18th Advanced ICFA
Beam Dynamics Workshop on

QUANTUM ASPECTS OF BEAM PHYSICS

CAPRI, ITALY 15 – 20, October 2000

editor

Pisin Chen

Stanford Linear Accelerator Center
Stanford University

World Scientific
New Jersey • London • Singapore • Hong Kong

Published by

World Scientific Publishing Co. Pte. Ltd.

P O Box 128, Farrer Road, Singapore 912805

USA office: Suite 1B, 1060 Main Street, River Edge, NJ 07661

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

QUANTUM ASPECTS OF BEAM PHYSICS
18th Advanced ICFA Beam Dynamics Workshop

ISBN 981-02-4950-0

Printed in Singapore by Uto-Print

CONTENTS

Preface v
P. Chen

A. Quantum Fluctuations in Beam Dynamics

Plenary Papers

The Diffractive Quantum Limits of Particle Colliders 3
C.T. Hill

Quantum Fluctuations in Free-Electron Lasers 19
C.B. Schroeder, C. Pellegrini, and P. Chen

Quantum Equation of Electron Motion 35
K.-J. Kim

Summary of Working Group A

Quantum Fluctuations in Beam Dynamics 46
H. Mais

Contributed Papers

Quantum-Mechanical Analysis of Optical Stochastic Cooling 57
S. Heifets

Quantum Ground State and Minimum Emittance of a
Fermionic Particle Beam in a Circular Accelerator 67
A.C. Kabel

Existence and Properties of an Eqilibrium
State with Beam–Beam Collisions 76
J.A. Ellison and R.L. Warnock

Possible Quantum Mechanical Effects on Beam Echos 90
A. Chao and B. Nash

Worldline Influence Functional: Abraham–Lorentz–Dirac–Langevin Equation from QED 95
P.R. Johnson and B.L. Hu

Quantization of the Coulomb Chain in an External Focusing Field 105
A.C. Kabel

Cooling of Particle Beams in Storage Rings 113
E.G. Bessonov

On the Darwin Lagrangian 125
E.G. Bessonov

B. Photon–Electron Interaction in Beam Production, Cooling, Monitoring

Plenary Papers

Coherent Atom Optics with Bose-Einstein Condensates 135
K. Bongs, S. Burger, S. Dettmer, K. Sengstock, and W. Ertmer

The Weakly-interacting Photon Gas in 2D: BEC, Superfluidity, and Vortices 158
R.Y. Chiao, C. McCormick, and J.M. Hickmann

Crystalline Beams 176
J. Wei and A.M. Sessler

Photo–electron Interactions in Beam Physics and Condense Beams—an Introduction 194
V. Telnov

Summary of Working Group B

Photon–Electron Interactions in Beam Production, Cooling, and Monitoring 208
F.V. Hartman

Contributed Papers

Theory of FEL using Two-Times Green Function Formalism 222
H. Takahashi

Harmonic Inverse Free Electron Laser Micro-Buncher 232
S. Pottorf and X.J. Wang

The Role of Quantum Mechanics in Neutrino Factories 242
J.C. Gallardo, A.M. Sessler, and J. Wurtele

Terawatt Laser Pulse Multiplexing for
Compton Scattering X-Ray Sources 255
D.U.L. Yu, D. Newsham, J. Zeng, A Smirnov, F.V. Hartmann
A.L. Troha, A. LeFoll, D.J. Gibson, and H.A. Baldis

Extremely Low-Emittance, Low-Current RFQ
with Ion-Trap Injector 269
A. Ogata and H. Okamoto

Laser Cooling of Ortho-Positronium: Toward
Realization of Bose-Einstein Condensation 279
T. Hirose, T. Asonuma, H. Iijima, M. Irako, K. Kadoya,
T. Kumita, B. Matsumoto, N.N. Mondal, K. Wada and H. Yabu

High Energy Electrons and Synchrotron Radiation
from a Photonic Band-Gap Fiber Accelerator 290
R.Y. Chiao, D. Solli, and J.M. Hickmann

C. Beam Phenomena Under Strong EM Fields; Astro-Beam Physics and Laboratory Astrophysics

Plenary Papers

Recent Results in Crystal Channeling Experiments 301
E. Uggerhøj

Nonlinear QED in Heavy Ion Collisions 318
S.R. Klein

Unruh Effect in Storage Rings 336
J.M. Leinaas

Relativistic Jets in Microquasars 353
F. Mirabel

BEPPOSAX and EUSO: Two Approaches for the
Exploration of the 'Extreme Universe' 363
L. Scarsi

Gamma Ray Bursts and Vacuum Polarization
Process in Electromagnetic Black Holes 387
R.J. Ruffini

Summary of Working Group C

Introduction and Summary of Working Group C: Part I 410
J.S.T. Ng

Contributed Papers

Is There Emmitted Radiation in the Unruh Effect? 414
B.L. Hu and A. Raval

Fermilab A0 Channeling Program 424
R.A. Carrigan, Jr., J. Freudenberger, S. Fritzler, H. Genz, A. Richter, A. Ushakov, A. Zilges, and J.P.F. Sellschop

Integral Characteristics of Bremsstrahlung and
Pair Photoproduction in a Medium 433
V.N. Baier and V.M. Katkov

The Coulomb Corrections to e^+e^- Pair Production
in Ultrarelativistic Heavy-Ion Collisions 443
R.N. Lee

Spin Depolarization Due to Beam–Beam Interaction
in Linear Colliders 450
K.A. Thompson

Gravitational Čerenkov Radiation and Scalar Stars 460
S. Capozziello, G. Lambiase and D.F. Torres

D. Quantum Methodologies in Beam Physics

Plenary Papers

Supersymmetry and Beam Dynamics 473
J.D. Bjorken and P. Chen

Landau Damping in Nonlinear Schrödinger Equations 483
R. Fedele, S. DeNicola, V.G. Vaccaro, D. Anderson, and M. Lisak

Summary of Working Group D

Quantum Methodology in Beam Physics 499
A. Dragt and M. Pusterla

Contributed Papers

Controlled Stochastic Collective Dynamics of Particle Beams
in the Stability Regime 507
N.C. Petroni, S. DeMartino, S. DeSiena, and F. Illuminati

Quantum Mechanical Formalism of Particle Beam Optics 517
S.A. Khan

Localized Coherent Structures and Patterns Formation
in Collective Models of Beam Motion 527
A. Fedorova and M. Zeitlin

Quasiclassical Calculations for Wigner Functions
via Multiresolution 539
A. Fedorova and M. Zeitlin

Single-Particle Quantum Dynamics in a Magnetic Lattice 551
M. Venturini and R.D. Ruth

Quantum-like Approach to Beam Dynamics—Application to the LHC and HIDIF Projects 561
M. Pusterla

Quantum Mechanics of Dirac Particle Beam Optics: Single-Particle Theory 568
R. Jaganathan

Quantum Models in Beam Physics and Signal Analysis 578
M. Manko

Radiative Corrections in Symmetrized Classical Electrodynamics 588
J.R. Van Meter, A.L. Troha, D.J. Gibson, A.K. Kerman, P. Chen, and F.V. Hartemann

Beyond Unruh Effect: Nonequilibrium Quantum Dynamics of Moving Charges 612
B.L. Hu and P.R. Johnson

Appendices

Committees 625

List of Participants 626

Conference Program 629

Conference Photos 633

Author Index 643

CONTROLLED STOCHASTIC COLLECTIVE DYNAMICS OF PARTICLE BEAMS IN THE STABILITY REGIME

N. CUFARO PETRONI
Dipartimento Interateneo di Fisica dell'Università e del Politecnico di Bari;
INFN Sezione di Bari and INFM Unità di Bari;
Via G. Amendola 173, 70126 Bari, Italy;
Email: cufaro@ba.infn.it

S. DE MARTINO, S. DE SIENA, F. ILLUMINATI
Dipartimento di Fisica dell'Università di Salerno;
INFM Unità di Salerno and INFN Sezione di Napoli, Gruppo collegato di Salerno;
Via S. Allende, I-84081 Baronissi (SA), Italy;
Email: demartino@sa.infn.it, desiena@sa.infn.it, illuminati@sa.infn.it

Introducing a description of the collective transverse dynamics of charged (proton) beams in the stability regime by suitable classical stochastic fluctuations, we show that the transition probabilities associated to Nelson processes can be exploited to model evolutions suitable to control the transverse beam dynamics. In particular we show how to control, in the quadrupole approximation to the beam–field interaction, both the focusing and the transverse oscillations of the beam, either together or independently.

1 Introduction

In this paper we study the intermediate, but physically relevant, regime of beam dynamics in which a balance is realized, on the average, between the energy dissipation and the external RF energy pumping [1]. We thus describe the beam dynamics exploiting the theory of *classical* stochastic dynamical systems with time–reversal invariance, which has been introduced and extensively studied in the context of Nelson stochastic mechanics [2]. The study of these dynamical systems is based on an extension of the variational principles of classical mechanics to include the case of a diffusive kinematics replacing the deterministic one [3]. This is remarkable since variational principles are a very powerful tool in the description of physical systems. Here the stochastic variational principle yields two coupled hydrodynamic equations, respectively for the density and for the forward drift, which provide an effective description of the transverse oscillations of the beam profile in the regime of stability.

On the other hand, it is also interesting to remark that the two real, nonlinearly coupled hydrodynamic equations of the stochastic mechanics are equivalent to one complex, linear equation of the form of a Schrödinger equation, with the Planck action constant replaced by the diffusion coefficient of

the random kinematics. This fact connects our stochastic approach, which is developed in full detail elsewhere [4], to the recently developed quantum–like approaches to beam dynamics [5]. Moreover, since this description involves not only a Fokker–Planck equation but also a dynamical prescription, i.e. the specification of the external potential, it allows to implement the powerful techniques of active control [6] also to beam dynamics. This is at variance with the case of a purely dissipative Fokker–Planck dynamics which only describes a passive, irreversible evolution of the state [7].

In fact, once we obtain the description of the collective dynamics of the beam in terms of the hydrodynamic equations of Nelson stochastic mechanics with the proper diffusion coefficient, we can implement techniques of control already developed in the general context of stochastic dynamical systems [6]. These techniques exploit the transition probabilities, a fundamental object in the theory of diffusion processes, in order to drive the beam toward a specified and controlled evolution. In particular, we construct time–dependent potentials which drive the system from an initial state with a certain degree of collimation towards a final state characterized by a better focusing. At the same time, and independently, also the transverse betatron oscillations can be controlled and varied.

2 Stochastic collective dynamics in the stability regime

In this section we model the spatial fluctuations through the random kinematics performed by a representative particle that oscillates, in a reference frame comoving with the bunch, around the closed ideal orbit. This representative particle is identified with the collective degree of freedom by letting the associated probability density coincide with the real density of particles in the bunch. This last step is achieved by rescaling the normalization of the total number of particles. Before proceeding, we establish the notations that will be used in the following, according to the standard conventions.

We denote by $\mathbf{r} \equiv (x, y)$ a point in the transverse section orthogonal to the beam direction. We then measure the time in units of length through the arc length s along the design orbit (curvilinear coordinate). We now consider the (two–dimensional) diffusion process $\mathbf{q}(s)$ which describes the transverse motion of the representative particle and whose probability density coincides with the particle density of the bunch in the transverse direction. The evolution in the "time" s of the process $\mathbf{q}$ is described by the Itô stochastic differential equation

$$d\mathbf{q}(s) = \mathbf{v}_{(+)}(\mathbf{q}(s), s)ds + \sqrt{\mathcal{E}}d\mathbf{w}(s)\,, \tag{1}$$

where $\mathbf{v}_{(+)}$ is the (forward) drift, and $d\mathbf{w}(s) \equiv \mathbf{w}(s + ds) - \mathbf{w}(s)$ is the δ–correlated time increment of the standard white noise, and where we have

fixed the diffusion coefficient to be the characteristic transverse emittance. Equation (1) defines the random kinematics performed by the collective degree of freedom.

In the stability regime the energy lost by photonic emissions is regained in the RF cavities, and on average the dynamics is time–reversal invariant. We are thus in a situation in which there are both a random kinematics and time reversal invariance. Therefore the dynamics must be independently added to the kinematics (at variance with the purely dissipative Fokker–Planck or Langevin case) by introducing a stochastic generalization of the least action principle[3]. The latter is obtained as a generalization of the variational principle of classical mechanics, by replacing the classical deterministic kinematics, $d\mathbf{q}_c(s) = \mathbf{v}_c(s)ds$, with the random diffusive kinematics of equation (1). The equations of motion thus obtained take the form of two coupled hydrodynamic equations describing the evolution in time of the beam density and of the velocity field of the beam profile.

As a first consequence of the stochastic variational principle[3] we find that the current velocity has a gradient form:

$$m\mathbf{v}(\mathbf{r}, s) = \nabla S(\mathbf{r}, s)\,. \tag{2}$$

The two (nonlinearly coupled) Lagrange equations of motion for the density ρ and for the current velocity $\mathbf{v}$, of the form (2) are: the continuity equation

$$\partial_s \rho = -\nabla \cdot (\rho \mathbf{v})\,, \tag{3}$$

and a dynamical equation

$$\partial_s S + \frac{m}{2}\mathbf{v}^2 - 2m\mathcal{E}^2 \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} + V(\mathbf{r}, s) = 0\,. \tag{4}$$

This dynamical equation is typical for time–reversal invariant diffusion processes (Nelson processes). It has the same form of the Hamilton–Jacobi–Madelung (HJM) equation, originally introduced in the hydrodynamic description of quantum mechanics by Madelung[8]. It can also be shown that (3) it is equivalent to the standard Fokker–Planck equation

$$\partial_s \rho = -\nabla \cdot [\mathbf{v}_{(+)}\rho] + \mathcal{E}\,\nabla^2 \rho\,. \tag{5}$$

The time–reversal invariance is assured by the fact that the forward drift velocity $\mathbf{v}_{(+)}(\mathbf{r}, s)$ is not a field given *a priori*, as usual for diffusion processes of the Langevin type. On the contrary, given a certain initial condition, it is dynamically determined at any instant of time by the HJM evolution equation (4).

The equations (3) and (4) describe the collective behaviour of the beam at each instant of time through the evolution of both the beam profile and the velocity field of the beam.

It is finally worth noticing that, introducing the trivial representation [8]

$$\psi(\mathbf{r}, s) = \sqrt{\rho(\mathbf{r}, s)}\, \mathrm{e}^{iS(\mathbf{r},s)/2m\mathcal{E}}\,, \tag{6}$$

the coupled equations (3) and (4) are equivalent to a single linear equation of the form of the Schrödinger equation in the function ψ, with the Planck action constant replaced by the emittance $\mathcal{E}$:

$$i2m\mathcal{E}\partial_s\psi = -2m\mathcal{E}^2\nabla^2\psi + V\psi\,. \tag{7}$$

In this formulation the "wave function" ψ carries the information on both the dynamics of the bunch density ρ, and of the velocity field of the bunch, where the velocity field is determined through equation (2) by the phase function $S(\mathbf{r}, s)$. This shows, as previously claimed, that our procedure, starting from a different point of view, leads to a description formally analogous to that of the quantum–like approaches to beam dynamics [5].

3 Controlled beam dynamics in the quadrupole approximation

We now move on to construct explicit examples of controlled beam dynamics. In considering an accelerating machine we assume, as usual, that the longitudinal and the transverse dynamics can be deemed independent with a high degree of approximation. We will work in the framework of the quadrupole approximation, with the further simplification of considering decoupled evolutions along the radial direction x and the vertical direction y in the local reference frame.

Under these conditions, we can separate the original, two–dimensional diffusion process into two independent, one–dimensional processes respectively along x and y, each ruled by a harmonic potential. The configurational variable ξ of the previous section can here indifferently be either x or y depending on the considered transverse direction. The potential in each transverse direction has, in units of mass, the general form:

$$V(\xi, s) = \frac{1}{2}m\omega^2(s)\xi^2 - mf(s)\xi + mU(s)\,. \tag{8}$$

We have considered here a time–dependent frequency (parametric oscillator) in order to describe also the effects due to strong focusing [1]. Our aim is now to exploit the hydrodynamic equations (3) and (4) as control equations for the

beam dynamics. In particular, we will show how to compute a controlling, time–dependent potential which allows to drive a bunch prepared in a state with a certain degree of collimation towards a final state with better focusing.

We consider a Gaussian shape for the initial density profile of a bunch in each transverse direction, with constant dispersion, and with the centre of the profile which performs a classical harmonic motion with the same frequency associated to the initial potential (8). The motion of the centre models the betatron oscillations of the bunch. In our quantum–like approach, the state of the bunch is thus formally represented by a coherent state. As anticipated at the end of the previous section, we will now consider an instance of controlled evolution that does not require an extra smoothing procedure for the driving velocity field, i.e. the transition between pairs of Gaussian densities. In particular we will describe transitions from a coherent oscillating packet to another Gaussian state with a better collimation (smaller dispersion). It is worth noticing that we can also implement a procedure that allows to vary independently the dispersion (collimation) of the bunch density and the motion of the centre of the density profile (characteristics of the betatron oscillations).

To this end we will recall that if the velocity field of a Fokker-Planck equation (5) with constant diffusion coefficient $\mathcal{E}$ (the transverse emittance) has the linear form $v_{(+)}(\xi, s) = A(s) + B(s)\xi$, with $A(s)$ and $B(s)$ continuous functions of s, then there are always Gaussian solutions $\mathcal{N}\big(\mu(s), \nu(s)\big)$, where $\mu(s)$ is the displacement of the centre of the Gaussian distribution and $\nu(s)$ is the variance of the Gaussian distribution.

As previously stated, all along the time evolution our states keep a Gaussian shape for the density, and the centre of the density profile performs an arbitrarily assigned motion. Then, if we adopt the concise quantum–like representation of the bunch state (6) it is straightforward to show that the general form for the wave packet will be:

$$\psi(\xi, s) = (2\pi\nu)^{-1/4} \exp\left[-\frac{(\xi-\mu)^2}{4\nu} + \frac{i}{2m\mathcal{E}}\left(m\mu'\xi + m\frac{\nu'}{4\nu}(\xi-\mu)^2 + \theta\right)\right], \tag{9}$$

while the forward velocity field reads

$$v_{(+)}(\xi, s) = \mu' + \frac{\nu' - 2\mathcal{E}}{2\nu}(\xi - \mu). \tag{10}$$

Here the s-dependent functions $\mu(s)$ and $\nu(s)$ describe respectively the motion of the centre of the density profile and the spreading of the bunch density in the chosen transverse direction; on the other hand $\theta(s)$ plays the role of an arbitrary integration constant. Of course a suitable potential must also be

tailored in order to keep the evolution of the wave function (9) on the right track: we will show that in fact this control potential has the form suggested in (8).

The equation (9) represents the most general Gaussian packet, with a given generic motion $\mu(s)$ of its centre and with a given dispersion $\nu(s)$, associated to a linear form of the forward velocity in the Fokker–Planck equation (5). This also allows us to keep independent the initial and the final motion of the centre of the packet from the dispersions. As a first example let us now consider the transitions between two states of the form (9) with constant dispersion and with a harmonic motion of the centre of the profile. If initially (namely for $s \ll \tau$, where from now on τ is the transition instant) we start with $\nu(s) = \nu_1$ and $\mu(s) = a_1 \cos(\omega_1 s)$, we will have an initial Gaussian density profile with spreading ν_1 and with harmonic betatron oscillation of frequency $\omega_1 = \mathcal{E}/\nu_1$. We now want to drive the system towards a final (for $s \gg \tau$) state of the form (9), but with a spreading $\nu_2 < \nu_1$ (better collimation) and a new betatron oscillation $\mu_2(s)$. To this end we only need to put in the solution $\mathcal{N}(\mu(s), \nu(s))$ two functions $\mu(s), \nu(s)$ which interpolate between the corresponding initial and final functions of the motion of the centre, and of the spreading respectively. Moreover, with a suitable choice of the ξ–independent part of the phase function in (9), the forward velocity field will also smoothly interpolate between the initial and the final velocity fields [6]. The control potential which drives the solution toward the required end is finally obtained with $\tilde{\rho}$ given by the interpolating solution $\mathcal{N}(\mu(s), \nu(s))$, and with $v_{(+)}$ given by the associated forward velocity. Of course there is a large number of possible choices for the interpolating functions $\mu(s), \nu(s)$: this will allow us to single out the forms that better realize specific requirements. For example, it is possible to choose a characteristic transition time (the time needed to go from the initial to the final state) by inserting exponential relaxation terms in the interpolating functions.

We will now present a few explicit examples of transitions. Our initial ($s \ll \tau$) Gaussian, coherent, oscillating wave function has the form

$$\psi_1(\xi, s) = (2\pi\nu_1)^{-1/4} \exp\left[\frac{-(\xi - a_1 \cos\omega_1 s)^2}{4\nu_1}\right] \times \tag{11}$$

$$-i \exp\left[\frac{4a_1\xi \sin\omega_1 s - a_1^2 \sin 2\omega_1 s + 4\nu_1\omega_1 s}{8\nu_1}\right],$$

where we must also remember that

$$\omega_1 = \frac{\mathcal{E}}{\nu_1}. \tag{12}$$

The relation (12) means that our initial potential is purely harmonic with frequency ω_1. From the wave function (11) we have

$$\mu(s) = a_1 \cos\omega_1 s = a_1 \cos\left(\frac{\mathcal{E}s}{\nu_1}\right), \qquad \nu(s) = \nu_1, \qquad (s \ll \tau). \quad (13)$$

As for the initial phase function, by inspection of equations (11) and (6), and by taking (12) into account, we immediately get

$$S(\xi, s) = m\omega_1 \left(\frac{a_1^2}{4}\sin 2\omega_1 s - \mathcal{E}s - a_1\xi \sin\omega_1 s\right), \qquad (s \ll \tau). \quad (14)$$

First of all we want to describe the (smooth) transition of our initial wave function to a final one of the same form but characterized by a new set of parameters:

$$a_1 \to a_2, \qquad \nu_1 \to \nu_2, \qquad \omega_1 = \frac{\mathcal{E}}{\nu_1} \to \omega_2 = \frac{\mathcal{E}}{\nu_2}. \quad (15)$$

The choice (15) also means that the final potential is still purely harmonic with a new frequency ω_2. In order to achieve that we consider for example the function

$$\Gamma(s) = \frac{1}{1 + e^{-(s-\tau)/\gamma}} \quad (16)$$

which smoothly goes from 0 (for $s \ll \tau$) to 1 (for $s \gg \tau$) with a flex point in $s = \tau$ and a transition velocity equal to $1/\gamma$. Of course here τ and γ are completely free parameters: a suitable choice of them will allow to fine tune the timing and the velocity of the transition. Now the required transition is implemented by choosing $\mu(s) = a_1 \cos(\mathcal{E}s/\nu_1)(1 - \Gamma(s)) + a_2 \cos(\mathcal{E}s/\nu_2)\Gamma(s)$, and $\nu(s) = \nu_1(1 - \Gamma(s)) + \nu_2\Gamma(s)$, which clearly interpolates between the two initial and final Gaussian, coherent, oscillating states.
The phase function can now be calculated from (9) and we have

$$S(\xi, s) = m\left[\alpha(s)\xi^2 + \beta(s)\xi + H(s) + \theta(s)\right] \quad (17)$$

$$\alpha(s) = \frac{\nu'}{4\nu}, \qquad \beta(s) = \mu' - \frac{\mu\nu'}{2\nu}, \qquad H(s) = \frac{\nu'\mu^2}{4\nu}. \quad (18)$$

Since α, β and H are now fixed by the chosen interpolating $\mu(s)$ and $\nu(s)$, a comparison between (17) and (14), and in particular between the asymptotic ($s \to \pm\infty$) expressions of the ξ–independent term of the phase, will suggest the following form for the arbitrary $\tilde{\theta}(s)$ function (where $\tilde{\theta}(s) \equiv \theta(s) + H(s)$):

$$\theta(s) = \left[\frac{\mathcal{E}a_1^2}{4\nu_1}\sin\left(\frac{2\mathcal{E}s}{\nu_1}\right) - \frac{\mathcal{E}^2 s}{\nu_1}\right](1 - \Gamma(s)) + \left[\frac{\mathcal{E}a_2^2}{4\nu_2}\sin\left(\frac{2\mathcal{E}s}{\nu_2}\right) - \frac{\mathcal{E}^2 s}{\nu_2}\right]\Gamma(s). \quad (19)$$

Finally the potential will have the form

$$V_c(\xi, s) = m\left[\frac{1}{2}G(s)\xi^2 - F(s)\xi + W(s)\right] \ , \ G(s) = \frac{\mathcal{E}^2}{\nu^2} - \frac{\nu''}{2\nu} + \frac{\nu'^2}{4\nu^2}, \quad (20)$$

$$F(s) = \mu'' + \mu G \ , \ W(s) = \frac{G\mu^2}{2} - \frac{\mu'^2}{2} - \frac{\mathcal{E}^2}{\nu} - \theta'(s)\,, \quad (21)$$

where now all the terms are given by the previous relations.
As already remarked this potential has exactly the form (8). The functions $\alpha(s)$, $\beta(s)$, $G(s)$, $F(s)$ and $W(s)$, which determine the potential, can now be explicitly calculated. However their analytic expressions are by far too long (albeit elementary), and we do not report them here. They are plotted in [4].

The potential V_c has the required time behaviour since it is a simple harmonic potential for $s \ll \tau$ and $s \gg \tau$ (albeit with two different frequencies), and shows some extra terms only in a limited interval around the transition. Of course this does not constitute the only potential we can obtain in this way. For example the function $\mu(s)$, instead, could be chosen in such a way that the oscillation of the centre of the profile be slower than the initial one, despite the fact that the better collimation requires a final potential associated to a frequency $\omega_2 = \mathcal{E}/\nu_2$ larger than the initial one and then to a stronger betatron oscillation. This can be achieved by keeping a suitable forcing part $F(s)$ different from zero also for $s \gg \tau$: namely in this case the final potential does not reduces itself to a simple harmonic one. It is easy to show that if the final oscillation has the generalized form

$$\mu(s) = a\,\cos(\omega s) + \frac{b}{m}\,\sin(\omega s), \quad (22)$$

with ω not coincident with $\mathcal{E}/\nu$, the forcing function $F(s)$ calculated from (20) will correspondingly be

$$F(s) = m\left(\omega^2 - \frac{\mathcal{E}^2}{\nu^2}\right)\left(a\,\cos\omega s + \frac{b}{m}\,\sin\omega s\right). \quad (23)$$

In this case the potentials are more complicated but can still be suitably explored by means of our method. As an example we consider the case where the final state is characterized by two independent parameters: ω_2 for the frequency and ν_2 for the packet spreading. Now a relation similar to (12) will be no longer satisfied. As a consequence the original choice of interpolating $\mu(s)$ and $\nu(s)$ will be changed in $\mu(s) = a_1 \cos\left(\frac{\mathcal{E}s}{\nu_1}\right)(1 - \Gamma(s)) + a_2 \cos(\omega_2 s)\,\Gamma(s)$ and $\nu(s) = \nu_1(1 - \Gamma(s)) + \nu_2\Gamma(s)$, while we get a new determination for the

arbitrary $\tilde{\theta}(s)$ function:

$$\tilde{\theta}(s) = \left[\frac{\mathcal{E}a_1^2}{4\nu_1}\sin\left(\frac{2\mathcal{E}s}{\nu_1}\right) - \frac{\mathcal{E}^2 s}{\nu_1}\right](1-\Gamma(s)) + \left[\frac{\omega_2 a_2^2}{4}\sin(2\omega_2 s) - \mathcal{E}\omega_2 s\right]\Gamma(s). \tag{24}$$

The functions defining the time evolution of both the phase and the potential can now be calculated once more and we find that, the functions $\alpha(s)$, $\beta(s)$ keep a form very similar to the previous one. Instead the new $G(s)$ displays an opposite behaviour, since the final frequency ω_2 is smaller than the initial frequency ω_1, and thus the betatron oscillations are suppressed. Regarding the functions $F(s)$ and $W(s)$, they do not disappear any more for $s \gg \tau$, so that asymptotically we do not have a purely harmonic potential since now in (8) both the term linear and that constant in ξ will be present for every $s > \tau$. However it is clear that other choices are always possible: for example the arbitrary function $\theta(s)$ could be defined so that in (20) the ξ-independent term $W(s)$ of the potential V_c be identically zero. Of course there would be a price to pay for that: in fact now in the phase function S the ξ-independent term will no more follow an asymptotic behaviour of the type (14) since the relation (24) will no more be satisfied.

In the most general case of transitions between states with non constant dispersion (strong focusing) it is clear that the procedure can also be suitably extended. In fact it is sufficient to exploit for instance the expressions for the interpolating dispersion, but with time dependent initial and final dispersions $\nu_1(s)$ and $\nu_2(s)$. The general form (20) of the controlling potential is thus calculated, but with a new expression for $\nu(t)$. Finally, also the initial and final laws of motion of the profile centre, $\mu_1(s)$ and $\mu_2(s)$, can always be chosen as in the previously discussed example. However, in this case, a forcing part $F(s)$ is needed to retain the oscillatory motion (22) for $s \gg \tau$.

In future work we will study the extension of these control techniques beyond the quadrupole approximation and address in detail problems related to dynamical instabilities and halo formation. This latter problem has recently been addressed in the framework of a quantum–like approach [9].

1. M. Conte and W. W. MacKay, *An Introduction to the Physics of Particle Accelerators* (World Scientific, Singapore, 1991);
   D. A. Edwards and M. J. Syphers, *An Introduction to the Physics of High Energy Accelerators* (Wiley, N. Y., 1993);
   H. Wiedemann, *Particle accelerator physics* (Springer, Berlin, 1998).
2. E. Nelson, *Quantum Fluctuations* (Princeton University Press, Princeton N. J., 1985);
   F. Guerra, Phys.Rep. **77**, 263 (1981).

3. F. Guerra and L. M. Morato, Phys. Rev. **D 27**, 1774 (1983).
4. N. Cufaro Petroni, S. De Martino, S. De Siena and F. Illuminati, Phys. Rev. **E 63** (2001), in press.
5. R. Fedele, G. Miele and L. Palumbo, Phys. Lett. **A 194**, 113 (1994), and references therein;
   S.I.Tzenov, Phys.Lett. **A 232**, 260 (1997).
6. N. Cufaro Petroni, S. De Martino, S. De Siena, and F. Illuminati, J. Phys. A: Math. Gen. **32**, 7489 (1999).
7. S. Chattopadhyay, AIP Conf. Proc. **127**, 444 (1983);
   F. Ruggiero, Ann. Phys. (N.Y.) **153**, 122 (1984);
   J. F. Schonfeld, Ann. Phys. (N.Y.) **160**, 149 (1985);
   F. Ruggiero, E. Picasso and L. A. Radicati, Ann. Phys. (N. Y.) **197**, 396 (1990).
8. E. Madelung, Z. Physik **40**, 332 (1926);
   D. Bohm, Phys. Rev. **85**, 166, 180 (1952).
9. S.A.Khan and M.Pusterla, Eur.Phys.J. **A 7**, 583 (2000).