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Quantum Mechanical States as Attractors for Nelson Processes[1]

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In this paper we reconsider, in the light of the Nelson stochastic mechanics, the idea originally proposed by Bohm and Vigier that arbitrary solutions of the evolution equation for the probability densities always relax in time toward the quantum mechanical density $|\psi|^2$ derived from the Schrödinger equation. The analysis of a few general propositions and of some physical examples show that the choice of the L^1 metrics and of the Nelson stochastic flux is correct for a particular class of quantum states, but cannot be adopted in general. This indicates that the question if the quantum mechanical densities attract other solution of the classical Fokker–Planck equations associated to the Schrödinger equation is physically meaningful, even if a classical probabilistic model good for every quantum state is still not available. A few suggestion in this direction are finally discussed.

1. INTRODUCTION

In an important old paper[1] Bohm and Vigier have discussed the possibility that some criticisms to the assumptions of the causal interpretation of the quantum mechanics[2] could be overcome by means of an extension of the hydrodynamical model initially proposed by Madelung[3] in the direction of allowing that the Madelung fluid "undergoes more or less random fluctuations in its motion." In particular this model, given in terms of a fluid with irregular fluctuations, was supposed to answer a criticism of Pauli and others[4] about the hypothesis, made in the causal interpretation, that, if $\psi(\mathbf{r}, t)$ satisfies the nonrelativistic Schrödinger equation, then the

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probability density function (pdf) in an ensemble of particles with this wave function is $f(\mathbf{r}, t) = |\psi(\mathbf{r}, t)|^2$. The physical idea of Bohm and Vigier was that, even if our ensemble of quantum systems is described by an arbitrary initial pdf it will decay in time to an ensemble with pdf $|\psi|^2$, because of the random fluctuations arising from the interactions with a subquantum medium: "no matter what the initial probability distribution may have been (for example, a delta function) it will eventually be given by $P = |\psi|^2$." In the work cited above, however, some mathematical difficulties made the general proof of this property less than complete.

On the other hand this paper can historically be considered as a stepping stone on the way to the understanding of the deep relations connecting the quantum mechanics with the world of the classical random phenomena. Researches in this field eventually led to the formulation of the stochastic mechanics[5]: a classical model where the particles follow continuous random trajectories in space-time and all the observable predictions of the quantum mechanics can be completely reproduced. This theory is "by no means a causal theory, but probabilistic concepts enter in a classical way."[5]

The aim of the present paper is to review, in the light of the stochastic mechanics, the old idea of Bohm and Vigier about the decay of every initial pdf toward the quantum mechanical pdf: in Sec. 2 we will briefly recall the fundamentals of the Bohm and Vigier model and the principles of the stochastic mechanics; in Sec. 3 we will discuss a few general properties of the time evolution of the pdf's of the Markov processes in the particular metric induced by the usual norm in $L^1(\mathbf{R})$; in Sec. 4 we discuss a few specific examples of quantum systems described in stochastic mechanics and we show that, in the chosen metrics, the trend of all the initial pdf's to decay in time toward the quantum mechanical pdf is not a general property since it holds only for a wide but particular class of wave functions; finally in Sec. 5 a short discussion follows about these results and the possibility of their generalization.

2. THE CAUSAL INTERPRETATION AND THE STOCHASTIC MECHANICS

The causal interpretation of the quantum mechanics is based on the idea that a nonrelativistic particle of mass m, whose wave function obeys the Schrödinger equation

$$i\hbar\partial_t\psi(\mathbf{r}, t) = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}, t) + V(\mathbf{r}, t)\,\psi(\mathbf{r}, t) \tag{1}$$

is a classical object following a continuous and causally defined trajectory with a well-defined position and accompanied by a physically real wave field ψ which contributes to determine its motion. In fact, if we write down (1) in terms of the real functions $R(\mathbf{r}, t)$ and $S(\mathbf{r}, t)$ with

$$\psi(\mathbf{r}, t) = R(\mathbf{r}, t)\, e^{iS(\mathbf{r}, t)/\hbar} \tag{2}$$

and separate real and imaginary parts, we have

$$\partial_t R^2 + \nabla\left(R^2 \frac{\nabla S}{m}\right) = 0 \tag{3}$$

$$\partial_t S + \frac{(\nabla S)^2}{2m} + V - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} = 0 \tag{4}$$

where $R^2(\mathbf{r}, t) = |\psi(\mathbf{r}, t)|^2$ is interpreted as the density of a fluid with stream velocity

$$v(\mathbf{r}, t) = \frac{\nabla S}{m} \tag{5}$$

Thus Eq. (3) expresses the conservation of the fluid while Eq. (4) plays the role of a Hamilton–Jacobi equation for the velocity potential S in the presence of a quantum potential

$$-\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}$$

which depends on the form of the wave function. In the causal interpretation the particle follows deterministic trajectories dictated by (5) when v is identified with the velocity of a particle passing through $\mathbf{r}$ at the time t.

It is important to remark now that, if we define

$$v_{(+)}(\mathbf{r}, t) = \frac{\nabla S}{m} + \frac{\hbar}{2m}\frac{\nabla R^2}{R^2} \tag{6}$$

the continuity equation (3) takes the form

$$\partial_t R^2 = \frac{\hbar}{2m}\nabla^2 R^2 - \nabla(R^2 v_{(+)}) \tag{7}$$

so that R^2 can also be considered as a particular solution of the evolution equation of the pdf's of a Markov process (Fokker–Planck equation)

$$\partial_t f = \nu\nabla^2 f - \nabla(f v_{(+)}) \tag{8}$$

characterized by the velocity field $v_{(+)}$ and by a diffusion coefficient

$$\nu = \frac{\hbar}{2m} \tag{9}$$

This points out a possible connection between the density R^2 of the Madelung fluid and the pdf of a suitable Markov process describing the random motion of a classical particle. As a matter of fact this connection is not at all compulsory at this point since the causal interpretation is a deterministic theory with no randomness involved in its fundamentals so that the analogy between (7) and (8) could also be considered purely formal. Moreover it must be remarked that, while for a given $v_{(+)}$ we can determine an infinity of solutions of (8) (one for every initial condition $f(\mathbf{r}, 0) = f_0(\mathbf{r})$) which are pdf's of Markov processes, the quantum mechanics are characterized by the selection of just one particular solution $f = R^2$ among all the possibilities. In fact it must be emphasized that R^2 and $v_{(+)}$ are not independent: they are both derived from a ψ solution of (1) and hence are locked together by their common origin. In other words: not every couple of f, solution of (8), and $v_{(+)}$ can be considered as derived from the same solution of the Schrödinger equation through the relation (6) and $f = R^2$.

That notwithstanding, the causal interpretation is obliged to add some randomness to its deterministic description in order to reproduce the statistical predictions of the quantum mechanics and hence it identifies the function $R^2 = |\psi|^2$ with the pdf of an ensemble of particles. But, since this addition is made by hand, is it easy for the critics of the model to argue that "it should be possible to have an arbitrary probability distribution (a special case of which is the function $P = \delta(x - x_0)$ representing a particle in a well-defined location) that is at least in principle independent of the ψ field and dependent only on our degree of information concerning the location of the particle."[1]

A more convincing connection between quantum mechanics and classical random phenomena was achieved only later by means of the stochastic mechanics[6]: here the particle position is promoted to a stochastic Markov process $\xi(t)$ defined on some probabilistic space $(\Omega, \mathscr{F}, \mathbf{P})$ and taking values (for our limited purposes) in $\mathbf{R}^3$. This process is characterized by a pdf $f(\mathbf{r}, t)$ and a transition pdf $p(\mathbf{r}, t \mid \mathbf{r}', t')$ and satisfies an Itô stochastic differential equation of the form

$$d\xi(t) = v_{(+)}(\xi(t), t)\, dt + d\eta(t) \tag{10}$$

where $v_{(+)}$ is a velocity field which plays the role of a dynamical variable not given *a priori* but subsequently determined on the basis of a variational principle, and $\eta(t)$ is a Brownian process independent of ξ and such that

$$\mathbf{E}(d\eta(t)) = 0, \qquad \mathbf{E}(d\eta(t)\, d\eta(t)) = 2\nu \mathbf{I}\, dt$$

where $d\eta(t) = \eta(t+dt) - \eta(t)$ (for $dt > 0$), ν is the diffusion coefficient, and $\mathbf{I}$ is the 3×3 identity matrix. We know that under fair analytical conditions on the velocity field $v_{(+)}$, the solution of (10) exists and is unique if we supplement our equation with the initial condition $\xi(0) = \xi_0$; moreover, the pdf of the process satisfies the evolution equation (8) associated with the initial condition $f(\mathbf{r}, 0) = f_0(\mathbf{r})$ if $f_0(\mathbf{r})$ is the pdf of ξ_0. An important role is played by the family of the transition pdf's $p(\mathbf{r}, t \mid \mathbf{r}', s)$ which are defined as the conditional pdf's of our process under the hypothesis that $\xi(s) = \mathbf{r}'$: in particular $p(\mathbf{r}, t \mid \mathbf{r}', 0)$ will be the solutions of (8) if we choose as initial condition $\xi_0 = \mathbf{r}'$ ($\mathbf{P}$-a.s.). The relevance of the transition pdf is well appreciated when we realize that every other solution of (8) (which satisfies the boundary and the non-negativity conditions to be a pdf) is propagated from its initial condition $f_0(\mathbf{r})$ following the prescription

$$f(\mathbf{r}, t) = \int_{\mathbf{R}^3} p(\mathbf{r}, t \mid \mathbf{r}', 0)\, f_0(\mathbf{r}')\, d^3\mathbf{r}', \qquad t > 0 \tag{11}$$

or more generally

$$f(\mathbf{r}, t) = \int_{\mathbf{R}^3} p(\mathbf{r}, t \mid \mathbf{r}', s)\, f(\mathbf{r}', s)\, d^3\mathbf{r}', \qquad t > s \tag{12}$$

In other words the transition pdf's, also called *fundamental solutions* of (8), play the role of the propagators and are the solutions of (8) which satisfy (in the sense of the distributions) the initial conditions

$$p(\mathbf{r}, t \mid \mathbf{r}', 0) \to \delta(\mathbf{r} - \mathbf{r}'), \qquad t \to 0^+ \tag{13}$$

A suitable definition of the Lagrangian and of the stochastic action functional for the sytem described by means of the dynamical variables f and $v_{(+)}$ allows us to select, by means of the principle of stationarity of the action, the particular processes which reproduce the quantum mechanics. More precisely the selected processes will have a drift velocity

$$v = v_{(+)} - \nu \frac{\nabla f}{f}$$

which, as required in (5), is always the gradient of a particular function $S(\mathbf{r}, t)$ solution of (4) with $R=\sqrt{f}$. Moreover, it is possible to show that from f and S selected in this way we can always build a wave function

$$\psi(\mathbf{r}, t)=\sqrt{f(\mathbf{r}, t)}\, e^{iS(\mathbf{r}, t)/\hbar}$$

which satisfies the Schrödinger equation (1).

In this formulation the foundations to interpret R^2 as a particular solution of a Fokker–Planck equation for the pdf of Markov processes are well established. Of course we pay for this by abandoning the idea of deterministic trajectories even if the stochastic mechanics keeps intact the description by means of continuous trajectories in space-time and recovers the paths of the causal interpretation as averages on the stochastic trajectories. In this perspective the idea proposed by Bohm and Vigier of a relaxation in time of arbitrary pdf's solutions of (8) toward the quantum mechanical pdf $|\psi|^2$ can be checked as a property of the solutions of the Fokker–Planck equations with the field $v_{(+)}$ derived according to (6) from the wave functions solutions of (1). In other words, in this paper we will analyze an updated version of the Bohm and Vigier idea that for every quantum wave function ψ there exists a stochastic flux, described by a family of transition pdf's $p(\mathbf{r}, t \mid \mathbf{r}', s)$, such that: (a) the quantum pdf $|\psi|^2$ is correctly propagated by p; (b) every other pdf propagated by p approximates, in a suitable sense, the quantum pdf $|\psi|^2$ for $t \to +\infty$. In particular, we will explore the possibility that the p associated by the Nelson stochastic mechanics to a quantum state ψ can be interpreted as the origin of the Bohm and Vigier stochastic flux, namely we will examinate if and how the solutions of (8) selected by the stochastic mechanics to reproduce the quantum predictions attract other solutions which do not satisfy the stationary stochastic action principle and hence cannot be considered as describing quantum systems.

3. TIME EVOLUTION OF THE MARKOV PROCESSES

In what follows we will limit ourselves to the case of the one-dimensional trajectories, so that the Markov processes $\xi(t)$ considered will always take values in $\mathbf{R}$. The set of all the probability density functions of the absolutely continuous real random variables defined on a probability space $(\Omega, \mathscr{F}, \mathbf{P})$ coincides with the set $\mathscr{D}$ of all the non-negative functions $f(x)$ of the hypersphere of norm 1 in the Banach space $L^1(\mathbf{R})$ with norm

$$\|f\|=\int_{-\infty}^{+\infty}|f(x)|\, dx$$

and hence the time-dependent pdf $f(x, t)$ of the stochastic processes $\xi(t)$ will be considered as trajectories on this subset $\mathscr{D}$. For Markov processes the transition pdf's $p(x, t \mid y, s)$ classified by means of the initial condition $\xi(s) = y$ (with $s < t$) are particular trajectories with nonabsolutely continuous initial conditions. In $\mathscr{D}$ we can then introduce a metric induced by the norm in $L^1(\mathbf{R})$:

$$\mathbf{d}(f, g) = \frac{1}{2} \int_{-\infty}^{+\infty} |f(x) - g(x)| \, dx$$

Here the factor 1/2 guarantees that we always have $0 \leqslant \mathbf{d}(f, g) \leqslant 1$: the value 1 is attained when f and g have disjoint supports, and the value 0 when they coincide (Lebesgue almost everywhere).

Definition 1. We will say that the pdf $f(x, t)$ *L^1-approximates* the pdf $g(x, t)$ (for $t \to +\infty$), and we will write

$$f(x, t) \overset{L^1}{\sim} g(x, t), \qquad t \to +\infty$$

when

$$\mathbf{d}(f, g) \to 0, \qquad t \to +\infty$$

In particular we will say that f *L^1-converges* toward g (for $t \to +\infty$) if the pdf $g(x)$ does not depend on the time t.

This means that the two trajectories on the unit hypersphere tend to approximate one another in the L^1-norm when $t \to +\infty$. If the stochastic processes $\xi(t)$ under examination are Markov processes (as happens in stochastic mechanics) satisfying the stochastic differential equation (10) with initial condition $\xi(0) = \xi_0$, their pdf will satisfy the one-dimensional evolution equation

$$\partial_t f(x, t) = \nu \partial_x^2 f(x, t) - \partial_x (v_{(+)}(x, t) f(x, t)) \tag{14}$$

with the initial condition $f(x, 0) = f_0(x)$ if $f_0(x)$ is the pdf of ξ_0.

We will examinate next a few properties of the concept of L^1-approximation for processes satisfying Eq. (10).

Proposition 1. If f and g are solutions of (14), the distance $\mathbf{d}(f, g)$ is a monotonic nonincreasing function of the time t.

Proof. If we write $\mathbf{d}(t)=\mathbf{d}(f, g)$ to put in evidence the dependence on time, we have from (12) that (for $t>s$)

$$
\begin{aligned}
\mathbf{d}(t) &= \tfrac{1}{2}\int_{-\infty}^{+\infty} |f(x, t) - g(x, t)|\, dx \\
&= \tfrac{1}{2}\int_{-\infty}^{+\infty} \left|\int_{-\infty}^{+\infty} p(x, t \mid y, s)[f(y, s) - g(y, s)]\, dy\right| dx \\
&\leqslant \tfrac{1}{2}\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} p(x, t \mid y, s)\, |f(y, s) - g(y, s)|\, dy\right) dx \\
&= \tfrac{1}{2}\int_{-\infty}^{+\infty} |f(y, s) - g(y, s)| \left(\int_{-\infty}^{+\infty} p(x, t \mid y, s)\, dx\right) dy \\
&= \tfrac{1}{2}\int_{-\infty}^{+\infty} |f(y, s) - g(y, s)|\, dy = \mathbf{d}(s)
\end{aligned}
$$

where we have used the Fubini theorem to exchange the order of integration and the obvious fact that

$$\int_{-\infty}^{+\infty} p(x, t \mid y, s)\, dx = 1$$

for every real y and for $t>s$. □

Of course, even if this general proposition for Markov processes states that the distance $\mathbf{d}(f, g)$ among the solutions of (14) is a nonincreasing function of time, this is not enough to derive the consequence that this distance actually decreases, let alone the fact that it is infinitesimal when $t \to +\infty$. However, this property is sufficient to prove that, since $\mathbf{d}(t)$ is a monotone and bounded function of t, the limit of $\mathbf{d}(t)$ for $t \to +\infty$ always exists and is finite.

In order to examinate the conditions that are sufficient to make the distance $\mathbf{d}(f, g)$ actually tend to zero when $t \to +\infty$, let us now introduce the following definition:

Definition 2. We will say that the family of the transition pdf's $p(x, t \mid y, 0)$ *L^1-approximates* the pdf $g(x, t)$ in a *locally uniform way in y* (y-l.u.) for $t \to +\infty$, and we will write

$$p(x, t \mid y, 0) \overset{L^1}{\sim} g(x, t) \quad y\text{-l.u.}, \qquad t \to +\infty$$

when for every $K>0$ and for every $\varepsilon>0$ we can find a $T>0$ such that

$$\mathbf{d}(p, g) = \mathbf{d}(p(x, t \mid y, 0), g(x, t)) < \varepsilon$$

for every $t>T$ and for every y such that $|y| \leqslant K$.

The meaning of this definition is the following: the transition pdf's which L^1-approximate the same pdf g for $t \to +\infty$ progressively forget their dependence on the initial condition y, in the sense that, for every y, they approximate the same pdf g. Moreover, the local uniformity in y requires something more than the simple L^1-approximation of every p to the same g independently from y, albeit something less than the *global* uniformity which would ask that the inequality $\mathbf{d}(p, g) < \varepsilon$ be verified for every $t > T$ and for every real y without limitations. Of course the global uniformity implies the local uniformity, but the converse is not in general true.

Proposition 2. If the transition pdf's $p(x, t \mid y, 0)$ L^1-approximate y-l.u. the pdf $g(x, t)$ for $t \to +\infty$, then every $f(x, t)$ solution of the evolution equation (14) L^1-approximates $g(x, t)$ for $t \to +\infty$.

Proof. Let $f(x, t)$ be an arbitrary solution of (14), corresponding to the initial condition $f(x, 0) = f_0(x)$, and $\varepsilon > 0$ an arbitrary positive number. Since f_0 is in $L^1(\mathbf{R})$ we will always be able to find a $K > 0$ such that

$$\int_{|y| > K} f_0(y)\, dy < \frac{\varepsilon}{2}$$

Moreover, since the approximation is y-l.u., for the given ε and K we can always find $T > 0$ such that

$$\mathbf{d}(p, g) = \frac{1}{2} \int_{-\infty}^{+\infty} |p(x, t \mid y, 0) - g(x, t)|\, dx < \frac{\varepsilon}{2}$$

for every $t > T$ and for every real y such that $|y| \leqslant K$. Then, since we always have $\mathbf{d}(p, g) \leqslant 1$, we get for every $t > T$

$$\begin{aligned}
\mathbf{d}(f, g) &= \frac{1}{2} \int_{-\infty}^{+\infty} |f(x, t) - g(x, t)|\, dx \\
&= \frac{1}{2} \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{+\infty} [p(x, t \mid y, 0) - g(x, t)]\, f_0(y)\, dy \right| dx \\
&\leqslant \frac{1}{2} \int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} |p(x, t \mid y, 0) - g(x, t)|\, f_0(y)\, dy \right) dx \\
&= \int_{-\infty}^{+\infty} f_0(y) \left(\frac{1}{2} \int_{-\infty}^{+\infty} |p(x, t \mid y, 0) - g(x, t)|\, dx \right) dy \\
&= \int_{-\infty}^{+\infty} f_0(y)\, \mathbf{d}(p, g)\, dy
\end{aligned}$$

$$= \int_{|y| > K} f_0(y)\, \mathbf{d}(p, g)\, dy + \int_{|y| \leqslant K} f_0(y)\, \mathbf{d}(p, g)\, dy$$

$$< \int_{|y| > K} f_0(y)\, dy + \frac{\varepsilon}{2} \int_{|y| \leqslant K} f_0(y)\, dy < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

where we used all the previous limitations and the Fubini theorem to exchange the order of integration. □

Let us remark that in the proof we nowhere used the hypothesis that $g(x, t)$ is a solution of (14): in fact, it is enough to suppose that g is the time-dependent pdf of a generic Markov process. However, even if the transition pdf's p L^1-approximate a g which is not a solution of (14) the triangular inequality for the metric $\mathbf{d}$ allows us to show that all the solutions of (14) L^1-approximate one another as stated in the following proposition:

Corollary 1. If the transition pdf's L^1-approximate y-l.u. an arbitrary pdf g, then

$$\mathbf{d}(f_1, f_2) \to 0, \qquad t \to +\infty$$

for every f_1, f_2 solutions of (14).

Proof. From Proposition 2 we have $f_1 \overset{L^1}{\sim} g$ and $f_2 \overset{L^1}{\sim} g$ so that from the triangular inequality

$$\mathbf{d}(f_1, f_2) \leqslant \mathbf{d}(f_1, g) + \mathbf{d}(f_2, g) \to 0, \qquad t \to +\infty$$

for every f_1 and f_2 solutions of (14). □

The meaning of this Corollary is that, under the conditions of Proposition 2, all the solutions of (14) globally tend to L^1-approximate one another after a sufficiently long time. Vice-versa, if we can find two solutions f_1 and f_2 of (14) such that $\mathbf{d}(f_1, f_2)$ is not infinitesimal for $t \to +\infty$, then no pdf g can be L^1-approximated y-l.u. by the family of the transition pdf's p.

4. EXAMPLES FROM QUANTUM MECHANICS

In order to discuss our examples in detail it will be useful to derive a formula to calculate the L^1-distance among the pdf's $\mathcal{N}(m, \sigma^2)$ of *normal* random variables, namely pdf's of the form

$$g_{m,\sigma}(x) = \frac{e^{-(x-m)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}$$

with real m and $\sigma > 0$. In the following we will indicate with the symbol

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$$

the usual *error function* and we will also pose

$$\mathbf{d}(a, b; p, q) = \mathbf{d}(g_{a,p}, g_{b,q})$$

Proposition 3. With the previous notations, if $p > q$ we have that

$$\mathbf{d}(a, b; p, q) = \left[\Phi\left(\frac{x_1 - b}{q}\right) - \Phi\left(\frac{x_2 - b}{q}\right) \right] - \left[\Phi\left(\frac{x_1 - a}{p}\right) - \Phi\left(\frac{x_2 - a}{p}\right) \right]$$

where

$$x_1 = \frac{aq^2 - bp^2 - qp\sqrt{(a-b)^2 + 2(q^2 - p^2)\ln(q/p)}}{q^2 - p^2}$$

$$x_2 = \frac{aq^2 - bp^2 + qp\sqrt{(a-b)^2 + 2(q^2 - p^2)\ln(q/p)}}{q^2 - p^2}$$

If $p = q$ and $a \neq b$ we have that

$$\mathbf{d}(a, b; p, p) = 2\Phi\left(\frac{|b-a|}{2p}\right) - 1$$

Finally, if $p = q$ and $a = b$ we have that $\mathbf{d}(a, a; p, p) = 0$.

Proof. The points where the difference between the two normal pdf's change its sign are the solutions of the equation

$$(q^2 - p^2)\, x^2 - 2(aq^2 - bp^2)\, x + [a^2q^2 - b^2p^2 + 2q^2p^2 \ln(p/q)] = 0 \qquad (15)$$

If $p \neq q$ (in particular, to fix the ideas, if $p > q$) the solutions are the numbers x_1 and x_2 indicated in our proposition which are always real since

$$(q^2 - p^2) \ln\frac{q}{p} = p^2 \left[\left(\frac{q}{p}\right)^2 - 1 \right] \ln\frac{q}{p}$$

and $(t^2-1)\ln t>0$ for every $t>0$. Moreover, since $p>q$, it is immediately seen that $x_1>x_2$. On the other hand, if $p=q$ and $a\neq b$, Eq. (15) has just one solution

$$x_0=\frac{a+b}{2}$$

We will finally just neglect the case $p=q$ and $a=b$ since this means that the two pdf's coincide so that we immediately get $\mathbf{d}(a, a; p, p)=0$. Let us remark now that, in the case $p\neq q$ (in particular $p>q$ and $x_1>x_2$), we have $g_{a,p}(x)\geqslant g_{b,q}(x)$ for $x\leqslant x_2$ and $x\geqslant x_1$, and $g_{a,p}(x)\leqslant g_{b,q}(x)$ for $x_2\leqslant x\leqslant x_1$, so that we easily have

$$\begin{aligned}\mathbf{d}(a, b; p, q)&=\frac{1}{2}\int_{-\infty}^{x_2}[g_{a,p}(x)-g_{b,q}(x)]\,dx+\frac{1}{2}\int_{x_2}^{x_1}[g_{b,q}(x)-g_{a,p}(x)]\,dx\\&\quad+\frac{1}{2}\int_{x_1}^{+\infty}[g_{a,p}(x)-g_{b,q}(x)]\,dx\\&=\left[\Phi\left(\frac{x_1-b}{q}\right)-\Phi\left(\frac{x_2-b}{q}\right)\right]\\&\quad-\left[\Phi\left(\frac{x_1-a}{p}\right)-\Phi\left(\frac{x_2-a}{p}\right)\right]\end{aligned}$$

On the other hand, if $p=q$ (but $a\neq b$ and, to fix our ideas, $b>a$) we have $g_{a,p}(x)\geqslant g_{b,p}(x)$ for $x\leqslant x_0$ and $g_{a,p}(x)\leqslant g_{b,p}(x)$ for $x\geqslant x_0$ so that

$$\begin{aligned}\mathbf{d}(a, b; p, p)&=\frac{1}{2}\int_{-\infty}^{x_0}[g_{a,p}(x)-g_{b,p}(x)]\,dx+\frac{1}{2}\int_{x_0}^{+\infty}[g_{b,p}(x)-g_{a,p}(x)]\,dx\\&=2\Phi\left(\frac{b-a}{2p}\right)-1\end{aligned}$$

Of course, with our conventions, these formulas never have negative values. □

The usefulness of the formulas in Proposition 3 are put in evidence by the remark that in the examples discussed in this paper both the transition pdf's and the pdf's derived from the quantum mechanical wave functions are normal, so that an application of Proposition 2 requires the calculation of distances among normal pdf's. That this is actually the case is due to the following proposition which indicates a very simple way to find the fundamental solutions of a class of evolution equations (14) which contain all the situations of our future examples.

Proposition 4. If the velocity field of the evolution equation (2) has the form

$$v_{(+)}(x, t) = -b(t)\, x - c(t)$$

with $b(t)$ and $c(t)$ continuous functions of time, then the fundamental solutions $p(x, t \mid y, 0)$ are normal pdf's $\mathcal{N}(\mu(t), \beta(t))$ where $\mu(t)$ and $\beta(t)$ are solutions of the equations

$$\mu'(t) + b(t)\, \mu(t) + c(t) = 0$$

$$\beta'(t) + 2b(t)\, \beta(t) - 2\nu = 0$$

with initial conditions $\beta(0) = 0$ and $\mu(0) = y$.

Proof. With the given velocity field the evolution equation takes the form

$$\partial_t f = \nu \partial_x^2 f + (bx + c)\, \partial_x f + bf$$

so that it is easy to verify that a normal pdf of the form

$$p(x, t \mid y, 0) = \frac{e^{[x - \mu(t)]^2/2\beta(t)}}{\sqrt{2\pi\beta(t)}} \tag{16}$$

will be a solution if $\mu(t)$ and $\beta(t)$ satisfy the two first-order, ordinary differential equations indicated in the proposition. The initial conditions $\beta(0) = 0$ and $\mu(0) = y$ are then imposed in order to satisfy the relation

$$p(x, t \mid y, 0) \to \delta(x - y), \qquad t \to 0^+$$

namely the one-dimensional analogs of (13), so that their role is to select the fundamental solutions p among all the other possible solutions of the form (16). □

We will discuss now our particular examples for systems reduced to a single nonrelativistic particle with a mass m, by remembering that the connection between the quantum mechanics and the stochastic mechanics is guaranteed if the diffusion coefficient and the Planck constant satisfy the relation (9). Let us consider first of all a simple harmonic oscillator with elastic constant k and classical (circular) frequency $\omega = \sqrt{k/m}$ and two possible wave functions obeying the Schrödinger equation: the (stationary) wave function of the ground state

$$\psi_0(x, t) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/4} e^{-x^2/4\sigma^2} e^{-i\omega t/2}$$

and the (nonstationary) wave function of the oscillating coherent wave packet with initial displacement a

$$\psi_C(x, t) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/4} \exp\left[-\frac{(x - a\cos\omega t)^2}{4\sigma^2} - i\left(\frac{4ax\sin\omega t - a^2\sin 2\omega t}{8\sigma^2} + \frac{\omega t}{2}\right)\right]$$

where we have defined

$$\sigma^2 = \frac{\nu}{\omega}$$

From the position (2) we find for our wave functions that

$$R_0^2(x, t) = f_0(x, t) = \frac{e^{-x^2/2\sigma^2}}{\sigma\sqrt{2\pi}}$$

$$R_C^2(x, t) = f_C(x, t) = \frac{e^{-(x - a\cos\omega t)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}$$

$$S_0(x, t) = -\frac{1}{2}\hbar\omega t$$

$$S_C(x, t) = -\frac{1}{2}\hbar\omega t - \hbar\frac{4ax\sin\omega t - a^2\sin\omega t}{8\sigma^2}$$

and hence we can calculate from (6) the corresponding velocity fields

$$v^0_{(+)}(x, t) = -\omega x$$

$$v^C_{(+)}(x, t) = -\omega x + \omega a(\cos\omega t - \sin\omega t)$$

This means that f_0 and f_C are respectively of the form $\mathcal{N}(0, \sigma^2)$ and $\mathcal{N}(a\cos\omega t, \sigma^2)$, and that the fundamental solutions of the corresponding evolution equation (14) can be calculated by means of Proposition 4 with

$$b_0(t) = \omega, \qquad c_0(t) = 0$$

$$b_C(t) = \omega, \qquad c_C(t) = -\omega a(\cos\omega t - \sin\omega t)$$

so that $p_0(x, t \mid y, 0)$ and $p_C(x, t \mid y, 0)$ will respectively be the normal pdf's $\mathcal{N}(\mu_0(t), \beta_0(t))$ and $\mathcal{N}(\mu_C(t), \beta_C(t))$, where

$$\beta_0(t) = \sigma^2(1 - e^{-2\omega t}), \qquad \mu_0(t) = y\, e^{-\omega t}$$

$$\beta_C(t) = \sigma^2(1 - e^{-2\omega t}), \qquad \mu_C(t) = a\cos\omega t + (y - a)\, e^{-\omega t}$$

A second class of examples can be drawn from the wave functions of a free particle of mass m. In particular we will choose to examinate the behavior of the (nonstationary) wave function of a wave packet of minimal uncertainty centered around $x=0$ with initial dispersion $\sigma^2>0$:

$$\psi_F(x,t)=\left(\frac{1}{2\pi\sigma^2\chi^2(t)}\right)^{1/4} e^{-x^2/4\sigma^2\chi(t)}$$

where

$$\chi(t)=1+i\omega t, \qquad \omega=\frac{\nu}{\sigma^2}$$

In this case we have from (2)

$$R_F(x,t)=f_F(x,t)=\frac{e^{-x^2/2\sigma^2\alpha^2(t)}}{\sqrt{2\pi}\,\sigma\alpha(t)}$$

$$S_F(x,t)=\frac{h}{2}\left(\frac{\omega t x^2}{2\sigma^2\alpha^2(t)}-\arctan\omega t\right)$$

where

$$\alpha(t)=|\chi(t)|=\sqrt{1+\omega^2t^2}$$

This means that f_F is normal of the form $\mathcal{N}(0,\sigma^2\alpha^2(t))$; moreover, we get from (6) the velocity field

$$v_{(+)}^F(x,t)=-\frac{1-\omega t}{1+\omega^2t^2}\,\omega x$$

and the fundamental solutions of the corresponding evolution equations (14) can then be calculated by means of Proposition 4 with

$$b_F(t)=\frac{1-\omega t}{1+\omega^2t^2}\,\omega, \qquad c_F(t)=0$$

As a consequence $p_F(x,t\,|\,y,0)$ is a normal pdf $\mathcal{N}(\mu_F(t),\beta_F(t))$, where

$$\mu_F(t)=y\sqrt{1+\omega^2t^2}\,e^{-\arctan\omega t}$$

$$\beta_F(t)=\sigma^2(1+\omega^2t^2)(1-e^{-2\arctan\omega t})$$

We can now use the results of Proposition 3 in order to calculate $\mathbf{d}(p_0,f_0)$, $\mathbf{d}(p_C,f_C)$ and $\mathbf{d}(p_F,f_F)$: a long but simple calculation will show that (y-l.u.)

$$p_0\overset{L^1}{\sim}f_0, \qquad p_C\overset{L^1}{\sim}f_C, \qquad t\to+\infty$$

in the examples drawn from the harmonic oscillator, but that p_F will not L^1-approximate f_F since $\mathbf{d}(p_F, f_F)$ turns out to be different from zero and still dependent on y in the limit $t \to +\infty$:

$$\begin{aligned}\mathbf{d}(p_F, f_F) \to\ & \Phi(e^{\pi/2}[y - \sqrt{1-e^{-\pi}}\sqrt{y^2 - \ln(1-e^{-\pi})}]) \\ & - \Phi(e^{\pi/2}[y + \sqrt{1-e^{-\pi}}\sqrt{y^2 - \ln(1-e^{-\pi})}]) \\ & - \Phi(e^{\pi/2}[y\sqrt{1-e^{-\pi}} - \sqrt{y^2 - \ln(1-e^{-\pi})}]) \\ & + \Phi(e^{\pi/2}[y\sqrt{1-e^{-\pi}} + \sqrt{y^2 - \ln(1-e^{-\pi})}])\end{aligned}$$

For example, if $y = 0$ (so that both p_F and f_F will remain centered around $x = 0$ along all their evolution) we get in the limit $t \to +\infty$:

$$\begin{aligned}\mathbf{d}(p_F, f_F) \to\ & 2[\Phi(e^{\pi/2}\sqrt{-\ln(1-e^{-\pi})}) \\ & - \Phi(e^{\pi/2}\sqrt{1-e^{-\pi}}\sqrt{-\ln(1-e^{-\pi})})] \sim 0.011\end{aligned}$$

It is also possible to show that in this case two transition pdf's with different initial conditions $y \neq y'$ will never L^1-approximate one another as $t \to +\infty$, since

$$\mathbf{d}(p(x, t \mid y, 0), p(x, t \mid y', 0)) \to 2\Phi\left(\frac{|y - y'|}{2\sqrt{e^\pi - 1}}\right) - 1$$

which is zero if and only if $y = y'$. Hence on the basis of Corollary 1 we can state that every solution of the evolution equation (14) L^1-approximates the quantum mechanical pdf (for $t \to +\infty$) only in the examples of the harmonic oscillator but not in that of the free particle.

5. DISCUSSION AND CONCLUSIONS

It is apparent from our examples that the Markov processes associated to the quantum mechanical wave functions by the stochastic mechanics do not always exhibit the behavior required by the Bohm and Vigier hypothesis. In fact the calculations show that, in order to recover the property of a global relaxation in time of the pdf's toward the quantum mechanical solution, we must restrict ourselves to a particular set of physical systems.

The different behaviors of our examples are in fact inscribed in the form of the time dependence of the parameters of the normal pdf's involved in our calculations. It is easy to see that, in the case of the harmonic oscillator, for every real y we have (for $t \to +\infty$)

$$\mu_0(t) \to 0, \qquad \beta_0(t) \to \sigma^2$$

$$|\mu_C(t) - a\cos\omega t| \to 0, \qquad \beta_C(t) \to \sigma^2$$

On the other hand, $\mu_F(t)$ and $\beta_F(t)$ behave differently from the corresponding parameters of the quantum mechanical pdf f_F, since (for $t \to +\infty$)

$$|\mu_F(t) - e^{-\pi/2} y\omega t| \to 0, \qquad |\beta_F(t) - (1 - e^{-\pi})\,\sigma^2\omega^2 t^2| \to 0$$

while the quantum mechanical f_F is a normal pdf which remains centered around $x = 0$ with a variance which diverges as $\sigma^2\omega^2 t^2$. It is also useful to point out that in this case it is of no avail to remark that both p_F and f_F will flatten to zero when $t \to +\infty$: the relevant fact is that this flattening happens at rates different enough to make the L^1-distance remain nonzero even in the limit $t \to +\infty$.

Of course the difference between the cases of the harmonic oscillator and the free particle can also be traced back to the behaviors of the corresponding velocity fields $v_{(+)}$. In fact, while on the one hand $v^0_{(+)}$ is always directed toward the origin of the x axis (namely the equilibrium point of the oscillator) for every x and $t > 0$ and $v^C_{(+)}$ behaves in the same way for $t > 0$ and $|x| \geqslant \sqrt{2}\,a$ (but oscillates between inward and outward directions for $|x| \leqslant \sqrt{2}\,a$), on the other hand the velocity field $v^F_{(+)}$ of the free particle is (everywhere in x) directed toward the center only for $t < 1/\omega$, but becomes and remains everywhere directed in the outward direction when $t > 1/\omega$. Physically this indicates that, while in the two examples from the harmonic oscillator the velocity field always drags the process toward the center $x = 0$ (with the possible exception of a limited region around the origin), in the free particle case, after a time $1/\omega = \sigma^2/\nu$, the velocity field always carries the process away from this center. It is remarkable, moreover, that in the formulation chosen in the original Bohm and Vigier paper not one of our three examples would have shown the correct property: our L^1-metric plays here an important role in discriminating the well-behaved systems among all the possibilities.

The fact that the Nelson transition pdf's do not always L^1-approximate one another also means that it is impossible to find a unique pdf g L^1-approximated by them independently from y, and hence that the solutions of (14) in the discussed free particle case will not globally tend to L^1-approximate one another in time. Of course nothing forbids *a priori*, even in this case, that particular subsets of solutions can show the tendency to mutually L^1-approximate and hence the field is open to investigations about, for instance, the possibility that some particular solution of (14) can be stable with respect to small perturbations of their initial conditions: which in some minimal sense was the essential intention of the Bohm and Vigier proposal. In any case our examples show that, at least for a significant set of systems and wave functions the Bohm and Vigier property holds in the L^1-metrics if we adopt the transition pdf suggested by the Nelson stochastic mechanics, and hence it can be surely stated that their original idea posed an interesting and physically well-grounded problem. It is not possible at present to state clearly and in a general way in which cases we realize the conditions for a global (or at least local) mutual L^1-approximation of the solutions of (14). The examples discussed show that the discriminating property is not the stationarity of the quantum mechanical wave function since also the square modulus of the nonstationary, coherent, oscillating wave packet of the harmonic oscillator attracts in L^1 every other solution of (14). An indication can perhaps be found in the fact that the main difference between the two systems seems to be principally in the fact that their energy spectra are very different: the harmonic oscillator has a completely discrete spectrum and the free particle a completely continuous one. Hence a first idea can be to distinguish between bound states, which exhibit the Bohm and Vigier property, and scattering states, which do not. An interesting suggestion in this direction comes, in fact, from the papers of Shucker[7] where, for systems with zero potential, it is shown that the sample paths of the processes of the stochastic mechanics behave asymptotically (for $t \to +\infty$) like the paths of the classical mechanics. Of course the settlement of this question will require the discussion of further examples and the investigation of more general properties. However, it must be pointed out that in this paper we have made the very particular choice of selecting the transition pdf's of the Nelson stochastic mechanics as a good candidate to the generation of the right stochastic flux exhibiting the Bohm and Vigier property in some suitable sense. As a consequence another possible conclusion of this article could also be that the Nelson flux is not the right candidate to represent, in the general case, the interpretative scheme of Bohm and Vigier. Hence we consider wide open the possibility that the right transition pdf's can be built in a different way. For example it is well known that in the Nelson stochastic mechanics the diffusive part of the stochastic differential equation (10) is given *a priori*. Hence, since the transition pdf which propagates a given time-dependent pdf $f(\mathbf{r}, t)$ is not uniquely determined (and are not, in general, observable in the stochastic mechanics), nothing forbids one to find a diffusive flux, different from that of Nelson, which exhibites the Bohm and Vigier property for every possible quantum wave function. In particular a possibility lies in the generalization of the stochastic mechanics where also the diffusive part of the stochastic differential equation controlling the process is dynamically determined in a way such that the Bohm and Vigier property is always satisfied.

REFERENCES

1. D. Bohm and J.-P. Vigier, *Phys. Rev.* **96**, 208 (1954).
2. L. de Broglie, *C. R. Acad. Sci. Paris* **183**, 447 (1926); L. de Broglie, *C. R. Acad. Sci. Paris* **184**, 273 (1927); L. de Broglie, *C. R. Acad. Sci. Paris* **185**, 380 (1927); D. Bohm, *Phys. Rev.* **85**, 166, 180 (1952).
3. E. Madelung, *Z. Phys.* **40**, 332 (1926).
4. W. Pauli, in *Louis de Broglie, Physicien et Penseur* (Albin Michel, Paris, 1953); J. B. Keller, *Phys. Rev.* **89**, 1040 (1953).
5. E. Nelson, *Phys. Rev.* **150**, 1079 (1966); E. Nelson, *Dynamical Theories of Brownian Motion* (Princeton University Press, Princeton, 1967); E. Nelson, *Quantum Fluctuations* (Princeton University Press, Princeton, 1985).
6. F. Guerra, *Phys. Rep.* **77**, 263 (1981); F. Guerra and L. Morato, *Phys. Rev. D* **27**, 1774 (1983); F. Guerra and R. Marra, *Phys. Rev. D* **28**, 1916 (1983); F. Guerra and R. Marra, *Phys. Rev. D* **29**, 1647 (1984).
7. D. Shucker, *J. Funct. Anal.* **38**, 146 (1980); D. Shucker, *J. Math. Phys.* **22**, 491 (1981).