An Alternative Derivation of the Spin-Dependent Quantum Potential.

N. CUFARO PETRONI

Istituto Nazionale di Fisica Nucleare
Dipartimento di Fisica dell'Università - 70126 Bari, Italia

P. GUERET

Institut de Mathémathiques Pures et Appliées
Université P. et M. Curie - 4, Pl. Jussieu, 75230 Paris, France

A. KYPRIANIDIS and J. P. VIGIER

Institut H. Poincaré, Laboratoire de Physique Théorique
11, rue P. et M. Curie, 75231 Paris, France

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Summary. – A new derivation, avoiding the difficulties of the old demonstrations, is given for the quantum potential form of spin-$\frac{1}{2}$ fields.

The derivation of the form of a spin-dependent Quantum Potential for spin-$\frac{1}{2}$ particles, presented in recent papers (1), was based on the use of the second-order Feynman and Gell-Mann equation (2) for 4-component spinors ($\hbar = c = 1$):

$$(1)\qquad [(i\hat{\partial}-e\hat{A})(i\hat{\partial}-e\hat{A})-m^2]\psi = \left[(i\partial_\mu - eA_\mu)(i\partial^\mu - eA^\mu) - \frac{e}{2}F_{\mu\nu}\delta^{\mu\nu} - m^2\right]\psi = 0\,.$$

In order to get the result, the following decomposition of the spinor ψ was adopted (as an analog of the decomposition of a complex number in a modulus and a phase factor):

$$(2)\qquad \psi = Qw \quad \text{with} \quad Q = |\bar{\psi}\psi|\,, \quad w = \frac{\psi}{Q}\,,$$

where Q is a real scalar positive function and w a spinor with $|\bar{w}w| = 1$.

(1) N. CUFARO PETRONI, PH. GUERET and J. P. VIGIER: *Nuovo Cimento B*, **81**, 243 (1984); *Phys. Rev. D*, **30**, 495 (1984).
(2) R. P. FEYNMAN and M. GELL-MANN: *Phys. Rev.*, **109**, 193 (1958).

Of course $\bar{w}w$ jumps between $+1$ and -1 in a discontinuous way everywhere changes its sign. As a consequence the relations

$$\Box(\bar{w}w) = 0\,, \qquad \partial_\mu(\bar{w}w) = 0\,, \tag{3}$$

that are essential in the derivation [1], are not everywhere defined in space-time. The same problem arises for the final form of the quantum potential [1], which contains second derivatives of Q.

In order to avoid these problems, we suggest here a decomposition of ψ different from (2), but leading to the same final result: so that the proposed form of the quantum potential [1] can be considered correct. In fact, by writing down a spinor ψ in terms of two component quantities

$$\psi = \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \tag{4}$$

we can define the (4×4)-matrix

$$R = pI + q\gamma_5 \quad \text{with} \quad p = (\xi^\dagger \xi)^{\frac{1}{2}}\,, \quad q = (\eta^\dagger \eta)^{\frac{1}{2}}\,. \tag{5}$$

Of course we will have

$$\bar{R} = \gamma_0 R^\dagger \gamma_0 = pI - q\gamma_5\,, \tag{6}$$

so that

$$\bar{R}R = R\bar{R} = p^2 - q^2 - \bar{\psi}\psi \tag{7}$$

is a real scalar nonpositive function. It can be shown that

$$R^{-1} = \frac{\bar{R}}{p^2 - q^2} \tag{8}$$

and hence we can define

$$u = R^{-1}\psi = \frac{\bar{R}}{p^2 - q^2} \quad \text{with} \quad \bar{u}u = 1\,. \tag{9}$$

As a consequence we can always put

$$\psi = Ru\,, \tag{10}$$

where R is a (4×4)-matrix defined by means of the everywhere continuous functions p, q and u is a 4-component quantity with $\bar{u}u$ everywhere equal to $+1$.

We must remark here that, differently from the decomposition (2), the new decomposition is not covariant. In fact it is well known that, for a general 4-spinor ψ, the 2-component quantities ξ, η of (4) are not 2-spinors. Hence p, q are not scalar functions and u of (9) is not a 4-spinor. However, we can adopt (10) because we will see, at the end of the calculation, that the meaningful physical quantities (as quantum potential, currents etc.) can be expressed only by means of covariant quantities (ψ, $\bar{\psi}\psi$ etc.). Hence our noncovariant decomposition can be considered as a formal trick in order to work with everywhere continuous functions.

The derivation follows now this line: the Lagrangian density leading to (1) is

$$\mathscr{L} = \overline{(i\hat{\partial} - e\hat{A})\psi}(i\hat{\partial} - e\hat{A})\psi - m^2\bar{\psi}\psi\,, \tag{11}$$

so that the conserved current density with its decomposition in a drift and a polarization part (that are separately conserved) is

$$J_\mu = \frac{1}{m}\operatorname{Re}\left[\bar\psi\gamma_\mu(i\hat\partial - e\hat A)\psi\right] = \frac{1}{m}\operatorname{Re}\left[\bar\psi(i\partial_\mu - eA_\mu)\psi\right] + \frac{1}{2m}\partial^\nu(\bar\psi\sigma_{\mu\nu}\psi) = j_\mu + j'_\mu\,. \tag{12}$$

If now we use (10) in the wave equation (1) we get

$$\bar u\frac{\bar R\Box R}{\bar R R}u + \bar u\Box u + 2\bar u\frac{\bar R\partial^\mu R}{\bar R R}\partial_\mu u - e^2A_\mu A^\mu + ie\partial_\mu A^\mu + 2ieA_\mu\bar u\frac{\bar R\partial^\mu R}{\bar R R}u + {} + 2ieA_\mu\bar u\partial^\mu u + \frac{e}{2}F_{\mu\nu}\bar u\sigma^{\mu\nu}u + m^2 = 0\,. \tag{13}$$

The imaginary part of (13) coincides with the conservation equation for the drift current: $\partial^\mu j_\mu = 0$, whereas the real part

$$\frac{\bar R\Box R + \Box\bar R R}{2\bar R R} - \partial_\mu\bar u\partial^\mu u + \frac{\overline{u\bar R\partial_\mu R\partial^\mu u} + \partial_\mu\bar u\partial^\mu\bar R R u}{\bar R R} - e^2A_\mu A^\mu + {} + 2ieA_\mu\bar u\frac{\bar R\partial^\mu R - \partial^\mu\bar R R}{2\bar R R}u + 2ieA_\mu\bar u\partial^\mu u + \frac{e}{2}F_{\mu\nu}\bar u\partial^{\mu\nu}u + m^2 = 0 \tag{14}$$

can be put in the form

$$g_\mu g^\mu - V - \frac{e}{2}F_{\mu\nu}\bar u\sigma^{\mu\nu}u - m^2 = 0\,, \tag{15}$$

where

$$g^\mu = m\frac{j^\mu}{\bar R R} \tag{16}$$

is a momentum density and

$$V = \frac{\bar R\Box R + \Box\bar R R}{2\bar R R} - \partial_\mu\bar u\partial^\mu u - \bar u\partial_\mu u\bar u\partial^\mu u + \frac{\overline{u\bar R\partial_\mu R\partial^\mu u} + \partial_\mu\bar u\partial^\mu\bar R R u}{\bar R R} - {} - \bar u\partial_\mu u\bar u\frac{\bar R\partial^\mu R - \partial^\mu\bar R R}{\bar R R}u - \bar u\frac{\bar R\partial^\mu R - \partial^\mu\bar R R}{2\bar R R}u\bar u\frac{\bar R\partial_\mu R - \partial_\mu\bar R R}{2\bar R R}u \tag{17}$$

is the quantum potential. A little bit long calculation leads to the following form of V built only by means of covariant quantities:

$$V = \frac{\Box(\bar\psi\psi)^{\frac12}}{(\bar\psi\psi)^{\frac12}} + \frac{\partial_\mu\bar\psi\psi\bar\psi\partial^\mu\psi - \bar\psi\psi\partial_\mu\bar\psi\partial^\mu\psi}{(\bar\psi\psi)^2} \tag{18}$$

or, in a more compact form,

$$V = \frac{\operatorname{Tr}(\bar\Omega\Box\Omega)}{\operatorname{Tr}(\bar\Omega\Omega)} \tag{19}$$

with $\Omega = (P - T)^{\frac12}$ and $P = \psi\bar\psi$, $T = \bar\psi\psi - \psi\bar\psi$; so that our old result [1] must be considered now as completely proved.