \phi(f(t)) + \sigma_0)\eta(t)}_{stochastic\ term} + \underbrace{\phi(f(t))f(t)r(t)}_{reflexive\ term} \tag{7}$$

Central hypothesis: the "weight" $\phi(f(t))$ of the factorized manipulative function $\Phi(y, x) = \phi(f(t))G(x(t))$: we suppose that $\phi(f(t)) = |f(t)|$, this means that there are conditions where

- $f(t) \to \pm 1$
- all agents believe in the linear model
- the influence of all agents on the price makes the linear model to work!

We only let the possibility of a residual stochastic term parametrized by σ_0 due to agents and to the economical reasons that are not related to the reflexive effect.

The complete model equations

$$\begin{cases} f(t) = \dfrac{\sum\limits_{\tau=0}^{N-2} r(t-\tau)r(t-\tau-1)}{\left(\sum\limits_{\tau=0}^{N-2} r(t-\tau)^2\right)^{1/2}\left(\sum\limits_{\tau=1}^{N-1} r(t-\tau)^2\right)^{1/2}} \quad \phi(f(t)) = |f(t)| \\ r(t+1) = r_0(1-\phi(f(t))+\sigma_0)\{\eta(t)+\gamma[x_c(t)-x(t)]\} \\ \qquad\qquad +\phi(f(t))f(t)r(t) \\ x(t+1) = x(t) + r(t) \\ x_c(t+1) = x_c(t) + r_0(1+\sigma_0)\eta(t), \end{cases} \tag{8}$$

the stochastic term $\eta(t)$ is an i.i.d. Gaussian variable with $\langle\eta\rangle = 0$ and $\langle\eta(t)\eta(t')\rangle = \delta_{t,t'}$.
$x(t) = \log S(t); x_c(t) = \log S_c(t)$

The last ingredient: the dynamics of the *correct price* (i.e. without reflexive effect) and the reversion to its value. The dynamics of $x_c(t)$ is obtained putting $\phi(f(t)) = 0$. The reversion is parametrized by $\gamma \ll 1$.

$$x_c(t+1) = x_c(t) + r_0(1+\sigma_0)\eta(t),$$

Values of parameters chosen: $\gamma \in [0, 10^{-4}]$, $N = 3, 5, 6, 7, 12, 15$. r_0 depends on the time scale used, $0.01 < \sigma_0 < 0.1$.

The behavior of the model: equilibrium points

Let us start with $\sigma_0 = 0$. The fixed point equation:

$f(t+1) = f(t) \forall t \Rightarrow \phi(f(t)) = 1, f(t) = \pm 1.$

In this case, for arbitrary values of $\bar{r}$, we have two equilibrium points for $f(t)$:

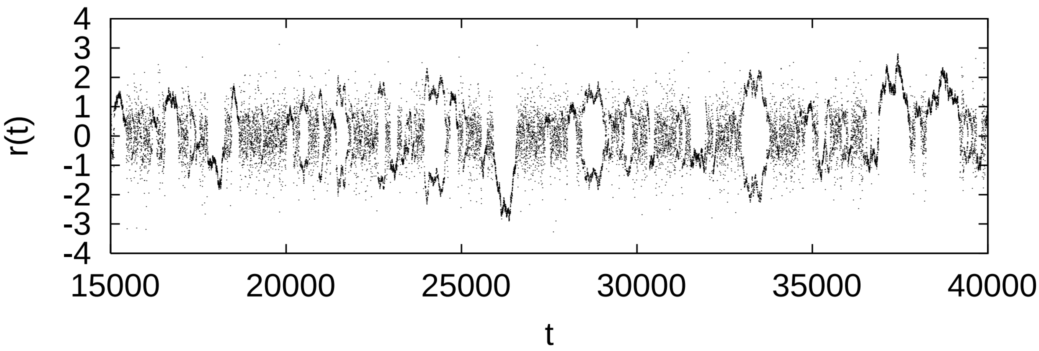
$$
\begin{aligned}
&r(t+1) = r(t) = \bar{r}, \;\; f(t) = 1 \;\; \text{(trend)} \\
&r(t+1) = -r(t) = -\bar{r}, \;\; f(t) = -1 \;\; \text{(lateral movement)}.
\end{aligned}
$$

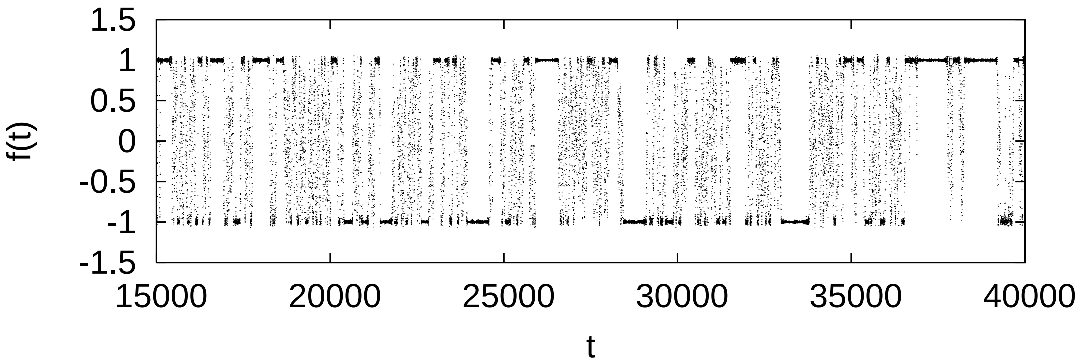
The first point corresponds to a *trend* state where the return of the price shows a persistent trend of increase (if $r(t) > 0$) or decrease (if $r(t) < 0$).

The second point corresponds to what technical analysts call *lateral movement*, where the price goes up and down without significant displacement from a reference level.

If $\sigma_0 \neq 0$, these equilibrium points become unstable and the model shows an intermittent behavior.

The behavior of the model: intermittency





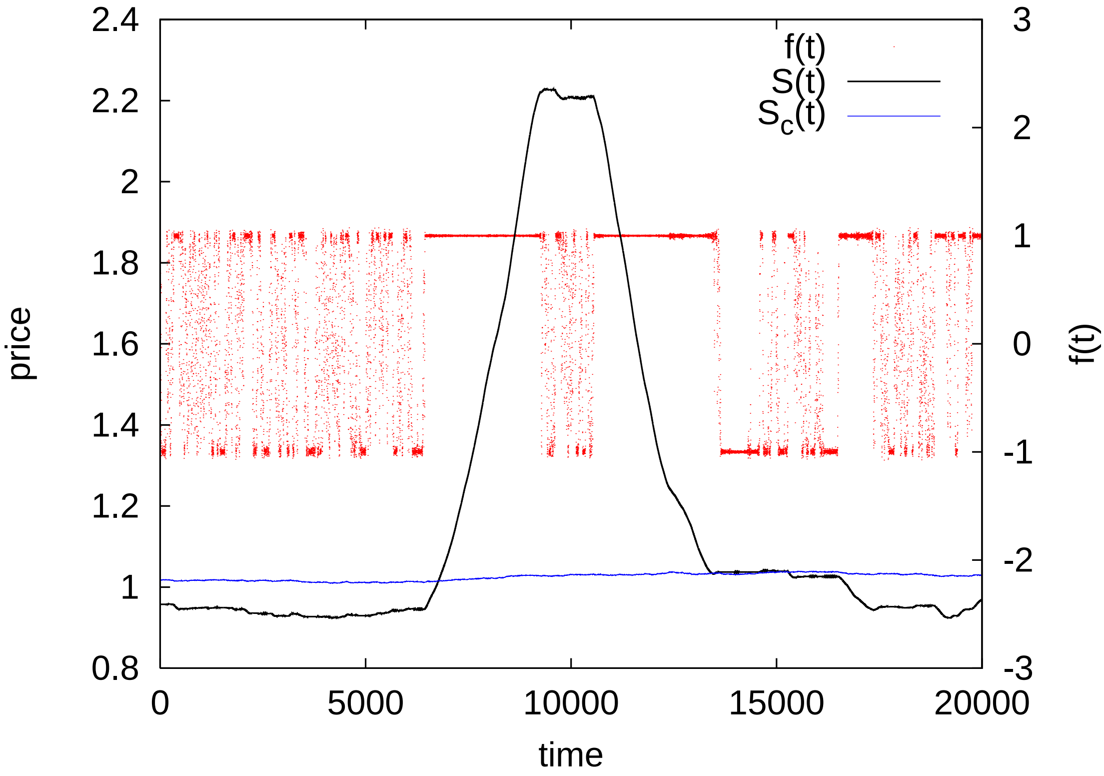
The function $f(t)$ oscillates in an intermittent way between the values ± 1

During each "laminar zone" the value of the return $r(t)$ slowly changes.

The "diffusion" of $\bar{r}$ is due to the fact that going slowly (with respect to N) from $\bar{r} \to \lambda \bar{r}$ the value of $f(t) = \pm 1$ is not affected.

The laminar zone becomes unstable when $|r(t)| \to 0$

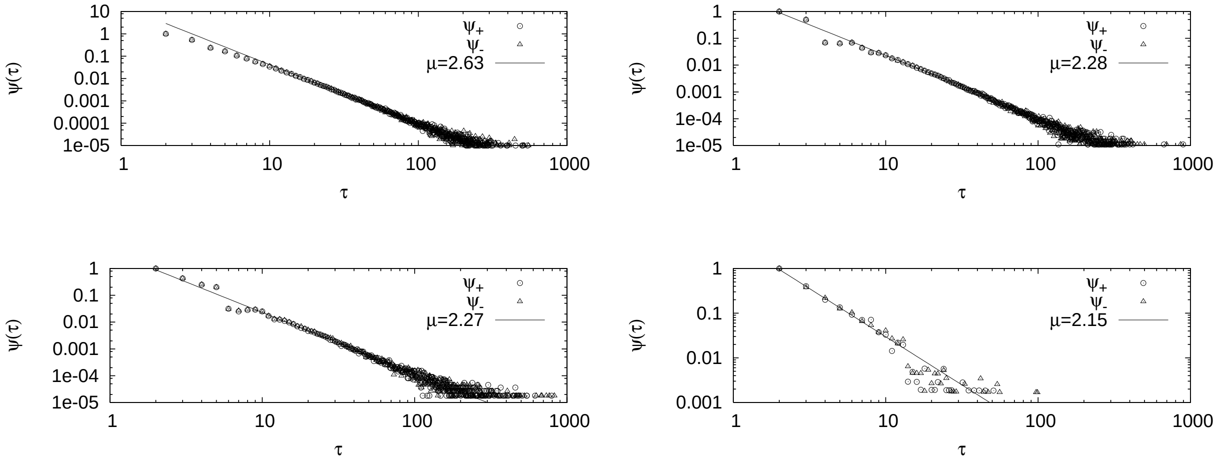
The behavior of the model: boom-bust cycle



We first observe a laminar zone with $f(t) = 1$ and $r(t) > 0$ leading to a boom.

At the end of the boom (not necessarily due to the reverting force γ) we observe a return to the "right" price that induces itself a reflexive decay with $f(t) = 1$ but $r(t) < 0$.

The behavior of the model: waiting time statistics in $f(t) = \pm 1$



We numerically compute the waiting time statistics in the two states $f(t) = \pm 1$, $\psi_{\pm}(t)$.

We observe a clear asymptotic power law tail behavior.

Figures show results for $N = 3, 5, 7, 15$ (from top to bottom) and $\sigma_0 = 0.05$

The fit gives for the exponent of the tail $\mu = 2.63, 2.28, 2.27, 2.15$, respectively.

There is a large interval of values for N and σ_0 where $2.0 \leq \mu \leq 2.5$

Comments on the model: connection with real financial market time series

There are several points in common between this model behavior and some properties of real market dynamics

1. Observed over a long time interval the $r(t)$ series is uncorrelated because there are the same number of situations where $r(t+1)$ is approximatively equal to $\pm r(t)$. At the same time there are long periods of trends and of "lateral movement" on a shorter time interval (but still longer than N).

2. The reflexive effects can lead the price *outside of equilibrium* very far from the correct price value.

3. The diffusion generated by the time series $r(t)$ has fat tails.

4. The exponent of the power law tail μ is in agreement with the long time scaling observed in a previous work on real time financial time series (US$-DM futures and DJIA, $\mu \simeq 2.2$) using diffusion entropy technique †

†L. Palatella, J. Perelló, M. Montero, J. Masoliver, Activity autocorrelation in financial market,Eur. Phys. J. B **38**, 671-677 (2004).

Conclusion and discussion: from toy- to real-model

Several steps are needed before using this model directly on real market dynamics:

- Parameter estimation: we need to calibrate the model to obtain the values of σ_0 and N and to account for the presence of different N-agents.

- Probably we need to eliminate perfect symmetry, increasing the reflexive effect during the decreasing trends (like leverage effect in GARCH-like models)

- After parameter estimation one could perform risk-neutral evaluation for option pricing using the linear reflexive model and compare with BS results.

For pure mathematicians and for people who are not interested to economical and financial applications:

- We show that a white noise random system with an observer that tries to model the system dynamics and that at the same time can affect it may lead to an intermittent behavior.

- We propose an alternative route to power law tail intermittency different from the standard Manneville map. In LRM intermittency is due to the existence of a "line" of indifferent equilibrium points (from $r(t) = \bar{r}$ to $r(t) = \lambda\bar{r}$).

An alternative route towards power law tail intermittency

The LRM model, also in the simplified version with $\gamma = 0$, leads to power law tail distribution of permanence times with asymptotic power law index $\mu \simeq 2$.

This value is important for different reasons and for various topics like earthquakes, financial time series, DNA sequences, heartbeat, etc.

Why does it happen in the LRM?

We show [‡] that the LRM in the laminar zones follows the approximated dynamics

$$dx = -\frac{\alpha}{x}dt + \sigma_0 dW(t), \;\; x(0) = x_0$$

where $dW(t)$ are the increments of a Wiener process. The system enters in the laminar zone with a value of $|r(t)| \simeq 1$ and can exit only when it comes back to the origin (First Passage Time problem).

We *analitically* prove that the FPTs of this simplified model are distributed according to a probability density function $\psi(t)$ that asymptotically behaves like

$$\psi(t) \simeq t^{-\mu}, \; \mu = 3/2 + \beta/2 \text{ as } t \to +\infty. \quad (9)$$

where $\beta \equiv 2\alpha/\sigma_0$. Notice that if $\beta = 0$ we have $\mu = 3/2$. α is obtained numerically but it is proportional to σ_0 thus we prove that the value of μ is robust under additive noise disturbance.

[‡] L. Palatella, *Intermittency in an invariant deterministic dynamical system perturbed by non-invariant noise*, submitted to Physical Review

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