Quantum Communications and Measurement

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Plenum Press • New York and London

Library of Congress Cataloging-in-Publication Data

International Workshop on Quantum Communications and Measurement (1994 : Nottingham, England)
Quantum communications and measurement / edited by V.P. Belavkin, O. Hirota, and R.L. Hudson.
p. cm.
"Proceedings of an International Workshop on Quantum Communications and Measurement, held July 11-16, 1994, in Nottingham, England"--T.p. verso.
Includes bibliographical references and index.
ISBN 0-306-45128-X
1. Quantum optics--Congresses. 2. Optical communications--Congresses. 3. Quantum electronicy--Congresses. I. Belavkin, V. P. II. Hirota, O. (Osamu), 1948- . III. Hudson, R. L. (Robin Lyth), 1960- .
QC446.15.I68 1995
621.382'7--dc20 95-32969
CIP

Proceedings of an International Workshop on Quantum Communications and Measurement, held July 11–16, 1994, in Nottingham, England

ISBN 0-306-45128-X

A Division of Plenum Publishing Corporation
233 Spring Street, New York, N. Y. 10013

10 9 8 7 6 5 4 3 2 1

Printed in the United States of America

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ASYMPTOTIC BEHAVIOUR OF DENSITIES FOR NELSON PROCESSES[1]

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Keywords :
Nelson processes; Fokker-Planck equation.

In a recent paper[(1)] an idea of Bohm and Vigier[(2)] about the possible decay of every initial probability density function (pdf), whose evolution is ruled by the quantum Fokker-Planck equation, toward the quantum mechanical pdf has been discussed in the light of the stochastic mechanics. The causal interpretation of the quantum mechanics[(3)] is based on the idea that a non relativistic particle of mass m, whose wave function obeys the Schrödinger equation

$$i\hbar\partial_t\psi(\mathbf{r},t) = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) + V(\mathbf{r},t)\psi(\mathbf{r},t)\,, \tag{1}$$

is a classical object following a continuous and causally defined trajectory with a well defined position and accompanied by a physically real wave field ψ which contributes to determine its motion. If we write down (1) in terms of the real functions $R(\mathbf{r},t)$ and $S(\mathbf{r},t)$ with

$$\psi(\mathbf{r},t) = R(\mathbf{r},t)\,e^{iS(\mathbf{r},t)/\hbar} \tag{2}$$

[1]Paper contributed to the Workshop *Quantum Communications and Measurement*; Nottingham, July 10-16, 1994.

and separate real and imaginary parts, we have

$$\partial_t R^2 + \nabla\left(R^2 \frac{\nabla S}{m}\right) = 0, \tag{3}$$

$$\partial_t S + \frac{(\nabla S)^2}{2m} + V - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} = 0, \tag{4}$$

where $R^2(\mathbf{r},t) = |\psi(\mathbf{r},t)|^2$ is interpreted as the density of a fluid with stream velocity

$$v(\mathbf{r},t) = \frac{\nabla S}{m}. \tag{5}$$

It is important to remark now that, if we define

$$v_{(+)}(\mathbf{r},t) = \frac{\nabla S}{m} + \frac{\hbar}{2m}\frac{\nabla R^2}{R^2} \tag{6}$$

the continuity equation (3) takes the form

$$\partial_t R^2 = \frac{\hbar}{2m}\nabla^2 R^2 - \nabla(R^2 v_{(+)}) \tag{7}$$

so that R^2 can also be considered as a particular solution of the evolution equation of the pdf's of a Markov process (Fokker-Planck equation)

$$\partial_t f = \nu \nabla^2 f - \nabla(f v_{(+)}) \tag{8}$$

characterized by the velocity field $v_{(+)}$ and by a diffusion coefficient

$$\nu = \frac{\hbar}{2m}. \tag{9}$$

This points out a possible connection between the density R^2 and the pdf of a suitable Markov process describing the random motion of a classical particle. As a matter of fact this connection is not at all compulsory at this point since the causal interpretation is a deterministic theory with no randomness involved in its fundamentals so that the analogy between (7) and (8) could also be considered purely formal. That notwithstanding the causal interpretation is obliged to add some randomness to its deterministic description in order to reproduce the statistical predictions of the quantum mechanics and hence it identifies the function $R^2 = |\psi|^2$ with the pdf of an ensemble of particles. But, since this addition is made by hand, is it easy for the critics of the model to argue that "it should be possible to have an arbitrary probability distribution (a special case of which is the function $P = \delta(x - x_0)$ representing a particle in a well defined location) that is at least in principle independent of the ψ field and dependent only on our degree of information concerning the location of the particle"[(2)]. The physical idea of Bohm and Vigier was that, even if our ensemble of quantum systems is described by an arbitrary initial pdf, this will decay in time to an ensemble with pdf $|\psi|^2$, because of the random fluctuations arising from the interactions with a subquantum medium: "no matter what the initial probability distribution may have been (for example a delta function) it will eventually be given by $P = |\psi|^2$".

A more convincing connection between quantum mechanics and classical random phenomena is achieved by means of the stochastic mechanics[(4)]: here the particle position is promoted to a stochastic Markov process $\xi(t)$ defined on some probabilistic space $(\Omega, \mathcal{F}, \mathbf{P})$ and taking values (for our limited purposes) in $\mathbf{R}^3$. This process is characterized by a pdf $f(\mathbf{r}, t)$ and a transition pdf $p(\mathbf{r}, t|\, \mathbf{r}', t')$ and satisfies an Itô stochastic differential equation of the form

$$d\xi(t) = v_{(+)}(\xi(t), t)dt + d\eta(t) \tag{10}$$

where $v_{(+)}$ is a velocity field which plays the role of a dynamical variable not given a priori but subsequently determined on the basis of a variational principle, and $\eta(t)$ is a Brownian process independent of ξ and such that

$$\mathbf{E}(d\eta(t)) = 0\,, \qquad \mathbf{E}(d\eta(t)\, d\eta(t)) = 2\nu\, \mathbf{I}\, dt$$

where $d\eta(t) = \eta(t + dt) - \eta(t)$ (for $dt > 0$), ν is the diffusion coefficient and $\mathbf{I}$ is the 3×3 identity matrix. A suitable definition of the Lagrangian and of the stochastic action functional for the system described by the dynamical variables f and $v_{(+)}$ allows us to select, by means of the principle of stationarity of the action, the particular processes which reproduce the quantum mechanics. In this formulation the foundations to interpret R^2 as a particular solution of a Fokker-Planck equation for the pdf of Markov processes are well established and the idea proposed by Bohm and Vigier of a relaxation in time of arbitrary pdf's solutions of (8) toward the quantum mechanical pdf $|\psi|^2$ can be checked as a property of the solutions of the Fokker-Planck equations with the field $v_{(+)}$ derived according to (6) from the wave functions solutions of (1). In other words we explore the possibility that the p associated by the Nelson stochastic mechanics to a quantum state ψ can be interpreted as the origin of the Bohm and Vigier stochastic flux and we examinate if and how the solutions of (8) selected by the stochastic mechanics to reproduce the quantum predictions attract other solutions which do not satisfy the stationary stochastic action principle and hence can not be considered as describing quantum systems.

In what follows we will limit ourselves to the case of the one dimensional trajectories, so that the Markov processes $\xi(t)$ considered will always take values in $\mathbf{R}$. The set of all the probability density functions (pdf's) of the absolutely continuous real random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ coincides with the set $\mathcal{D}$ of all the non negative functions $f(x)$ of the hypersphere of norm 1 in the Banach space $L^1(\mathbf{R})$ and hence the time dependent pdf $f(x, t)$ of the stochastic processes $\xi(t)$ will be considered as trajectories on this subset $\mathcal{D}$. For Markov processes the transition pdf's $p(x, t|\, y, s)$ classified by means of the initial condition $\xi(s) = y$ (with $s < t$) are particular trajectories (with non absolutely continuous initial conditions). In $\mathcal{D}$ we then introduce a metrics induced by the norm in $L^1(\mathbf{R})$:

$$\mathbf{d}(f, g) = \frac{1}{2} \int_{-\infty}^{+\infty} |f(x) - g(x)|\, dx\,.$$

Here the factor $1/2$ guarantees that we always have $0 \leq \mathbf{d}(f, g) \leq 1$: the value 1 is attained when f and g have disjoint supports, and the value 0 when they coincide (Lebesgue almost everywhere). If the stochastic processes $\xi(t)$ under examination are Markov processes (as happens in stochastic mechanics) satisfying the equation (10) with initial condition $\xi(0) = \xi_0$, their pdf will satisfy the one dimensional evolution equation

$$\partial_t f(x, t) = \nu \partial_x^2 f(x, t) - \partial_x (v_{(+)}(x, t) f(x, t))\,, \tag{11}$$

with the initial condition $f(x, 0) = f_0(x)$ if $f_0(x)$ is the pdf of ξ_0.

Definition 1: We will say that the pdf $f(x,t)$ L^1-*approximates* the pdf $g(x,t)$ (for $t \to +\infty$), and we will write

$$f(x,t) \approx g(x,t) \qquad (t \to +\infty)\,,$$

when

$$\mathbf{d}(f,g) \to 0 \qquad (t \to +\infty)\,.$$

In particular we will say that f L^1-*converges* toward g (for $t \to +\infty$) if the pdf $g(x)$ does not depend on the time t.

We will examine next a few properties of the concept of L^1-approximation for processes satisfying the equation (10). First of all we can prove[(1)] the following proposition

Proposition 1: If f and g are solutions of (11), the distance $\mathbf{d}(f,g)$ is a monotonic non-increasing function of the time t.

Of course, even if this proposition states that the distance $\mathbf{d}(f,g)$ among the solutions of (11) is a non-increasing function of time, this is not enough to derive the consequence that this distance actually decreases, let alone the the fact that it is infinitesimal, when $t \to +\infty$. However this property is sufficient to prove that, since $\mathbf{d}(t)$ is a monotone and bounded function of t, the limit of $\mathbf{d}(t)$ for $t \to +\infty$ always exists and is finite. In order to examinate the conditions that are sufficient to make the distance $\mathbf{d}(f,g)$ actually tend to zero when $t \to +\infty$, let us now introduce the following definition:

Definition 2: We will say that the family of the transition pdf's $p(x,t|\,y,0)$ L^1-*approximates* the pdf $g(x,t)$ in a *locally uniform way in* y (y - l.u.) for $t \to +\infty$, and we will write

$$p(x,t|\,y,0) \approx g(x,t) \qquad y\text{ - l.u.} \qquad (t \to +\infty)\,,$$

when for every $K > 0$ and for every $\epsilon > 0$ we can find a $T > 0$ such that

$$\mathbf{d}(p,g) = \mathbf{d}(p(x,t|\,y,0), g(x,t)) < \epsilon$$

for every $t > T$ and for every y such that $|y| \leq K$.

The local uniformity in y of the L^1-approximation of the p's to a pdf g allows us now to prove[(1)] the following proposition:

Proposition 2: If $p(x,t|\,y,0) \approx g(x,t)$, y - l.u., $(t \to +\infty)$, where p is the transition pdf of (11) and g an arbitrary pdf, then we have that $f(x,t) \approx g(x,t)$, $(t \to +\infty)$, for every $f(x,t)$ solution of (11).

In the proof we nowhere use the hypothesis that $g(x,t)$ is a solution of (11): in fact it is enough to suppose that g is a time dependent pdf. However, even if g is not a solution of (11), the triangular inequality for the metrics $\mathbf{d}$ allows us to show[(1)] that all the solutions of (11) L^1-approximate one another as stated in the following corollary:

Corollary 1: If $p(x,t|\,y,0) \approx g(x,t)$, y - l.u., $(t \to +\infty)$, where p is the transition pdf of (11) and g an arbitrary pdf, then we have that $f_1(x,t) \approx f_2(x,t)$, $(t \to +\infty)$, for every $f_1(x,t)$ and $f_2(x,t)$ solutions of (11).

Hence, under the conditions of Proposition 2, all the solutions of (11) globally tend to L^1-approximate one another after a sufficiently long time. On the other hand, if we can find two solutions f_1 and f_2 of (11) such that $\mathbf{d}(f_1, f_2)$ is not infinitesimal for $t \to +\infty$, then no pdf g can be L^1-approximated y - l.u. by the family of the transition pdf's p.

In order to discuss a few examples in detail it will be useful to derive a formula to calculate the L^1-distance among the pdf's $\mathcal{N}(m,\sigma^2)$ of *normal* random variables, namely pdf's of the form

$$g_{m,\sigma}(x) = \frac{e^{-(x-m)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}$$

with real m and $\sigma > 0$. In the following we will indicate with the symbol

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$$

the usual *error function* and we will also pose

$$\mathbf{d}(a,b;\, p,q) = \mathbf{d}(g_{a,p}, g_{b,q})\,.$$

With the previous notations we can now prove[1] that, if $p > q$ we have

$$\mathbf{d}(a,b;\, p,q) = \left[\Phi\left(\frac{x_1-b}{q}\right) - \Phi\left(\frac{x_2-b}{q}\right)\right] - \left[\Phi\left(\frac{x_1-a}{p}\right) - \Phi\left(\frac{x_2-a}{p}\right)\right] \qquad (12)$$

where

$$\begin{aligned} x_1 &= \frac{aq^2 - bp^2 - qp\sqrt{(a-b)^2 + 2(q^2-p^2)\ln(q/p)}}{q^2-p^2} \\ x_2 &= \frac{aq^2 - bp^2 + qp\sqrt{(a-b)^2 + 2(q^2-p^2)\ln(q/p)}}{q^2-p^2}\,; \end{aligned}$$

if $p = q$ and $a \neq b$ we have

$$\mathbf{d}(a,b;\, p,p) = 2\Phi\left(\frac{|b-a|}{2p}\right) - 1\,;$$

and finally, if $p = q$ and $a = b$ we have $\mathbf{d}(a,a;\, p,p) = 0$. This will be useful since in our examples both the transition pdf's and the pdf's derived from the quantum mechanical wave functions are normal. To show this we will use the following proposition[1] which indicates a very simple way to find the fundamental solutions of a class of evolution equations (11) which contains all the situations of our future examples.

Proposition 3: If the velocity field of the evolution equation (2) has the form

$$v_{(+)}(x,t) = -b(t)x - c(t)$$

with $b(t)$ and $c(t)$ continuous functions of time, then the fundamental solutions $p(x,t|\,y,0)$ are normal pdf's $\mathcal{N}(\mu(t),\beta(t))$ where $\mu(t)$ and $\beta(t)$ are solutions of the equations

$$\begin{aligned} \mu'(t) + b(t)\mu(t) + c(t) &= 0 \\ \beta'(t) + 2b(t)\beta(t) - 2\nu &= 0 \end{aligned}$$

with initial conditions $\beta(0) = 0$ and $\mu(0) = y$.

We will discuss now our examples for systems reduced to a single non relativistic particle with a mass m, by remembering that the connection between the quantum mechanics and the stochastic mechanics is guaranteed if the diffusion coefficient and the Planck constant satisfy the relation (9). Let us consider first of all a simple harmonic oscillator with elastic constant k and classical (circular) frequency $\omega = \sqrt{k/m}$ and two possible wave functions obeying the Schrödinger equation: the (stationary) wave function of the ground state

$$\psi_0(x,t) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/4} \mathrm{e}^{-x^2/4\sigma^2}\,\mathrm{e}^{-i\omega t/2}$$

and the (non stationary) wave function of the oscillating coherent wave packet with initial displacement a

$$\psi_C(x,t) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/4} \exp\left[-\frac{(x-a\cos\omega t)^2}{4\sigma^2} - i\left(\frac{4ax\sin\omega t - a^2\sin 2\omega t}{8\sigma^2} + \frac{\omega t}{2}\right)\right]$$

where we have defined

$$\sigma^2 = \frac{\nu}{\omega}\,.$$

From the position (2) we find for our wave functions that

$$\begin{aligned}
R_0^2(x,t) &= f_0(x,t) = \frac{\mathrm{e}^{-x^2/2\sigma^2}}{\sigma\sqrt{2\pi}} \\
R_C^2(x,t) &= f_C(x,t) = \frac{\mathrm{e}^{-(x-a\cos\omega t)^2/2\sigma^2}}{\sigma\sqrt{2\pi}} \\
S_0(x,t) &= -\frac{1}{2}\hbar\omega t \\
S_C(x,t) &= -\frac{1}{2}\hbar\omega t - \hbar\frac{4ax\sin\omega t - a^2\sin\omega t}{8\sigma^2}
\end{aligned}$$

and hence we can calculate from (6) the corresponding velocity fields

$$\begin{aligned}
v_{(+)}^0(x,t) &= -\omega x \\
v_{(+)}^C(x,t) &= -\omega x + \omega a(\cos\omega t - \sin\omega t)\,.
\end{aligned}$$

This means that f_0 and f_C are respectively of the form $\mathcal{N}(0,\sigma^2)$ and $\mathcal{N}(a\cos\omega t,\sigma^2)$, and that the fundamental solutions $p_0(x,t|\,y,0)$ and $p_C(x,t|\,y,0)$ of (11) can be calculated by means of Proposition 3 with

$$\begin{aligned}
b_0(t) &= \omega\,, & c_0(t) &= 0 \\
b_C(t) &= \omega\,, & c_C(t) &= -\omega a(\cos\omega t - \sin\omega t)
\end{aligned}$$

so that they will respectively have the form $\mathcal{N}(\mu_0(t),\beta_0(t))$ and $\mathcal{N}(\mu_C(t),\beta_C(t))$ where

$$\begin{aligned}
\beta_0(t) &= \sigma^2(1-\mathrm{e}^{-2\omega t})\,, & \mu_0(t) &= y\mathrm{e}^{-\omega t} \\
\beta_C(t) &= \sigma^2(1-\mathrm{e}^{-2\omega t})\,, & \mu_C(t) &= a\cos\omega t + (y-a)\mathrm{e}^{-\omega t}\,.
\end{aligned}$$

A second class of examples can be drawn from the wave functions of a free particle of mass m. In particular we will choose to examinate the behavior of the (non stationary) wave function of a wave packet of minimal uncertainty centered around $x = 0$ with initial dispersion $\sigma^2 > 0$:

$$\psi_F(x,t) = \left(\frac{1}{2\pi\sigma^2\chi^2(t)}\right)^{1/4} e^{-x^2/4\sigma^2\chi(t)}$$

where

$$\chi(t) = 1 + i\omega t\,, \qquad \omega = \frac{\nu}{\sigma^2}\,.$$

In this case we have from (2)

$$\begin{aligned} R_F(x,t) &= f_F(x,t) = \frac{e^{-x^2/2\sigma^2\alpha^2(t)}}{\sqrt{2\pi}\sigma\alpha(t)} \\ S_F(x,t) &= \frac{\hbar}{2}\left(\frac{\omega t x^2}{2\sigma^2\alpha^2(t)} - \arctan\omega t\right) \end{aligned}$$

where

$$\alpha(t) = |\chi(t)| = \sqrt{1+\omega^2t^2}\,.$$

This means that f_F is normal of the form $\mathcal{N}(0, \sigma^2\alpha^2(t))$. Moreover the velocity field is

$$v_{(+)}^F(x,t) = -\frac{1-\omega t}{1+\omega^2t^2}\omega x$$

and the fundamental solutions $p_F(x,t|\,y,0)$ of (11) can then be calculated by means of Proposition 3 with

$$b_F(t) = \frac{1-\omega t}{1+\omega^2t^2}\omega\,, \qquad c_F(t) = 0\,,$$

so that they will have the form $\mathcal{N}(\mu_F(t), \beta_F(t))$ where

$$\begin{aligned} \mu_F(t) &= y\sqrt{1+\omega^2t^2}e^{-\arctan\omega t} \\ \beta_F(t) &= \sigma^2(1+\omega^2t^2)(1-e^{-2\arctan\omega t})\,. \end{aligned}$$

We can now use (12) in order to calculate $\mathbf{d}(p_0, f_0)$, $\mathbf{d}(p_C, f_C)$ and $\mathbf{d}(p_F, f_F)$: a long but simple calculation will show that (y - l.u.)

$$p_0 \approx f_0\,, \qquad p_C \approx f_C \qquad (t \to +\infty)$$

in the examples drawn from the harmonic oscillator, but that p_F will not L^1-approximate f_F since $\mathbf{d}(p_F, f_F)$ turns out to be different from zero and still dependent on y in the limit $t \to +\infty$:

$$\begin{aligned} \mathbf{d}(p_F, f_F) \quad \to \quad & \Phi\Big(e^{\pi/2}[y - \sqrt{1-e^{-\pi}}\sqrt{y^2 - \ln(1-e^{-\pi})}\,]\Big) \\ & -\Phi\Big(e^{\pi/2}[y + \sqrt{1-e^{-\pi}}\sqrt{y^2 - \ln(1-e^{-\pi})}\,]\Big) \\ & \quad -\Phi\Big(e^{\pi/2}[y\sqrt{1-e^{-\pi}} - \sqrt{y^2 - \ln(1-e^{-\pi})}\,]\Big) \\ & \quad\quad +\Phi\Big(e^{\pi/2}[y\sqrt{1-e^{-\pi}} + \sqrt{y^2 - \ln(1-e^{-\pi})}\,]\Big)\,. \end{aligned}$$

For example, if $y = 0$ (so that both p_F and f_F will remain centered around $x = 0$ along all teir evolution) we get in the limit $t \to +\infty$:

$$\mathbf{d}(p_F, f_F) \to 2\Big[\Phi(e^{\pi/2}\sqrt{-\ln(1-e^{-\pi})}\,) - \Phi(e^{\pi/2}\sqrt{1-e^{-\pi}}\sqrt{-\ln(1-e^{-\pi})}\,)\Big] \sim 0.011\,.$$

It is also possible to show that in this case two transition pdf's with different initial conditions $y \neq y'$ will never L^1-approximate one another as $t \to +\infty$, since

$$\mathbf{d}(p(x,t|\,y,0),\, p(x,t|\,y',0)) \to 2\Phi\Big(\frac{|y-y'|}{2\sqrt{e^{\pi}-1}}\Big) - 1$$

which is zero if and only if $y = y'$. Hence on the basis of the Corollary 1 we can state that every solution of the evolution equation (11) L^1-approximates the quantum mechanical pdf (for $t \to +\infty$) only in the examples of the harmonic oscillator but not in that of the free particle.

It is apparent from our examples that the Markov processes associated to the quantum mechanical wave functions by the stochastic mechanics do not always exhibit the behavior required by the Bohm and Vigier hypothesis. In fact the calculations show that, in order to recover the property of a global relaxation in time of the pdf's toward the quantum mechanical solution, we must restrict ourselves to a particular set of physical systems. The different behaviors of our examples are in fact inscribed in the form of the time dependence of the parameters of the normal pdf's involved in our calculations. It is easy to see that, in the case of the harmonic oscillator, for every real y we have (for $t \to +\infty$)

$$\begin{aligned} \mu_0(t) &\to 0\,, & \beta_0(t) &\to \sigma^2 \\ |\mu_C(t) - a\cos\omega t| &\to 0\,, & \beta_C(t) &\to \sigma^2\,. \end{aligned}$$

On the other hand $\mu_F(t)$ and $\beta_F(t)$ behave differently from the corresponding parameters of the quantum mechanical pdf f_F, since (for $t \to +\infty$)

$$|\mu_F(t) - e^{-\pi/2} y\omega t| \to 0\,, \qquad |\beta_F(t) - (1-e^{-\pi})\sigma^2\omega^2 t^2| \to 0\,,$$

while the quantum mechanical f_F is a normal pdf which remains centered around $x = 0$ with a variance which diverges as $\sigma^2\omega^2 t^2$. In this case it is of no avail to remark that both p_F and f_F will flatten to zero when $t \to +\infty$: the relevant fact is that this flattening happens at rates different enough to make the L^1-distance remain non zero even in the limit $t \to +\infty$. It is remarkable, however, that in the formulation chosen in the original Bohm and Vigier paper no one of our three examples would have shown the correct property: our L^1-metrics plays here an important role in discriminating the well behaved systems among all the possibilities.

The fact that the Nelson transition pdf's do not always L^1-approximate one another also means that it is impossible to find a unique pdf g L^1-approximated by them independently from y, and hence that the solutions of (11) in the discussed free particle case will not globally tend to L^1-approximate one another in time. Of course nothing forbids a priori, even in this case, that particular subsets of solutions can show the tendency to mutually L^1-approximate and hence the field is open to investigations about, for instance, the possibility that some particular solution of (11) can be stable with respect to small perturbations of their initial conditions: which in some minimal sense was the essential intention of the Bohm and Vigier proposal. In any case our examples show that, at least for a significant set of systems and wave functions the Bohm and Vigier property holds in the L^1-metrics if we adopt the transition pdf suggested by the Nelson stochastic mechanics, and hence it can be surely stated that their original idea posed an interesting an physically well grounded problem. It is not possible at present to state clearly and in a general way in which cases we realize the conditions for a global (or at least local) mutual

L^1-approximation of the solutions of (11). The examples discussed show that the discriminating property is not the stationarity of the quantum mechanical wave function since also the square modulus of the non stationary, coherent, oscillating wave packet of the harmonic oscillator attracts in L^1 every other solution of (11). An indication can be perhaps found in the fact that the main difference between two systems seems to be principally in the fact that their energy spectra are very different: the harmonic oscillator has a completely discrete spectrum and the free particle a completely continuous one. Hence a first idea can be to distinguish between bound states, which exhibit the Bohm and Vigier property, and scattering states, which do not. Of course the settlement of this question will require the discussion of further examples and the investigation of more general properties. Finally it must be pointed out that we have made the very particular choice of selecting the transition pdf's of the Nelson stochastic mechanics as a good candidate to the generation of the right stochastic flux exhibiting the Bohm and Vigier property in some suitable sense. As a consequence another possible conclusion could also be that the Nelson flux is not the right candidate to represent, in the general case, the interpretative scheme of Bohm and Vigier. Hence we consider wide open the possibility that the right transition pdf's can be built in a different way. For example it is well known that in the Nelson stochastic mechanics the diffusive part of the stochastic differential equation (10) is given a priori. Hence, since the transition pdf which propagates a given time-dependent pdf $f(\mathbf{r}, t)$ is not uniquely determined (and are not, in general, observable in the stochastic mechanics), nothing forbids to find a diffusive flux, different from that of Nelson, which exhibites the Bohm and Vigier property for every possible quantum wave function. In particular a possibility lies in a generalization of the stochastic mechanics where also the diffusive part of the stochastic differential equation ruling the process is dynamically determined in a way such that the Bohm and Vigier property is always satisfied.

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