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Markov Process at the Velocity of Light: The Klein–Gordon Statistic

N. Cufaro Petroni

University of Bari, Italy

J. P. Vigier

Institut Henri Poincaré, Paris, France

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The Markovian random walk of a point at the velocity of light on a two-dimensional invariant space–time lattice is shown to yield the quantum statistic associated with the Klein–Gordon equation. Quantum mechanics thus appears as a particular case of Markovian processes in velocity space: and one justifies the introduction of Dirac's invariant "ether" as a possible physical stochastic subquantum level of matter which yields a realistic mechanical basis for recent attempts to reinterpret quantum mechanics in terms of material, causal, random behavior.

1. INTRODUCTION

Recent discussions on the Einstein-Podolsky-Rosen paradox (1935) have shown that quantum mechanics implies spacelike correlations between two linear polarizers which measure the rate of coincidence between the relative orientations of pairs of photons emitted in the S state. If a forthcoming crucial experiment of Aspect (1976) confirms this then the only possible "causal" (i.e., which preserves the fundamental fact that no individual particle can leave the light cone) way out of the resulting contradiction between relativity and the quantum theory of measurement seems to lie in the direction of an extension of the stochastic interpretations of quantum mechanics in terms of subquantum random fluctuations resulting from the action of a stochastic "hidden" invariant thermostat. Indeed these models (a) deduce the form of the quantum waves from the physical assumption that the stochastic jumps occur at the velocity of

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light; (b) interpret the preceding superluminal interaction in terms of superluminal propagation of a "quantum potential" (Vigier, 1979) which is not carried by individual particles but results from phaselike collective motions carried by the said thermostat.

The aim of the present paper is to analyze in a more precise way the physical and mathematical implications of these stochastic interpretations in the particular case of scalar particles.

In Section 2 we shall briefly discuss the physical properties of the only possible invariant undetectable relativistic thermostat known in the literature—i.e., Dirac's "ether" model: a model that provides a realistic physical basis for the above-mentioned interpretations.

In Section 3 we shall discuss the mathematical significance of the stochastic demonstrations already given in the literature starting among others with Bohm and Vigier (1954), Nelson (1966), de Broglie (1961), and the growing number of papers dealing with stochastic electrodynamics (De la Peña and Cetto, 1975).

2. THE SUBQUANTUM THERMOSTAT

All these models imply of course a modern revival of the old "ether" idea: a concept apparently definitively destroyed by the negative result of Michelson's experiment. As one knows, however, Dirac (1951) has shown that it is not so and that one can construct at least one material covariant "ether" perfectly compatible with relativity. It rests on the idea that through any point 0 passes a flow of stochastic particles and antiparticles (described in Figure 1 as particles moving backwards in time) whose momenta have the extremities of their four-vectors P^μ (with $P^\mu P_\mu = m^2c^2$) distributed with a uniform surface density on the two three-dimensional surfaces of the hyperboloids H_+ and H_-. They will thus remain invariant under all Lorentz transformations.

This stochastic relativistic distribution constitutes the only possible model for a physical undetectable thermostat for spin-zero particles into which we can study the relativistic analog of the classical nonrelativistic Brownian motion. Dirac has derived this from the indeterminacy principle. However, it differs from it by two new physical properties.

(a) Since the light cone behaves like an asymptotic accumulation manifold of Dirac's stochastic distribution we can assume that the corresponding stochastic jumps of a Brownian particle, submitted to its random action, occur practically at the velocity of light. Indeed, any given exchanged energy is statistically superseded by more energetic interactions.

(b) This ultrarelativistic Brownian motion includes the possibility of pair creation and/or annihilation. This is important since the mixture of

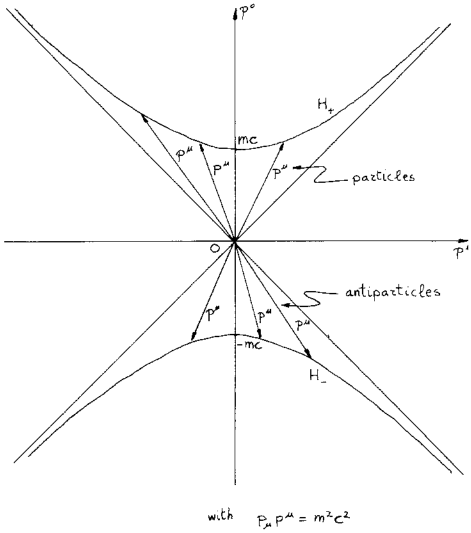

Fig. 1

particles and antiparticles has been shown to provide a realistic interpretation (Terletski and Vigier, 1961) of possible, negative, probability distributions.

The concrete analysis of this particular covariant case of stochastic motion can be carried along the two lines of demonstration utilized in nonrelativistic stochastic theory. The first line is just a relativistic generalization of the ideas introduced by Einstein and Smoluchowski into Brownian motion theory. Assuming that our particles are (1) carried along the lines of flow or a regular drift motion **v** of extended particles associated with a collective motion on the top of Dirac's thermostat, (characters in boldface type) denoting four-vectors, (2) jump stochastically at the velocity of light from one average drift line of flow to another and thus (for an ensemble of identical particles with arbitrary initial positions) reach an average mean conserved distribution $\rho(\mathbf{x})$; one can immediately demonstrate the stochastic force law first assumed by Nelson (1966), from which one deduces (Lehr and Park, 1977; Vigier, 1979; Guerra and Ruggiero, 1978) a stochastic wave $\psi(\mathbf{x}) = [\rho(\mathbf{x})]^{1/2} \exp[(iS(\mathbf{x})/\hbar]$ with $\mathbf{v} = (1/m)\nabla S$, which satisfies the Klein–Gordon equation.

This demonstration, however, being based on averages taken over four-dimensional volume elements, does not connect directly the underlying particle behavior with known statistical models discussed in the mathematical literature, such as Markovian processes.

The aim of the present work is thus to extend to the preceding relativistic case the second line of approach discussed in the nonrelativistic literature, i.e., to study the random walk of a moving point on a lattice discussed by Chandrasekhar (1943) in a famous paper and later extended from elliptical to hyperbolical equations by Avez (1976). This will be done in the next section.

3. RANDOM WALK ON A COVARIANT LATTICE

To simplify our demonstration we shall limit ourselves to the study of a two-dimensional space–time case x^0, x^1. Indeed, as will be shown later, its extension to four dimensions presents no conceptual difficulty.

First one can check immediately that the points P_{nm} located at the intersection of the set of curves

$$\begin{aligned} &x^{0^2} - x^{1^2} = \lambda_n^2, \qquad \lambda_0 > 0, \quad \lambda_n = \lambda_0 e^{n\delta}, \quad n, m = 0, \pm 1, \pm 2 \ldots \\ &x^1 = (\tanh\theta_m) x^0, \quad \theta_0 = 0, \quad \theta_m = m\delta, \quad \delta > 0 \end{aligned} \tag{3.1}$$

build an invariant discrete lattice (see Figure 2) in which the relativistic interval between $P_{n,m}$ and each $P_{n\pm 1, m\pm 1}$ is zero. The explicit expression for the $P_{n,m}$ coordinates is

$$x^0_{n,m} = \lambda_n \cosh\theta_m, \qquad x^1_{n,m} = \lambda_n \sinh\theta_m \tag{3.2}$$

The preceding lattice is clearly covariant since each point $P_{n,m}$ stands at the intersection of three intrisically invariant lines (i.e., a spacelike hyperbola and two isotropic light-cone-defining lines) which are transformed into themselves by any ortochronous Lorentz transformations.

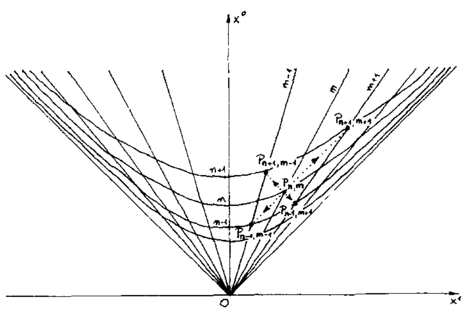

Fig. 2

The (finite) coordinate differences $\Delta_{t,s}x^0_{n,m}, \Delta_{t,s}x^1_{n,m}$ between the two points $P_{n,m}$ and $P_{n+t,m+s}$ ($t=\pm 1; s=\pm 1$) connected by one stochastic jump will satisfy

$$\frac{\Delta_{t,s}x^1_{n,m}}{\Delta_{t,s}x^0_{n,m}} = \frac{s}{t} = \pm 1 \tag{3.3}$$

In order to describe random walks on this lattice let us now define two sets of stochastic variables $\{\varepsilon_1, \varepsilon_2, \dots \varepsilon_j \dots\}; \{\eta_1, \eta_2, \dots \eta_k \dots\}$ with $\varepsilon_j = \pm 1, \eta_k = \pm 1$ for every j,k. The sign of ε_j (η_k) determines the fact that in the corresponding jump the velocity (the time orientation) has changed its sign, $\varepsilon_j = -1$ ($\eta_k = -1$) or has remained unchanged $\varepsilon_j = \pm 1$ ($\eta_k = \pm 1$) with respect to the preceding jump.

One then checks immediately that the general expression for the displacement $D_N^{t,s}(n,m)$, after N jumps from the initial point $P_{n,m}$ and a first jump in the direction defined by (t,s), can be written as the development

$$D_N^{t,s}(n,m) = \frac{s}{t}\Big(\Delta_{t,s}x^0_{n,m} + \varepsilon_1\Delta_{t\eta_1, s\eta_1\varepsilon_1}x^0_{n+t,m+s} + \varepsilon_1\varepsilon_2\Delta_{t\eta_1\eta_2, s\eta_1\eta_2\varepsilon_1\varepsilon_2}x^0_{n+t(1+\eta_1), m+s(1+\eta_1\varepsilon_1)} + \cdots\Big) \tag{3.4}$$

The probabilities for the realization of the signs of ε_j, η_k (with $j=k=1$) are given by Table I.

The functions $F_N^{t,s}(n,m) = \langle f(x^1_{n,m} + D_N^{t,s}(n,m))\rangle$ are the mean values of a function f (defined on the lattice) over all random walks of N jumps; they satisfy the following system of recurrence relations [one for each value of (t,s)]:

$$\begin{aligned}F_N^{t,s}(n,m) = {} & \big(1 - A\Delta_{-t,s}x^0_{n+t,m+s} - B\Delta_{-t,-s}x^0_{n+t,m+s} - C\Delta_{t,-s}x^0_{n+t,m+s}\big) \\ & \times F_{N-1}^{t,s}(n+t,m+s) + A\Delta_{-t,s}x^0_{n+t,m+s}F_{N-1}^{-t,s}(n+t,m+s) \\ & + B\Delta_{-t,-s}x^0_{n+t,m+s}F_{N-1}^{-t,-s}(n+t,m+s) \\ & + C\Delta_{t,-s}x^0_{n+t,m+s}F_{N-1}^{t,-s}(n+t,m+s)\end{aligned} \tag{3.5}$$

TABLE I

Probability	ε_1	η_1
$A\Delta_{-t,s}x^0_{n+t,m+s}$	-1	-1
$B\Delta_{-t,-s}x^0_{n+t,m+s}$	1	-1
$C\Delta_{t,-s}x^0_{n+t,m+s}$	-1	1
$1 - A\Delta_{-t,s}x^0_{n+t,m+s} - B\Delta_{-t,-s}x^0_{n+t,m+s} - C\Delta_{t,-s}x^0_{n+t,m+s}$	1	1

At the limit for $\delta \to 0$ (Heath, 1969; Kac, 1956) the lattice tends to recover all the interior of the light cone, the function $F_N^{t,s}(n,m)$ goes into the function $F^{t,s}(x^0,x^1)$ and relations (3.5) can be shown to go into a system of four differential equations $(t,s=\pm 1)$, i.e.,

$$\frac{\partial F^{t,s}}{\partial x^0}=-\frac{s}{t}\frac{\partial F^{t,s}}{\partial x^1}-A(F^{-t,s}-F^{t,s})-B(F^{-t,-s}-F^{t,s})+C(F^{t,-s}-F^{t,s}). \tag{3.6}$$

One then sees immediately that the function

$$\Phi=(F^{1,1}+F^{1,-1}-F^{-1,1}-F^{-1,-1})+i(F^{1,1}-F^{1,-1}+F^{-1,1}-F^{-1,1}) \tag{3.7}$$

is a solution of the free Klein–Gordon equation

$$\left[\left(\frac{\partial^2}{\partial x^{0^2}}-\frac{\partial^2}{\partial x^{1^2}}\right)-\frac{m^2c^2}{\hbar^2}\right]\Phi=0 \tag{3.8}$$

when one writes $C=2A+4B$ and $2(A+B)^2=m^2c^2/\hbar^2$ Q.E.D. (For details of deduction see the Appendix.)

In the preceding demonstration f is not arbitrary since it is correlated, through relation (3.7) with an average scalar density ρ and a scalar phase S (see Vigier, 1979) by the relation $\phi=\rho^{1/2}\exp(iS/\hbar)$. Indeed, one can demonstrate directly relation (3.8) with the help of a hydrodynamic picture which also yields Nelson's equation

$$m(D_c v-D_s u)=F^+ \tag{3.9}$$

This suggests three physical remarks.

(a) If one starts from a set of initial positions on a given hyperbola the function ϕ now represents an average relativistic diffusion process comparable to a sound wave (i.e., a regular collective motion) propagating within Dirac's "ether"-like vacuum and carrying a particle along v.

(b) Dirac's "ether," which creates stochastic jumps at the velocity of light, is apparently the only way to obtain such a covariant diffusion process.

(c) It also explains an essential characteristic of the said process, viz., its reversibility. As one knows, nonrelativistic stochastic processes are fundamentally irreversible, being associated with a steady loss of information about where the particle comes from. This situation is modified here by the minus sign in (3.9), which has been shown (Vigier, 1979) to result from the particle–antiparticle mixture included both in Dirac's "ether" and in our random walk. Of course our time-reversing steps just describe particle–antiparticle transitions and they both move forward in time; but they are necessary to recover an essential feature (viz., reversibility) of quantum mechanics.

4. GENERALIZATION AND CONCLUDING REMARKS

In order to achieve a generalization of our two-dimensional derivation we must remark that the preceding demonstration remains unchanged if we analyze a Markov process on simpler (not covariant) lattices. For example we can reproduce all formulas for the displacements and their consequences if we consider the following lattice:

$$\left.\begin{aligned} x^0_{n,m} &= n\Delta\tau \\ x^1_{n,m} &= m\Delta l \end{aligned}\right\} \qquad \frac{\Delta l}{\Delta\tau} = 1, \qquad n,m = 0, \pm 1, \pm 2, \ldots \tag{4.1}$$

As a consequence the covariance of the formulation at each step of a limiting process cannot be considered as a necessary requirement. However, the covariance of the result (Klein–Gordon equation) is obtained at the end of the process. This observation allows us to discuss formally the four-dimensional case as a straightforward generalization of lattices like (4.1), a procedure that evidently avoids all complications resulting from the construction of a four-dimensional covariant lattice.

To conclude, we want to make two remarks on the relation of the preceding stochastic derivation to demonstrations utilized until now in the literature.

(a) In opposition to the "classical" stochastic derivations of quantum statistics (Bohm and Vigier, 1954) it is not necessary here to consider a particle carried in a fluid wave described by the quantum mechanical equations. Indeed, our corpuscule is now just located in a sort of covariant "ether," not described a priori by a particular wave equation. A simple hypothesis on the stochastic behavior of the particle in this fluid is now sufficient to demonstrate directly the Klein–Gordon equation.

(b) The model yields a physical insight into the observed difference between classical Brownian motion and quantum Brownian motion (De la Pena and Cetto, 1975).

As one knows, in order to obtain the Schrödinger equation (nonrelativistic limit of Klein–Gordon equation) or the classical equations of Brownian motions, we must choose "a priori" a different sign in the dynamical equations obtained from Newton's law (Vigier, 1979). The present demonstration indicates that this choice has a physical basis: (1) in the existence of a Markov process at the velocity of light in velocity space, (2) in the simultaneous presence of particles and antiparticles (Feynman, 1948) in the observed statistics of quantum theory.

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APPENDIX

In order to demonstrate in detail the transition from relation (3.5) to (3.8) we first remark that in the $\delta \to 0$ limit, the lattice tends to recover all the surface of the forward light cone and the number N of the jumps in our light velocity random walks becomes infinite: provided we keep fixed the space–time limits of the initial diffusion process. In this way the discrete function $F_N^{t,s}(n,m)$ becomes a continuous function depending on the initial point coordinates, namely $F^{t,s}(x^0,x^1)$.

Starting from (3.5) on the x^0 axis ($m=0$), if we subtract from both sides $F_{N-1}^{t,s}(n+t,0)$ and then we divide by the time intervals, we have

$$\begin{aligned}\left.\frac{\Delta F_N^{t,s}}{\Delta x^0}\right|_{(n,0)} &= -\frac{s}{t} v \left.\frac{\Delta F_{N-1}^{t,s}}{\Delta x^1}\right|_{(n+t,0)} + \alpha A\left[F_{N-1}^{-t,s}(n+t,s) - F_{N-1}^{t,s}(n+t,s)\right] \\ &\quad \times \beta B\left[F_{N-1}^{-t,-s}(n+t,s) - F_{N-1}^{t,s}(n+t,s)\right] \\ &\quad + \gamma C\left[F_{N-1}^{t,-s}(n+t,s) - F_{N-1}^{t,s}(n+t,s)\right] \end{aligned} \tag{A.1}$$

where

$$\Delta_t(x^0)_{n,0} = (x^0)_{n+t,0} - (x^0)_{n,0} = \lambda_0 e^{n\delta}(e^{t\delta}-1)$$

$$\Delta_s(x^1)_{n+t,0} = (x^1)_{n+t,s} - (x^1)_{n+t,0} = \lambda_0 e^{n\delta} e^{t\delta} \sinh(s\delta)$$

$$\Delta_{t,s}(x^0)_{n,m} = (x^0)_{n+t,m+s} - (x^0)_{n,m} = t\lambda_0 e^{n\delta} e^{t(ms+1)\delta} \sinh\delta$$

$$\Delta_{t,s}(x^1)_{n,m} = (x^1)_{n+t,m+s} - (x^1)_{n,m} = s\lambda_0 e^{n\delta} e^{t(ms+1)\delta} \sinh\delta$$

with
$$\frac{s}{t} v = \frac{\Delta_s(x^1)_{n+t,0}}{\Delta_t(x^0)_{n,0}} \underset{\delta\to 0}{\longrightarrow} \frac{s}{t}$$

$$\left.\frac{\Delta F_N^{t,s}}{\Delta x^0}\right|_{(n,0)} = \frac{F_{N-1}^{t,s}(n+t,0) - F_N^{t,s}(n,0)}{\Delta_t(x^0)_{n,0}} \underset{\delta\to 0}{\rightarrow} \left.\frac{\partial F^{t,s}}{\partial x^0}\right|_{x^1=0} \tag{A.2}$$

$$\left.\frac{\Delta F_{N-1}^{t,s}}{\Delta x^1}\right|_{(n+t,0)} = \frac{F_{N-1}^{t,s}(n+t,s) - F_{N-1}^{t,s}(n+t,0)}{\Delta_s(x^1)_{n+t,0}} \underset{\delta\to 0}{\rightarrow} \left.\frac{\partial F^{t,s}}{\partial x^1}\right|_{x^1=0}$$

and

$$\alpha = \frac{\Delta_{-t,s}(x^0)_{n+t,s}}{\Delta_t(x^0)_{n,0}} \underset{\delta\to 0}{\rightarrow} -1$$

$$\beta = \frac{\Delta_{-t,-s}(x^0)_{n+t,s}}{\Delta_t(x^0)_{n,0}} \underset{\delta\to 0}{\rightarrow} -1$$

$$\gamma = \frac{\Delta_{t,-s}(x^0)_{n+t,s}}{\Delta_t(x^0)_{n,0}} \underset{\delta\to 0}{\rightarrow} 1;$$

so that, in the $\delta\to 0$ limit, (A.1) goes into (3.6) on the x^0 axis. From the four equations (A.1) we can now construct two complex equations:

$$\left.\frac{\Delta\varphi_N}{\Delta x^0}\right|_{(n,0)} = -v\left.\frac{\Delta\varphi_{N-1}}{\Delta x^1}\right|_{(n+t\rho)} - \left[\alpha A(\varphi_{N-1} - i\chi_{N-1}^*) + \beta B(\varphi_{N-1} - i\varphi_{N-1}^*)\right.$$
$$\left. - \gamma C(\chi_{N-1} - \varphi_{N-1})\right]\Big|_{(n+t,s)}$$

$$\left.\frac{\Delta\chi_N}{\Delta x^0}\right|_{(n,0)} = V\left.\frac{\Delta\chi_{N-1}}{\Delta x^0}\right|_{(n+t,0)} - \left[\alpha A(\chi_{N-1} - i\varphi_{N-1}^*) + \beta B(\chi_{N-1} - i\chi_{N-1}^*)\right.$$
$$\left. - \gamma C(\varphi_{N-1} - \chi_{N-1})\right]\Big|_{(n+t,s)} \tag{A.3}$$

with

$$\varphi_N = F_N^{1,1} + iF_N^{-1,-1}$$

and

$$\chi_N = F_N^{1,-1} + iF_N^{-1,1}.$$

Now, adding and subtracting (A.3) we obtain

$$\left.\frac{\Delta\xi_N}{\Delta x^0}\right|_{(n,0)} = -v\left.\frac{\Delta\xi_{N-1}}{\Delta x^1}\right|_{(n+t,0)} - \left[(\alpha A + \beta B)(\xi_{N-1} - i\xi_{N-1}^*)\right]\Big|_{(n+t,s)}$$

$$\left.\frac{\Delta\xi_N}{\Delta x^0}\right|_{(n,0)} = -v\left.\frac{\Delta\xi_{N-1}}{\Delta x^1}\right|_{(n+t,0)}$$
$$-\left[(\alpha A + \beta B + \gamma C)\xi_{N-1} + (\alpha A - \beta B)i\xi_{N-1}^*\right]\Big|_{(n+t,s)} \tag{A.4}$$

with

$$\xi_N = \varphi_N + \chi_N$$

and

$$\xi_N = \varphi_N - \chi_N.$$

Then we take the difference between (A.4) and obtain respectively,

$$\frac{\Delta\xi_{N-1}}{\Delta x^0}\bigg|_{(n+t',0)} = -v\frac{\Delta\xi_{N-2}}{\Delta x^1}\bigg|_{(n+t+t',0)} - [(\alpha A+\beta B)(\xi_{N-2}-i\xi^*_{N-2})]|_{(n+t+t',s)}$$

$$\frac{\Delta\xi_N}{\Delta x^0}\bigg|_{(n,s')} = -v\frac{\Delta\xi_{N-1}}{\Delta x^1}\bigg|_{(n+t,s')}$$

$$-[(\alpha A+\beta B+\gamma C)\xi_{N-1}+(\alpha A-\beta B)i\xi^*_{N-1}]|_{(n+t,s+s')} \quad \text{(A.5)}$$

Finally we divide them respectively by $\Delta_{t'}(x^0)_{n,0}$ and $\Delta_{s'}(x^1)_{n,0}$; so we have

$$\frac{\Delta^2\xi_N}{(\Delta x^0)^2}\bigg|_{(n,0)} = -av\frac{\Delta^2\xi_{N-1}}{\Delta x^0\Delta x^1}\bigg|_{(n+t,0)} - b(\alpha A+\beta B)\left[\frac{\Delta\xi_{N-1}}{\Delta x^0}\bigg|_{(n+t,s)} - i\frac{\Delta\xi^*_{N-1}}{\Delta x^0}\bigg|_{(n+t,s)}\right]$$

$$\frac{\Delta^2\xi_N}{\Delta x^0\Delta x^1}\bigg|_{(n,0)} = -cv\frac{\Delta^2\xi_{N-1}}{(\Delta x^1)^2}\bigg|_{(n+t,0)} - d\left[(\alpha A+\beta B+\gamma C)\frac{\Delta\xi_{N-1}}{\Delta x^1}\bigg|_{(n+t,s)}\right.$$

$$\left. + i(\alpha A-\beta B)\frac{\Delta\xi^*_{N-1}}{\Delta x^1}\bigg|_{(n+t,s)}\right] \quad \text{(A.6)}$$

where

$$\frac{\Delta^2\xi_N}{(\Delta x^0)^2}\bigg|_{(n,0)} = \left(\frac{\Delta\xi_{N-1}}{\Delta x^0}\bigg|_{(n+t',0)} - \frac{\Delta\xi_N}{\Delta x^0}\bigg|_{(n,0)}\right)\bigg/\Delta_{t'}(x^0)_{n,0} \underset{\delta=0}{\rightarrow} \frac{\partial^2\xi}{\partial x^{1^2}}\bigg|_{x^1=0}$$

$$\frac{\Delta^2\xi_{N-1}}{(\Delta x^1)^2}\bigg|_{(n+t,0)} = \left(\frac{\Delta\xi_{N-1}}{\Delta x^1}\bigg|_{(n+t,s')} - \frac{\Delta\xi_{N-1}}{\Delta x^1}\bigg|_{(n+t,0)}\right)\bigg/\Delta_{s'}(x^1)_{n+t,0} \underset{\delta\to 0}{\rightarrow} \frac{\partial^2\xi}{\partial x^{1^2}}\bigg|_{x^1=0}$$

$$\frac{\Delta^2\xi_N}{\Delta x^0\Delta x^1}\bigg|_{(n,0)} = \left(\frac{\Delta\xi_N}{\Delta x^0}\bigg|_{(n,s')} - \frac{\Delta\xi_N}{\Delta x^0}\bigg|_{n,0}\right)\bigg/\Delta_{s'}(x^1)_{n,0} \underset{\delta\to 0}{\rightarrow} \frac{\partial^2\xi}{\partial x^0\partial x^1}\bigg|_{x^1=0} \quad \text{(A.7)}$$

(and similarly for $N\to N^{-1}, N\to n+t$).

Moreover $a=\dfrac{\Delta_{t'}(x^0)_{n+t,0}}{\Delta_{t'}(x^0)_{n,0}} \underset{\delta\to 0}{\rightarrow} 1 \qquad c=\dfrac{\Delta_{s'}(x^1)_{n+t,0}}{\Delta_{s'}(x^1)_{n,0}} \underset{\delta\to 0}{\rightarrow} 1$

$$b=\frac{\Delta_{t'}(x^0)_{n+t,s}}{\Delta_{t'}(x^0)_{n,0}} \underset{\delta\to 0}{\rightarrow} 1 \qquad d=\frac{\Delta_{s'}(x^1)_{n+t,s}}{\Delta_{s'}(x^1)_{n,0}} \underset{\delta\to 0}{\rightarrow} 1$$

and the first derivative are defined as in (A.2). Now, in the limit $\delta \to 0$, relation (A.6) becomes of course

$$\left.\frac{\partial^2 \xi}{\partial x^{0^2}}\right|_{x^1=0} = -\left.\frac{\partial^2 \xi}{\partial x^1 \partial x^0}\right|_{x^1=0} + (A+B)\left.\left(\frac{\partial \xi}{\partial x^0} - i\frac{\partial \xi^*}{\partial x^0}\right)\right|_{x^1=0}$$

$$\left.\frac{\partial^2 \xi}{\partial x^1 \partial x^0}\right|_{x^1=0} = -\left.\frac{\partial^2 \xi}{\partial x^{1^2}}\right|_{x^1=0} + \left.\left[(A+B-C)\frac{\partial \xi}{\partial x^1} + i(A-B)\frac{\partial \xi^*}{\partial x^1}\right]\right|_{x^1=0} \tag{A.8}$$

and subtracting them we obtain

$$\left.\left(\frac{\partial^2 \xi}{\partial x^{0^2}} - \frac{\partial^2 \xi}{\partial x^{1^2}}\right)\right|_{x^1=0} = (A+B)\left.\left(\frac{\partial \xi}{\partial x^0} - i\frac{\partial \xi^*}{\partial x^0}\right)\right|_{x^1=0} - \left.\left[(A+B-C)\frac{\partial \xi}{\partial x^1} + i(A-B)\frac{\partial \xi^*}{\partial x^1}\right]\right|_{x^1=0} \tag{A.9}$$

In order to eliminate ξ from (A.9) we observe that, in the limit $\delta \to 0$, the first equation of (A.5) is

$$\left.\frac{\partial \xi}{\partial x^0}\right|_{x^1=0} = -\left.\frac{\partial \xi}{\partial x^1}\right|_{x^1=0} + (A+B)(\xi - i\xi^*) \tag{A.10}$$

so that we have

$$\left.\left(\frac{\partial^2 \xi}{\partial x^{0^2}} - \frac{\partial^2 \xi}{\partial x^{1^2}}\right)\right|_{x^1=0} = (2A+2B-C)\left.\frac{\partial \xi}{\partial x^0}\right|_{x^1=0} - 2iB\left.\frac{\partial \xi^*}{\partial x^0}\right|_{x^1=0} + (A+B)(C-2B)(\xi - i\xi^*)|_{x^1=0} \tag{A.11}$$

Finally, subtracting from (A.11) the conjugate equation multiplied by i

$$i\left.\left(\frac{\partial^2 \xi^*}{\partial x^{0^2}} - \frac{\partial^2 \xi^*}{\partial x^{1^2}}\right)\right|_{x^1=0} = (2A+2B-C)i\left.\frac{\partial \xi^*}{\partial x^0}\right|_{x^1=0} - 2B\left.\frac{\partial \xi}{\partial x^0}\right|_{x^1=0} - (A+B)(C-2B)(\xi - i\xi^*)|_{x^1=0} \tag{A.12}$$

and requiring that

$$C = 2A + 4B, \qquad 2(A+B)^2 = \frac{m^2c^2}{\hbar^2} \tag{A.13}$$

we have

$$\left[\left(\frac{\partial^2}{\partial x^{0^2}} - \frac{\partial^2}{\partial x^{1^2}}\right) - \frac{m^2c^2}{\hbar^2}\right]\Phi\bigg|_{x^1=0} = 0 \tag{A.14}$$

which is the Klein–Gordon equation on the x^0 axis of a Lorentz frame.

This evidently implies that (A.14) is valid over all space–time. Indeed if a scalar equation such as (A.14) is valid at a point in a given frame it remains valid for the same point in all frames. Moreover, since our lattice is covariant, if (A.14) is valid along a given line $x^1 = 0$, it is also valid on any lattice point of another of our lines (denoted $x^{1'} = 0$) which also plays the part of our x^0 axis in a different Lorentz frame, because we can evidently repeat the preceding demonstration in any Lorentz frame along any axis $x^1 = 0$.

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