Correlated Poisson processes and self-decomposable laws

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Abstract

We analyze a method to produce pairs of non independent Poisson processes $M(t), N(t)$ from positively correlated, self-decomposable, exponential renewals. In particular the present paper provides the family of copulas pairing the renewals, along with the closed form for the joint distribution $p_{m,n}(s,t)$ of the pair $\big(M(s), N(t)\big)$, an outcome which turns out to be instrumental to produce explicit algorithms for applications in finance and queuing theory. We finally discuss the cross-correlation properties of the two processes and the relative timing of their jumps

1 Introduction

Recent studies have shown that the spot dynamics of commodity markets displays mean reversion, seasonality and jumps [1], and some methodologies have been proposed to take dependency into account based on correlation and co-integration [2]. However, these approaches can become mathematically cumbersome and non-treatable when leaving the Gaussian-Itō world. In this context it has been indeed recently proposed [3] to consider 2-dimensional jump diffusion processes with a 2-dimensional Gaussian and a 2-dimensional compound Poisson component, and, as also suggested in different circumstances [4], we show here that a revealing approach to model the

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dependency of the 2-dimensional Poisson processes can be supplied on the ground of the *self-decomposability* of the exponential random variables used for its construction.

The present paper is in particular devoted to find both the copula function pairing our correlated renewals, and an explicit form for the joint distribution $p_{m,n}(s,t)$ of our pair of correlated Poisson processes $M(s), N(t)$: this will prove to be instrumental to produce the efficient algorithms that can be used in financial applications [3]. If indeed the pairs of correlated, exponential random variables (*rv*'s) (X_k, Y_k) – used to produce the renewals in our processes – are interpreted as random waiting times with *random delays*, the proposed model can help describing their co-movement and can answer some common questions arising in the financial context:

- Once a financial institution defaults how long should one wait for a dependent institution to default too?
- A market receives a news interpreted as a shock: how long should one wait to see the propagation of that shock onto a dependent market?
- What is the impact of the correlations among the shocks for different insurance companies on a fair assessment of the risk of losses [5]?

It is worth noticing, moreover, that we achieve our aim of producing a 2-dimensional Poisson process with *dependent* marginals without resorting to an *a priori* copula (distributional) approach: the dependence among arrival times will indeed be made explicit in terms of combinations of *rv*'s, and we only recover and discuss the corresponding copula functions as an outcome of this model. As a consequence, because of this $\boldsymbol{P}$-*a.s.* relationship between the random times, the two Poisson processes can be seen as linked with a form of *co-integration* between their jumps. Similar models – albeit rather less sophisticated – were also used in order to model a multi-component reliability system [4], while the so-called *Common Poisson Shock Models* [6] are in fact quite different from that presented here

The main practical consequence of our results is then that the price and the Greeks of the spread options considered in the applications [3] can be calculated in *closed form* using either the Margrabe formula (if the strike is zero), or some well known approximation [7]. In any case our model entails explicit algorithms for the simulation of correlated Poisson processes, and can be used in the *Monte Carlo* simulations. An extension to the multi-dimensional case, as well as to different dynamics other than Poisson, will be considered in future studies; but, under the assumption that only two underlyings have jump component, the price and the Greeks of spread options can be obtained even now by the moment-matching methodology recently proposed in [8]

The paper is organized as follows: in the Section 2 we first show how (hitherto *positively*) correlated exponential *rv*'s can be deduced from the self-decomposability of their laws; then in the Section 3 we briefly discuss the copula functions produced by this model. By using pairs of these exponential *rv*'s as correlated renewals, in the

Section 4 we subsequently produce a 2-dimensional Poisson process with correlated components, and in the Section 5 we explicitly deduce their joint distributions. Finally in the Section 6 the cross-correlation properties of the Poisson processes are briefly analyzed, and the relative timing of their jumps is used to shed new light on the dependence mechanism of a model allowing for the possibility of a delayed propagation of correlated shocks. We conclude by pointing out first that we would also be able to produce other correlated *rv*'s (Erlang, Gamma, *EPT*...) by making use, once more, of their self-decomposability; and then that the results of this paper should also be extended to *negatively* correlated renewals, a possibility – not open to other procedures – that will be postponed to future inquiries. Lengthy proofs are confined in the Appendices, together with a few details about the notation adopted throughout the paper

2 Correlation from self-decomposability

2.1 Joint distributions

A law with density (*pdf*) $f(x)$ and characteristic function (*chf*) $\varphi(u)$ is said to be *self-decomposable* (*sd*) [9, 10] when for every $0 < a < 1$ we can find another law with *pdf* $g_a(x)$ and *chf* $\chi_a(u)$ such that

$$\varphi(u) = \varphi(au)\chi_a(u)$$

This is a well known family of laws with many relevant properties. We will also say that a random variable (*rv*) X with *pdf* $f(x)$ and *chf* $\varphi(u)$ is *sd* when its law is *sd*: looking at the definition this means that for every $0 < a < 1$ we can always find two *independent* *rv*'s Y (with the same law of X) and Z_a (with *pdf* $g_a(x)$ and *chf* $\chi_a(u)$) such that

$$X \stackrel{d}{=} aY + Z_a$$

We can look at this, however, also from a different perspective: if Y is *sd*, and to the extent that, for $0 < a < 1$, an independent Z_a with the suitable law is known, we can define a third *rv*

$$X \equiv aY + Z_a \qquad\qquad \boldsymbol{P}\text{-}a.s.$$

being sure that it will have the same law as Y. In the following we will mainly adopt this second standpoint

We turn now, for later convenience, to give the joint laws of the triplet (X, Y, Z_a): for the *chf* $\psi(u, v, w)$ we easily find from the independence of Y and Z_a that

$$\begin{aligned}
\psi(u, v, w) &= \boldsymbol{E}\left[e^{i(uX+vY+wZ_a)}\right] \\
&= \varphi(au+v)\chi_a(u+w) = \varphi(au+v)\frac{\varphi(u+w)}{\varphi(a(u+w))}
\end{aligned}$$

while the marginal, joint *chf*'s of the pairs (X, Y) and (X, Z_a) respectively are

$$\begin{aligned}\phi(u,v) &= \psi(u,v,0) = \varphi(au+v)\frac{\varphi(u)}{\varphi(au)}\\ \omega(u,w) &= \psi(u,0,w) = \varphi(au)\frac{\varphi(u+w)}{\varphi(a(u+w))}\end{aligned}$$

As for the *pdf* $\kappa(x, y, z)$ on the other hand , by taking $s = au + v$, we have (we neglect the integration limits whenever they extend to all $\boldsymbol{R}$)

$$\begin{aligned}\kappa(x,y,z) &= \frac{1}{(2\pi)^3}\int du\int dv\int dw\, e^{-i(ux+vy+wz)}\varphi(au+v)\chi_a(u+w)\\ &= \frac{1}{(2\pi)^2}\int du\int dw\, e^{-i(ux+wz)}\chi_a(u+w)\frac{1}{2\pi}\int e^{-ivy}\varphi(au+v)\,dv\\ &= \frac{1}{(2\pi)^2}\int du\int dw\, e^{-i(ux+wz)}\chi_a(u+w)e^{-iyau}\frac{1}{2\pi}\int e^{-isy}\varphi(s)\,ds\\ &= f(y)\frac{1}{(2\pi)^2}\int du\int dw\, e^{-i[u(x-ay)+wz]}\chi_a(u+w)\end{aligned}$$

and again with $s = u + w,\ t = w$

$$\begin{aligned}\kappa(x,y,z) &= f(y)\frac{1}{(2\pi)^2}\int ds\int dt\, e^{-i[(x-ay)(s-t)+zt]}\chi_a(s)\\ &= f(y)\,\frac{1}{2\pi}\int ds\, e^{-is(x-ay)}\chi_a(s)\frac{1}{2\pi}\int dt\, e^{-it[z-(x-ay)]}\\ &= f(y)\,g_a(x-ay)\,\delta[z-(x-ay)]\end{aligned}$$

so that the marginal, joint *pdf*'s of (X, Y) and (X, Z_a) will respectively be

$$h(x,y) = f(y)\,g_a(x-ay) \qquad\qquad \ell(x,z) = \frac{1}{a}\,f\left(\frac{x-z}{a}\right)g_a(z) \tag{1}$$

Finally the joint cumulative distribution function (*cdf*) of (X, Y) is

$$H(x,y) = \int_{-\infty}^{y} f(y')\,G_a(x-ay')\,dy' \qquad\qquad G_a(z) = \int_{-\infty}^{z} g_a(z')\,dz'$$

where $G_a(z)$ is the *cdf* of Z_a. The particular form of $H(x, y)$ will be instrumental in finding the copula functions [11] eventually pairing X and Y

We can finally also calculate the correlation coefficients r_{XY} and r_{XZ_a}: if we put $\boldsymbol{E}[X] = \boldsymbol{E}[Y] = \mu$ and $\boldsymbol{V}[X] = \boldsymbol{V}[Y] = \sigma^2$, from the Y, Z_a independence we have

$$\boldsymbol{E}[XY] = \boldsymbol{E}[(aY+Z_a)Y] = a\sigma^2 + \mu^2$$

and hence $r_{XY} = a$. In a similar vein, to calculate r_{XZ_a} we first remark that $\boldsymbol{V}[X] = a^2\boldsymbol{V}[Y] + \boldsymbol{V}[Z_a]$, namely $\boldsymbol{V}[Z_a] = (1-a^2)\sigma^2$, and then from

$$\boldsymbol{E}[XZ_a] = \boldsymbol{E}[(aY+Z_a)Z_a] = (1-a^2)\sigma^2 + (1-a)\mu^2$$

we finally find $r_{XZ_a} = 1 - a^2$

2.2 An example: the exponential laws $\mathfrak{E}_1(\lambda)$

It is well known that the exponential laws $\mathfrak{E}_1(\lambda)$ with *pdf* and *chf* (see Appendix A for the notations adopted from now on)

$$\lambda f_1(\lambda x) = \lambda e^{-\lambda x}\vartheta(x) \qquad\qquad \varphi_1(u/\lambda) = \frac{\lambda}{\lambda - iu}$$

are a typical example of *sd* laws [9], and in this case we can explicitly give the law of Z_a: we have indeed

$$\chi_a(u) = \frac{\varphi_1(u/\lambda)}{\varphi_1(au/\lambda)} = \frac{\lambda - iau}{\lambda - iu} = a + (1-a)\frac{\lambda}{\lambda - iu} = a + (1-a)\varphi_1(u/\lambda) \tag{2}$$

which (for $0 < a < 1$) is a mixture of a law $\boldsymbol{\delta}_0$ degenerate in 0, and an exponential $\mathfrak{E}_1(\lambda)$, namely

$$Z_a \sim a\boldsymbol{\delta}_0 + (1-a)\mathfrak{E}_1(\lambda)$$

so that its *pdf* and *cdf* respectively are

$$\begin{aligned} g_a(z) &= a\delta(z) + (1-a)\lambda e^{-\lambda z}\vartheta(z) \\ G_a(z) &= \left[a + (1-a)(1-e^{-\lambda z})\right]\vartheta(z) \end{aligned}$$

It is also easy to prove on the other hand that this coincides with the law of the product of two other independent *rv*'s: an exponential $Z \sim \mathfrak{E}_1(\lambda)$, and a Bernoulli $B(1) \sim \mathfrak{B}(1, 1-a)$ with $a = \boldsymbol{P}\{B(1) = 0\}$, so that we can always write

$$Z_a = B(1)\, Z$$

In short, given two exponential *rv*'s $Y \sim \mathfrak{E}_1(\lambda)$ and $Z \sim \mathfrak{E}_1(\lambda)$, and a Bernoulli $B(1) \sim \mathfrak{B}(1, 1-a)$, all three mutually independent, the *rv* X defined as

$$X \equiv aY + B(1)Z \tag{3}$$

is again an exponential $\mathfrak{E}_1(\lambda)$. From (1) we also find that the joint *pdf* of X, Y is

$$h(x,y) = \lambda e^{-\lambda y}\vartheta(y)\left[a\delta(x-ay) + (1-a)\lambda e^{-\lambda(x-ay)}\vartheta(x-ay)\right]$$

and hence its joint *cdf* is

$$\begin{aligned} H(x,y) &= \int_{-\infty}^{y} \lambda e^{-\lambda y'}\vartheta(y')\left[a + (1-a)(1-e^{-\lambda(x-ay')})\right]\vartheta(x-ay')\,dy' \\ &= \vartheta\left(y \wedge \frac{x}{a}\right)\left[\left(1 - e^{-\lambda\left(y\wedge\frac{x}{a}\right)}\right) - e^{-\lambda x}\left(1 - e^{-\lambda(1-a)\left(y\wedge\frac{x}{a}\right)}\right)\right] \end{aligned}$$

Of course this is far from the only possible joint law with exponential marginals (see also Section 3.1), but it is noticeable because it traces its origins back to a model

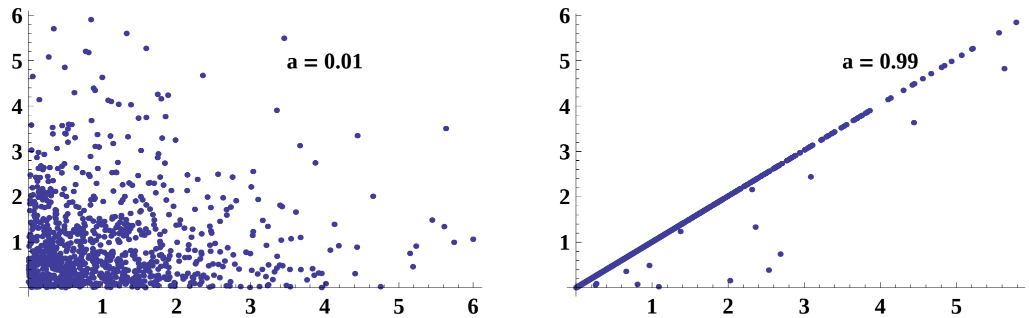


Figure 1: The pairs (X_k, Y_k) with correlation 0.01 and 0.99

of self-decomposability of the exponentials. As for the correlations among X, Y and Z, we already know that $r_{XY} = a$. For r_{XZ} we first find that

$$\boldsymbol{E}[XZ] = \boldsymbol{E}[(aY + Z_a)Z] = \frac{2-a}{\lambda^2}$$

and then that

$$r_{XZ} = 1 - a = 1 - r_{XY}$$

so that for our three exponentials in (3) we eventually have

$$r_{XY} + r_{XZ} = 1 \qquad r_{XY} = a \qquad r_{YZ} = 0$$

2.3 Positively correlated exponential *rv*'s

It is apparent now from the discussion in the previous section that the self-decomposability of the exponential laws $\mathfrak{E}_1(\lambda)$ can be turned into a simple procedure to generate identically distributed and correlated *rv*'s: given $Y \sim \mathfrak{E}_1(\lambda)$, in order to produce another $X \sim \mathfrak{E}_1(\lambda)$ with correlation $0 < a < 1$, it would be enough to take $Z \sim \mathfrak{E}_1(\lambda)$ and $B(1) \sim \mathfrak{B}(1, 1-a)$ independent from Y and define X as in (3). In other words X will be nothing else than the exponential Y *down a-rescaled*, plus another independent exponential Z *randomly intermittent with frequency* $1-a$. The self-decomposability of the exponential laws ensures then that, for every $0 < a < 1$, also X marginally is an $\mathfrak{E}_1(\lambda)$. Remark that we would not have the same result by taking more naive combinations of Y and Z. Consider for instance the sum $aY + (1-a)Z$ of our two independent, exponential *rv*'s: in this case, since $aY \sim \mathfrak{E}_1\left(\frac{\lambda}{a}\right)$ and $(1-a)Z \sim \mathfrak{E}_1\left(\frac{\lambda}{1-a}\right)$, the law of $aY + (1-a)Z$ would be $\mathfrak{E}_1\left(\frac{\lambda}{a}\right) * \mathfrak{E}_1\left(\frac{\lambda}{1-a}\right)$, which is neither an exponential $\mathfrak{E}_1(\lambda)$, nor even in general an Erlang $\mathfrak{E}_2$ because of the difference between the two parameters

The proposed procedure can now be adapted to generate a sequence of independent pairs of exponential *rv*'s, with correlated components, (X_k, Y_k), $k = 1, 2, \dots$ (or, if we prefer, X_k, Z_k) that will act in the subsequent sections as renewals for a two-dimensional point (Poisson) process: take indeed $0 < a < 1$, produce two independent *id* exponentials Y, Z and another independent Bernoulli $B(1)$, then define $X = aY + B(1)Z$ and take the pair X, Y. By independently replicating this procedure we will get a sequence of *iid* two-dimensional pairs (X_k, Y_k) that will be used later to generate a two-dimensional Poisson process with correlated renewals. Of course – not surprisingly – the case of uncorrelated pairs of renewals (X_k, Y_k), and hence of independent Poisson processes, is retrieved from our model in the limit $a \to 0$, because in this case we just have $X = Z$ which is by definition independent from Y. On the other hand it is also apparent that in the opposite limit $a \to 1$ (namely when r_{XY} goes to 1) we tend to have $X = Y$, $\boldsymbol{P}$-*a.s.* so that the time pairs will fall precisely on the diagonal of the two-dimensional time, and the two Poisson processes will simply $\boldsymbol{P}$-*a.s.* coincide. To see it from another standpoint we could look to some simulation of the pairs (X_k, Y_k): for small correlations a the scatter-plot of our pairs (X_k, Y_k) tends to evenly spread out within the first quadrant without any apparent hint to some forme of dependence; on the other hand for a near to 1 the points tend to cluster together along the diagonal, as can be seen in the Figure 1

3 Copulas for bivariate exponentials

3.1 A family of copula functions

From the discussion in the Section 2.2 we know that the pair X, Y of correlated *rv*'s deduced from their self-decomposability has the joint *cdf*

$$H(x,y) = \vartheta\left(y \wedge \frac{x}{a}\right)\left[\left(1 - e^{-\lambda\left(y\wedge\frac{x}{a}\right)}\right) - e^{-\lambda x}\left(1 - e^{-\lambda(1-a)\left(y\wedge\frac{x}{a}\right)}\right)\right] \tag{4}$$

with the exponential marginal *cdf*'s (the notation is here slightly simplified)

$$F(x) = \vartheta(x)\left(1 - e^{-\lambda x}\right) \qquad G(y) = \vartheta(y)\left(1 - e^{-\lambda y}\right) \tag{5}$$

To find out the *copula function* $C(u, v)$ [11] pairing X, Y we then first remark that

$$e^{-\lambda x} = 1 - F(x) \qquad e^{-\lambda y} = 1 - G(y) \qquad e^{-a\lambda y} = [1 - G(y)]^a$$

while

$$\begin{aligned} 0 \le \frac{x}{a} \le y \quad &\Longleftrightarrow \quad e^{-a\lambda y} \le e^{-\lambda x} \quad \Longleftrightarrow \quad [1 - G(y)]^a \le 1 - F(x) \\ 0 \le y \le \frac{x}{a} \quad &\Longleftrightarrow \quad e^{-a\lambda y} \ge e^{-\lambda x} \quad \Longleftrightarrow \quad [1 - G(y)]^a \ge 1 - F(x) \end{aligned}$$

and then that our joint *cdf* (4) takes the form

$$\begin{aligned} H(x,y) &= F(x) && \text{for } [1 - G(y)]^a \le 1 - F(x) \\ H(x,y) &= F(x) - [1 - G(y)]\left(1 - \frac{1 - F(x)}{[1 - G(y)]^a}\right) && \text{for } [1 - G(y)]^a \ge 1 - F(x) \end{aligned}$$

which can also be conveniently summarized as

$$H(x,y) = F(x) - [1-G(y)]\left(1 - \frac{1-F(x)}{[1-G(y)]^a}\right)^+$$

As a consequence we get the following family of copula functions

$$C_a(u,v) = u - (1-v)\left[1 - \frac{1-u}{(1-v)^a}\right]^+ = u - \frac{[(1-v)^a - (1-u)]^+}{(1-v)^{a-1}} \qquad (6)$$

which for $0 \le a \le 1$ runs between two extremal copulas

$$\begin{array}{rcll} C_0(u,v) & = & uv & \text{independent marginals} \\ C_1(u,v) & = & u \wedge v & \text{fully positively correlated marginals} \end{array}$$

It is easy to see that $C_1(u,v)$ also coincides with the Fréchet-Höffding upper bound $\overline{C}(u,v)$ for copulas (see Section 3.3)

3.2 Bivariate exponential distributions

Several examples – all different from (6) – of bivariate distributions with exponential marginals $\mathfrak{E}_1(\lambda)$ and $\mathfrak{E}_1(\mu)$ can be found in the literature [11, 12]. First we find the *Gumbel bivariate exponential distribution* [11, 13] with $0 \le a \le 1$ and

$$\begin{aligned} H(x,y) &= \vartheta(x)\vartheta(y)\left(1 - e^{-\lambda x} - e^{-\mu y} + e^{-\lambda x - \mu y - a\lambda\mu\, xy}\right) \\ h(x,y) &= \vartheta(x)\vartheta(y)\left[a(\lambda x + \mu y + a\lambda\mu\, xy) + 1 - a\right] e^{-\lambda x - \mu y - a\lambda\mu\, xy} \\ C_a(u,v) &= u + v - 1 + (1-u)(1-v)e^{-\frac{a}{\lambda\mu}\ln(1-u)\ln(1-v)} \end{aligned}$$

It is apparent that $C_0(u,v) = uv$ gives the independent exponentials, while

$$C_1(u,v) = u + v - 1 + (1-u)(1-v)e^{-\frac{1}{\lambda\mu}\ln(1-u)\ln(1-v)}$$

does not seem to correspond to some notable copula. Then there is the *Marshall-Olkin bivariate exponential distribution* [11, 14] with $0 \le a, b \le 1$ and

$$H(x,y) = \begin{cases} \vartheta(x)\vartheta(y)(1-e^{-\lambda x})^{1-a}(1-e^{-\mu y}) & \text{if } (1-e^{-\lambda x})^a \ge (1-e^{-\mu y})^b \\ \vartheta(x)\vartheta(y)(1-e^{-\lambda x})(1-e^{-\mu y})^{1-b} & \text{if } (1-e^{-\lambda x})^a \le (1-e^{-\mu y})^b \end{cases}$$

$$h(x,y) = \begin{cases} \frac{1-a}{(1-e^{-\lambda x})^a}\,\vartheta(x)\lambda e^{-\lambda x}\,\vartheta(y)\mu e^{-\mu y} & \text{if } (1-e^{-\lambda x})^a \ge (1-e^{-\mu y})^b \\ \frac{1-b}{(1-e^{-\mu y})^b}\,\vartheta(x)\lambda e^{-\lambda x}\,\vartheta(y)\mu e^{-\mu y} & \text{if } (1-e^{-\lambda x})^a \le (1-e^{-\mu y})^b \end{cases}$$

$$C_{a,b}(u,v) = (u^{1-a}v) \wedge (uv^{1-b}) \begin{cases} u^{1-a}v & \text{when } u^a \ge v^b \\ uv^{1-b} & \text{when } u^a \le v^b \end{cases}$$

In this case $C_{0,0}(u,v) = uv$ again is the independent copula, while $C_{1,1}(u,v) = u \wedge v$ is the Fréchet-Höffding upper bound $\overline{C}(u,v)$ (see Section 3.3): apart from these extremal values, however, also this Marshall-Olkin copula differs from (6). A third family of copulas can finally be traced back to the *Raftery bivariate exponential distribution* [11, 15]: in this case the copula functions are

$$C_a(u,v) = u \wedge v + \frac{a}{2-a}(uv)^{\frac{1}{a}}\left[1-(u\vee v)^{1-\frac{2}{a}}\right]$$

and correspond to the case of correlated exponential *rv*'s X, Y which are produced by three independent exponential *rv*'s U, V and Z according to the definitions

$$X \equiv aU + B(1)Z \qquad\qquad Y \equiv aV + B(1)Z$$

Here, at variance with our model based on self-decomposability, the correlation is apparently produced by the presence of the same exponential *rv* Z in both the right-hand sides of the definitions. In short, it results from these examples that our family of copulas (6) seems not to have been used in advance to couple pairs of marginal exponentials

3.3 Fréchet-Höffding bounds

It is well known known [11] that every copula function $C(u,v)$ falls between the Fréchet-Höffding bounds

$$\underline{C}(u,v) = (u+v-1)^+ \leq C(u,v) \leq u\wedge v = \overline{C}(u,v)$$

and we have also found in the Section 3.1 that the copula $C_1(u,v)$ for our fully correlated ($r_{XY} = 1$) exponential marginals coincides with the Fréchet-Höffding upper bound. By keeping in mind a possible generalization of our model to the case of *negatively correlated* exponentials, we will briefly recall in this section a few general features of the joint *cdf*'s $H(x,y) = C(F(x), G(y))$ produced by the pairing of two given *cdf*'s $F(x)$ and $G(x)$ by means of the Fréchet-Höffding lower and upper bounds

Let us suppose for simplicity that $F(x)$ and $G(x)$ are continuous and strictly increasing functions so that the inverse functions exist, and consider first the lower bound copula $\underline{C}(u,v) = (u+v-1)^+$: in that case the condition $F(x)+G(y) \geq 1$ is equivalent to both the inequalities

$$x \geq \beta(y) = F^{-1}(1-G(y)) \qquad\qquad y \geq \alpha(x) = G^{-1}(1-F(x))$$

and hence from $H(x,y) = (F(x)+G(y)-1)^+$ we first have

$$\begin{aligned}
\partial_x H(x,y) &= \begin{cases} f(x) & \text{if} \quad y \geq \alpha(x) \\ 0 & \text{if} \quad y < \alpha(x) \end{cases} \\
\partial_y H(x,y) &= \begin{cases} g(y) & \text{if} \quad x \geq \beta(y) \\ 0 & \text{if} \quad x < \beta(y) \end{cases}
\end{aligned}$$

where $f(x)$ and $g(y)$ are the corresponding marginal *pdf*'s, and then the joint *pdf* is

$$h(x,y) = \partial_x \partial_y H(x,y) = f(x)\delta(y-\alpha(x)) = g(y)\delta(x-\beta(y)) \tag{7}$$

As a consequence we can say that the joint laws produced by the copula $\underline{C}(u,v)$ describe pairs of coupled *rv*'s X, Y satisfying $\boldsymbol{P}$-*a.s.* the functional relations

$$X = \beta(Y) = F^{-1}(1-G(Y)) \qquad\qquad Y = \alpha(X) = G^{-1}(1-F(X)) \tag{8}$$

A formally identical result can be proved for the case of the upper bound 2-copula $\overline{C}(u,v) = u \wedge v$ but for the fact that now the functions $\alpha(x)$ and $\beta(y)$ must be redefined as

$$\alpha(x) = G^{-1}(F(x)) \qquad\qquad \beta(y) = F^{-1}(G(y))$$

In the case of the lower bound copula $\underline{C}(u,v)$ it is interesting to remark now that when for instance $F(x)$ and $G(y)$ are Gaussian *cdf*'s the functions $\alpha(x)$ and $\beta(y)$ are linear with negative proportionality coefficients, so that the pair X, Y is perfectly anti-correlated with $r_{XY} = -1$. The same happens also in the case of a pair of Student laws of the same order. This is true indeed for every other pair of marginal laws of the *same type* and with *support coincident with* $\boldsymbol{R}$. On the other hand when the marginals either are not of the same type, or have an unbounded support strictly contained in $\boldsymbol{R}$ (as happens for exponential laws), they apparently can not reciprocally be in a linear relation with negative proportionality coefficient, and hence can not be totally *linearly* anti-correlated. In this case it can still be proved by means of Höffding's Lemma (see [11] p. 190) that the minimal correlation is reached by means of the lower bound copula $\underline{C}$, but now $\alpha(x)$ and $\beta(y)$ can no longer be *linear* functions, and r_{XY} will be strictly larger than -1. By taking indeed the Fréchet-Höffding lower bound $\underline{C}(u,v) = (u+v-1)^+$ as the copula for our exponentials (5) we would find the *pdf* (7) and the functional relations (8) where now

$$\alpha(x) = -\frac{1}{\lambda}\ln\left(1-e^{-\lambda x}\right) \qquad\qquad \beta(y) = -\frac{1}{\lambda}\ln\left(1-e^{-\lambda y}\right)$$

and a short calculation would then show that in this case

$$r_{XY} = 1 - \frac{\pi^2}{6} \approx -0.645$$

so that this minimal anti-correlation allowed for exponential *rv*'s would in any case be larger than -1. It could in fact be proved in general (see [11] p. 30-32) that, when X and Y are continuous, Y is almost surely a decreasing function of X if and only if the copula of X and Y is $\underline{C}$. Random variables with copula $\underline{C}$ are often called countermonotonic. We postpone to a subsequent enquiry a detailed study of negatively correlated exponentials

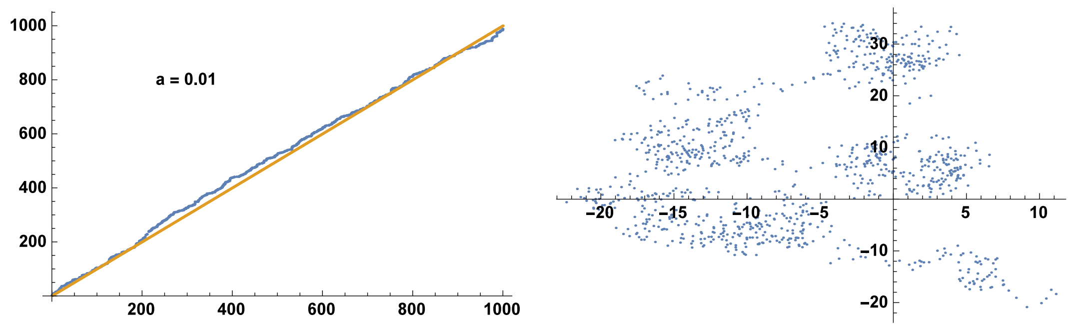


Figure 2: Sample pairs of the two-dimensional point process (T_n, S_n) with correlation $r_{XY} = 0.01$: on the left the points are compared with the average trend $\left(\frac{n}{\lambda}, \frac{n}{\mu}\right)$; on the right they are instead plotted after centering around these averages

4 Correlated Poisson processes

Following the discussion of Section 2 it is easy now to produce a sequence of *rv*'s by independently iterating the definition (3)

$$X_k = aY_k + B_k(1)Z_k \qquad k = 1, 2, \dots \tag{9}$$

in such a way that for every k: X_k, Y_k, Z_k are $\mathfrak{E}_1(\lambda)$, $B_k(1)$ are $\mathfrak{B}(1, 1-a)$, and $Y_k, Z_k, B_k(1)$ are mutually independent. The pairs (X_k, Y_k) instead will be a-correlated for every k. Add moreover $X_0 = Y_0 = Z_0 = 0$, $\boldsymbol{P}$-*a.s.* to the list, and take then the point processes for $n = 0, 1, 2, \dots$

$$T_n = \sum_{k=0}^{n} X_k \qquad S_n = \frac{\lambda}{\mu}\sum_{k=0}^{n} Y_k \qquad R_n = \sum_{k=0}^{n} Z_k \tag{10}$$

Since the $X_k \sim \mathfrak{E}_1(\lambda)$ are *iid rv*'s we know that $T_n \sim \mathfrak{E}_n(\lambda)$ are distributed as Erlang (gamma) laws with *pdf*'s $\lambda f_n(\lambda x)$ and *chf*'s $\varphi_n(u/\lambda)$ (see Appendix A for notations) where it is understood that $T_0 \sim \mathfrak{E}_0 = \boldsymbol{\delta}_0$. In a similar way we can argue that $S_n \sim \mathfrak{E}_n(\mu)$ and $R_n \sim \mathfrak{E}_n(\lambda)$. We will finally denote with $N(t) \sim \mathfrak{P}(\lambda t)$ and $M(t) \sim \mathfrak{P}(\mu t)$ the *correlated Poisson processes* associated respectively to T_n and S_n

In order to get a first look to these processes we generate $n = 1\,000$ pairs (X_k, Y_k) with the associated two dimensional point process (T_n, S_n), and then we simulate the corresponding Poisson processes $N(t)$ and $M(t)$. The pairs (T_n, S_n) are first plotted along with their average time increases $\left(\frac{n}{\lambda}, \frac{n}{\mu}\right)$, and then after centering around these averages, namely as

$$T_n - \frac{n}{\lambda} \qquad S_n - \frac{n}{\mu} \qquad n = 1, 2, \dots, 1\,000$$

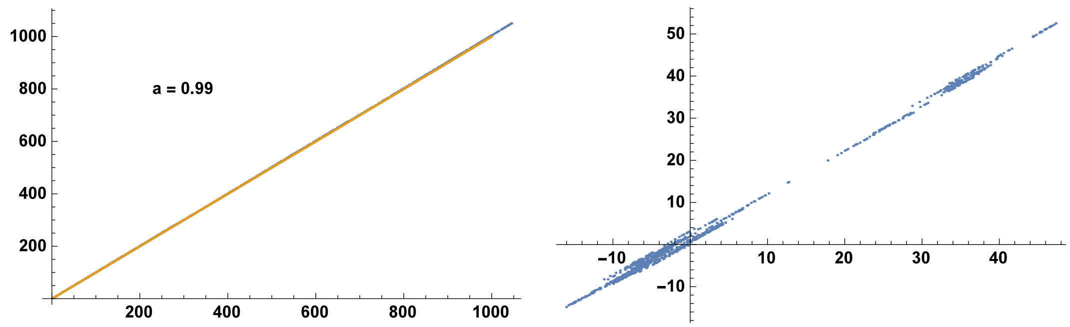


Figure 3: Sample pairs of the two-dimensional point process (T_n, S_n) with correlation $r_{XY} = 0.99$: on the left the points are compared with the average trend $\left(\frac{n}{\lambda}, \frac{n}{\mu}\right)$; on the right they are instead plotted after centering around these averages

In this second rendering the random behavior is magnified by consistently reducing the plot scale to a suitable size. In the same way for the Poisson processes we first show samples of the pair $N(t), M(t)$, and then that of their *compensated* versions $\widetilde{N}(t) = N(t) - \lambda t$ and $\widetilde{M}(t) = M(t) - \mu t$

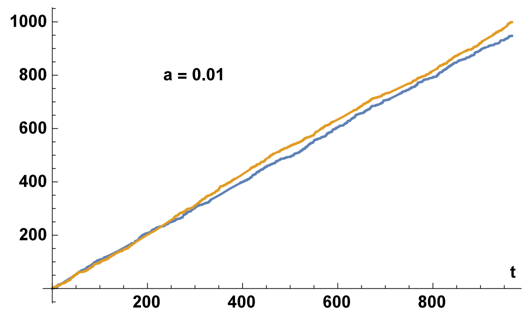
In the Figure 2 we plotted the two-dimensional point process (T_n, S_n) with $\lambda = \mu = 1$ and $a = r_{XY} = 0.01$: since the correlation among the renewals is negligible the right-hand plots (centered around the averages) apparently show a random behavior. In the Figure 3 instead we took $a = r_{XY} = 0.99$, namely we generated strongly and positively correlated renewals. In this second case, as it was to be expected, the centered time pairs fall into line among themselves. As for the Poisson processes themselves, in the Figure 4 the trajectories on the left hand side have $a = 0.01$ correlation and look fairly independent, after suitable compensation, on the right hand side. In the Figure 5 instead we took a correlation $a = 0.99$ and the compensated trajectories are now almost superimposed. Remark as on the left-hand sides of these figures both the Poisson processes and the time pairs appear to be quite near to one another, and to their averages because of a scale effect which is eliminated by compensation and centering in the corresponding right-hand sides

Proposition 4.1. *The rv's*

$$\zeta_n \equiv \sum_{k=0}^{n} B_k(1) Z_k$$

turn out to be the sum of a (random) binomial number $B(n) \sim \mathfrak{B}(n, 1-a)$ of iid exponentials $\mathfrak{E}_1(\lambda)$, and hence they follow an Erlang law with a random index $B(n)$ (here $B(0) = 0$), namely

$$\zeta_n = \sum_{k=0}^{B(n)} Z_k \sim \mathfrak{E}_{B(n)}(\lambda)$$



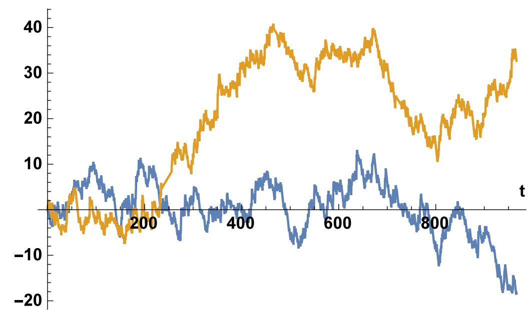


Figure 4: On the left sample paths are shown of the two Poisson processes $N(t)$ and $M(t)$ with correlation $r_{XY} = 0.01$; on the right we instead have the corresponding compensated Poisson processes $\widetilde{N}(t)$ and $\widetilde{M}(t)$

Proof: This is better seen from the point of view of the mixtures by remarking that, if $\varphi_1(u/\lambda)$ is the *chf* of $\mathfrak{E}_1(\lambda)$, we have from (2) (see also Appendix A)

$$\begin{aligned}\varphi_{\zeta_n}(u) &= \boldsymbol{E}\left[e^{iu\zeta_n}\right] = \boldsymbol{E}\left[\prod_{k=0}^{n} e^{iuB_k(1)Z_k}\right] = \prod_{k=0}^{n}\boldsymbol{E}\left[e^{iuB_k(1)Z_k}\right] \\ &= \left[a + (1-a)\varphi_1\left(\frac{u}{\lambda}\right)\right]^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}(1-a)^k\varphi_1^k\left(\frac{u}{\lambda}\right) \\ &= \sum_{k=0}^{n}\beta_k(n)\varphi_k\left(\frac{u}{\lambda}\right)\end{aligned}$$

which, if $\varphi_k(u) = [\varphi_1(u)]^k$ are the *chf* of $\mathfrak{E}_k(1)$, eventually is a mixture of Erlang laws $\mathfrak{E}_k(\lambda)$ with the binomial weights $\beta_k(n)$. It is understood here that $\varphi_1^0(u) = 1$, so that $\mathfrak{E}_0(\lambda) = \boldsymbol{\delta}_0$ and $f_0(x) = \delta(x)$ (see Appendix A) $\square$

A straightforward consequence of the previous proposition (which apparently just amounts to acknowledge a subordination) is that now from

$$\sum_{k=0}^{n} X_k = a\sum_{k=0}^{n} Y_k + \sum_{k=0}^{n} B_k(1)Z_k$$

we will also have

$$T_n = \frac{a\mu}{\lambda}S_n + \zeta_n = \frac{a\mu}{\lambda}S_n + \sum_{k=0}^{B(n)} Z_k = \frac{a\mu}{\lambda}S_n + R_{B(n)} \tag{11}$$

where $R_{B(n)} \sim \mathfrak{E}_{B(n)}(\lambda)$ is the point process R_n with a random index $B(n)$. It is worthwhile to notice that the previous results also substantiate the well known fact that the Erlang *rv*'s are self-decomposable too: the explicit knowledge of the ζ_n law allows indeed to construct pairs of dependent Erlang *rv*'s with correlation a

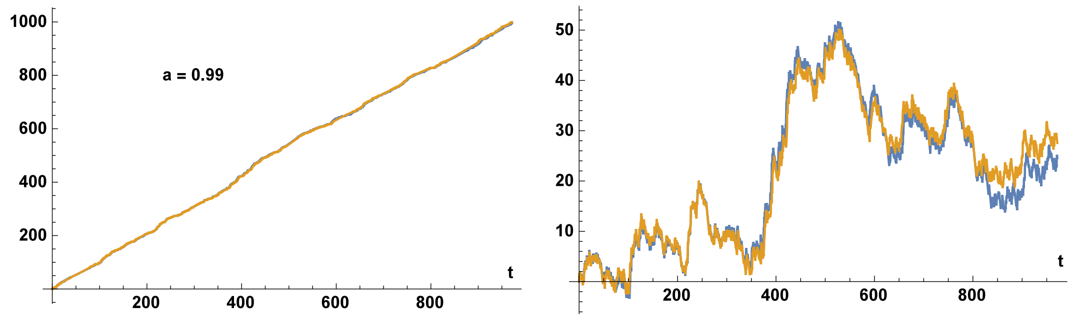


Figure 5: On the left sample paths are shown of the two Poisson processes $N(t)$ and $M(t)$ with correlation $r_{XY} = 0.99$; on the right we instead have the corresponding compensated Poisson processes $\widetilde{N}(t)$ and $\widetilde{M}(t)$

5 The joint distribution

Our main task is now to explicitly calculate the joint distribution of our Poisson processes at arbitrary times $s, t \geq 0$ and $n, m = 0, 1, 2, \ldots$

$$\begin{aligned} p_{m,n}(s,t) &= \boldsymbol{P}\{M(s) = m,\ N(t) = n\} \\ &= \boldsymbol{P}\{S_m \leq s < S_{m+1},\ T_n \leq t < T_{n+1}\} \end{aligned}$$

and to this effect we first remark (in a slightly simplified notation) that

$$\begin{aligned} p_{m,n} &= \boldsymbol{P}\{M(s) \geq m,\ N(t) \geq n\} - \boldsymbol{P}\{M(s) \geq m+1,\ N(t) \geq n\} \\ &\quad - \boldsymbol{P}\{M(s) \geq m,\ N(t) \geq n+1\} + \boldsymbol{P}\{M(s) \geq m+1,\ N(t) \geq n+1\} \\ &= q_{m,n} - q_{m+1,n} - q_{m,n+1} + q_{m+1,n+1} \end{aligned} \tag{12}$$

where

$$q_{m,n}(s,t) = \boldsymbol{P}\{M(s) \geq m,\ N(t) \geq n\} = \boldsymbol{P}\{S_m \leq s,\ T_n \leq t\}$$

so that by taking

$$w = \frac{\lambda r}{a} \qquad y = \frac{\lambda t}{a} \qquad z = \frac{\lambda t - a\mu s}{a} < y$$

from (11) we are reduced to calculate (see also Appendix A)

$$\begin{aligned} q_{m,n} &= \boldsymbol{P}\left\{S_m \le s,\ \frac{a\mu}{\lambda}S_n + R_{B(n)} \le t\right\} \\ &= \sum_{\ell=0}^{n}\beta_\ell(n)\int_0^\infty dr\lambda f_\ell(\lambda r) \\ &\qquad\qquad \boldsymbol{P}\left\{\, S_m \le s,\ \frac{a\mu}{\lambda}S_n + R_{B(n)} \le t \,\middle|\, R_\ell = r, B(n)=\ell\right\} \\ &= \lambda\sum_{\ell=0}^{n}\beta_\ell(n)\int_0^t dr f_\ell(\lambda r)\boldsymbol{P}\left\{S_m \le s,\ S_n \le \lambda\frac{t-r}{a\mu}\right\} \\ &= a\int_0^y dw\, h_n(aw)\boldsymbol{P}\left\{S_m \le \frac{y-z}{\mu},\ S_n \le \frac{y-w}{\mu}\right\} \end{aligned} \tag{13}$$

where $\lambda h_n(\lambda x)$ is the (Erlang binomial mixture) *pdf* of $R_{B(n)}$

Proposition 5.1. *For $n, m = 0, 1, 2, \dots$ and $\rho, \tau \ge 0$ we have*

$$\begin{aligned} \boldsymbol{P}\{S_m \le \rho,\ S_n \le \tau\} &= \Pi_{m\vee n}\big(\mu(\rho\wedge\tau)\big) \\ + \big[\Theta_{n-m}\vartheta(\tau-\rho) + \Theta_{m-n}\vartheta(\rho-\tau)\big] &\sum_{k=m\wedge n}^{(m\vee n)-1}\Pi_{(m\vee n)-k}\big(\mu|\rho-\tau|\big)\pi_k\big(\mu(\rho\wedge\tau)\big) \end{aligned}$$

with the notations adopted in the Appendix A for the Poisson laws

Proof: See Appendix B for a detailed proof □

Of course in (13) we take in particular

$$\rho = \frac{y-z}{\mu} = s \qquad\qquad \tau = \frac{y-w}{\mu} = \lambda\frac{t-r}{a\mu}$$

It is apparent that this result will be instrumental to calculate first $q_{m,n}(s,t)$ in (13), and then the distributions $p_{m,n}(s,t)$

Proposition 5.2. *If $a\mu s \ge \lambda t$, then $p_{m,n}(s,t) = 0$ whenever $m < n$*

Proof: Since our renewals X_k, Y_k, Z_k are all non-negative *rv*'s, the point processes are always non-decreasing

$$S_m \le S_{m+1} \qquad\qquad T_n \le T_{n+1} \qquad\qquad m, n = 0, 1, 2, \dots$$

while

$$T_n = \frac{a\mu}{\lambda}S_n + \sum_{k=0}^{B(n)} Z_k \ge \frac{a\mu}{\lambda}S_n$$

Now, if $M(s) = m$ and $N(t) = n$, we must have both $S_m \le s < S_{m+1}$ and $T_n \le t < T_{n+1}$. Suppose now $0 \le m < n$, namely $m+1 \le n$ and $S_{m+1} \le S_n$: then

$$\frac{a\mu}{\lambda}s < \frac{a\mu}{\lambda}\,S_{m+1} \le \frac{a\mu}{\lambda}\,S_n \le T_n \le t$$

which apparently contradicts the hypothesized inequality □

As a consequence when $a\mu s \geq \lambda t$ we can always restrict our calculations to the case $m \geq n \geq 0$. We can now finally state our complete results about the joint distributions $p_{m,n}(s,t)$

Proposition 5.3. *Take for short*

$$y = \frac{\lambda t}{a} > 0 \qquad\qquad z = \frac{\lambda t - a\mu s}{a} < y$$

Then, when $a\mu s > \lambda t$*, namely* $z < 0$*, we have*

$$p_{m,n}(y,z) = \begin{cases} 0 & n > m \geq 0 \\ Q_{n,n}(y,z) & m = n \geq 0 \\ Q_{m,n}(y,z) - Q_{m,n+1}(y,z) & m > n \geq 0 \end{cases} \tag{14}$$

where we defined

$$Q_{m,n}(y,z) = a\int_0^y dw\, h_n(aw) \sum_{k=n}^{m} \pi_{m-k}(w-z)\pi_k(y-w) \qquad m \geq n \geq 0 \tag{15}$$

When instead $a\mu s < \lambda t$*, and hence* $0 < z < y$*, we have*

$$\begin{aligned} & p_{m,n}(y,z) \\ = & \begin{cases} A_{m,n}(y,z) - A_{m,n+1}(y,z) + B_{m,n}(y,z) - B_{m,n-1}(y,z) & n > m \geq 0 \\ A_{n,n}(y,z) - A_{n,n+1}(y,z) + B_{n,n}(y,z) + C_{n,n}(y,z) & m = n \geq 0 \\ A_{m,n}(y,z) - A_{m,n+1}(y,z) + C_{m,n}(y,z) - C_{m,n+1}(y,z) & m > n \geq 0 \end{cases} \end{aligned} \tag{16}$$

where we defined

$$A_{m,n}(y,z) = a\int_0^z dw\, h_n(aw)\pi_m(y-z) \qquad n,m \geq 0 \tag{17}$$

$$B_{m,n}(y,z) = a\int_0^z dw\, h_{n+1}(aw) \sum_{k=0}^{n-m} \pi_k(z-w)\pi_m(y-z) \quad n \geq m \geq 0 \tag{18}$$

$$C_{m,n}(y,z) = a\int_z^y dw\, h_n(aw) \sum_{k=n}^{m} \pi_{m-k}(w-z)\pi_k(y-w) \quad m \geq n \geq 1 \tag{19}$$

while for $m \geq n = 0$ *we always have* $C_{m,0}(y,z) = 0$*. Moreover both the results for* $z < 0$*, and for* $0 < z < y$ *connect with continuity in* $z = 0$ *in the sense that*

$$p_{m,n}(y,0^-) = p_{m,n}(y,0^+) \qquad\qquad m,n \geq 0$$

Proof: Take first the case $a\mu s > \lambda t$, namely $z < 0$, and recall that for the integration variable in (13) it is $0 \leq w \leq y$. As a consequence, when the Proposition 5.1 in used in (13), we will always have

$$0 \leq \tau = \frac{y-w}{\mu} \leq \frac{y-z}{\mu} = s = \rho \tag{20}$$

On the other hand, since the conditions of the Proposition 5.2 are met, we can also restrict ourselves to evaluate $p_{m,n}(s,t)$ for $0 \le n \le m$. Then, by considering separately the cases $m = n \ge 0$ and $m > n \ge 0$, from (13) and from the Proposition 5.1 we first calculate $q_{m,n}, q_{m+1,n}, q_{m,n+1}$ and $q_{m+1,n+1}$, and finally (lengthy algebraic details can be found in the Appendix C) from (12) we find (32).

When on the other hand $a\mu s < \lambda t$ (namely $y > z > 0$ and $0 < w < y$) and we use Proposition 5.1 in (13), instead of (29) we find

$$0 \le \tau = \frac{y-w}{\mu} \qquad\qquad 0 \le \rho = s = \frac{y-z}{\mu} \tag{21}$$

so that ρ and τ can now be in an order whatsoever. As a consequence Proposition 5.2 does not hold, and we must consider all the possible orderings of m, n. Following then the same line of reasoning as before, and always taking separately the different n, m orderings, a tedious calculation (see Appendix C) gives first the q's from (13), and eventually the p's of our proposition from (12)

We finally show that the values of $p_{m,n}(y,z)$ separately listed in the Proposition 5.3 for $z < 0$ and $z > 0$ connect with continuity in $z = 0$, in the sense that for every $y > 0$

$$p_{m,n}(y,0^-) = p_{m,n}(y,0^+)$$

For $z < 0$ (namely $a\mu s > \lambda t$) the results are given in (32) and (15), so that for $z \uparrow 0^-$ and every $m, n \ge 0$, we simply have

$$p_{m,n}(y,0^-) = 0 \qquad\qquad n > m \ge 0 \tag{22}$$

$$p_{n,n}(y,0^-) = a\int_0^y dw\, h_n(aw)\,\pi_0(w)\,\pi_n(y-w) \qquad\qquad n = m \ge 0 \tag{23}$$

$$\begin{aligned} p_{m,n}(y,0^-) = a\int_0^y dw \Bigg\{ & h_{n+1}(aw)\pi_{m-n}(w)\pi_n(y-w) \qquad m > n \ge 0 \\ & + \big[h_n(aw) - h_{n+1}(aw)\big]\sum_{k=n}^{m} \pi_{m-k}(w)\pi_k(y-w)\Bigg\} \end{aligned} \tag{24}$$

On the other hand, when $z > 0$ (namely $a\mu s < \lambda t$) we have (16), (17), (18) and (19), so that now z appears also as an integration limit, and some care should be exercised for $z \downarrow 0^+$. When indeed the integrand contains the distribution $\delta(x)$, as in fact happens in every first term of $h_n(x)$ which is $\beta_0(n)\delta(x) = a^n\delta(x)$ (see also Appendix A), we have for every regular function $\xi(x)$

$$\lim_{z\downarrow 0^+}\int_0^z \xi(x)\delta(x)\,dx = \xi(0) \qquad\qquad \lim_{z\downarrow 0^+}\int_z^y \xi(x)\delta(x)\,dx = 0$$

As a consequence we have

$$\begin{aligned} \lim_{z\downarrow 0^+}\int_0^z dx\,\xi(x)h_n(x) &= a^n\xi(0) \\ \lim_{z\downarrow 0^+}\int_z^y dx\,\xi(x)h_n(x) &= \int_0^y dx\,\xi(x)\sum_{k=1}^{n}\beta_k(n)f_k(x) \end{aligned}$$

With this provisions in mind, it is then only a question of sheer calculation (see Appendix C) to show that the $p_{m,n}(y,z)$ for $z>0$ as given in (16), (17), (18) and (19) correctly converge to the values (37), (38) and (39) for every possible ordering of n,m. For instance for $n>m\geq 0$, with $x=aw$ and recalling also that $\pi_k(0)=\delta_{k,0}$ (so that $\pi_{n-m}(0)=0$ because $n>m$), in the limit $z\downarrow 0^+$ we immediately have

$$\begin{aligned}
&p_{m,n}(y,0^+)\\
&\quad=\pi_m(y)\int_0^{0^+}dx\left[a^n-a^{n+1}+a^n\pi_{n-m}(0)-(a^n-a^{n+1})\sum_{k=0}^{n-m}\pi_k(0)\right]\delta(x)\\
&\quad=\pi_m(y)\left[a^n-a^{n+1}-(a^n-a^{n+1})\sum_{k=0}^{n-m}\delta_{k,0}\right]=0
\end{aligned}$$

and so on for the other two cases □

Proposition 5.4. *The terms Q,A,B and C in the Proposition 5.3 can be expressed in terms of finite combinations of elementary functions: when $z<0$ (namely $a\mu s>\lambda t$) we have for $m\geq n\geq 0$*

$$\begin{aligned}
Q_{m,n}(y,z) &= \sum_{k=n}^{m}\sum_{j=k}^{m}\frac{(-1)^{j-k}}{a^j}\binom{j}{k}\sum_{\ell=0}^{n}\beta_\ell(n)\\
&\qquad\qquad \pi_{m-j}(y-z)\pi_{j+\ell}(ay)\,\Phi(j+1;j+\ell+1;ay)
\end{aligned}\tag{25}$$

Here and in the following $\Phi(\alpha;\beta;x)$ are confluent hypergeometric functions. When instead $z>0$ (namely $a\mu s<\lambda t$) we have for every $n,m\geq 0$

$$A_{m,n}(y,z)=\pi_m(y-z)\sum_{k=0}^{n}\beta_k(n)\left[1+\pi_k(az)-\sum_{j=0}^{k}\pi_j(az)\right]\tag{26}$$

while for $n\geq m\geq 0$ it is

$$\begin{aligned}
B_{m,n}(y,z) &= \pi_m(y-z)\sum_{k=0}^{n-m}\pi_k(z)\sum_{\ell=0}^{n+1}\beta_\ell(n+1)\\
&\qquad\qquad \frac{(az)^\ell k!}{(k+\ell)!}\,\Phi\left(\ell,k+\ell+1,(1-a)z\right)
\end{aligned}\tag{27}$$

and for $m\geq n\geq 1$ (for $m\geq n=0$ we always have $C_{m,0}(y,z)=0$) it is

$$\begin{aligned}
C_{m,n}(y,z)&=\frac{e^{-(1-a)(y-z)}}{a^m}\sum_{\ell=1}^{n}\beta_\ell(n)\sum_{k=n}^{m}\sum_{j=0}^{\ell-1}(-1)^{\ell-1-j}\binom{k+\ell-j-1}{k}\\
&\quad\pi_j(ay)\pi_{m+\ell-j}(a(y-z))\Phi(k+\ell-j,m+\ell-j+1,a(y-z))
\end{aligned}\tag{28}$$

Finally, since the parameters α, β of the $\Phi(\alpha, \beta, x)$ involved in the previous equations are integer numbers with $0 \leq \alpha < \beta$, our confluent hypergeometric functions are just finite combinations of powers and exponentials according to the following formulas

$$\Phi(0\,, \beta, x) = 1 \qquad\qquad \beta > \alpha = 0$$
$$\Phi(\alpha, \beta, x) = e^x \sum_{\gamma=1}^{\alpha} (-1)^{\alpha-\gamma} \binom{\beta-\gamma-1}{\beta-\alpha-1} \frac{\pi_{\gamma-1}(x)}{\pi_{\beta-1}(x)}\, \Pi_{\alpha-\gamma+1}(x) \qquad \beta > \alpha \geq 1$$

Proof: The detailed proof unfolds along a sequence of integrations based on known results and is here omitted for the sake of brevity (see Appendix D) □

We end this section with a short list of a few explicit examples of joint probabilities holding in the region $a\mu s \geq \lambda t$:

$$\begin{aligned}
p_{0,0}(s,t) &= e^{-\mu s} \\
p_{1,0}(s,t) &= \frac{e^{-\mu s}}{a}\big[(1-a)(1-e^{-\lambda t}) + a\mu s - \lambda t\big] \\
p_{1,1}(s,t) &= \frac{e^{-\mu s}}{a}\big[\lambda t - (1-a)(1-e^{-\lambda t})\big] \\
p_{2,0}(s,t) &= \frac{e^{-\mu s}}{2a^2}\big[2(1-a)(1+a\mu s)(1-e^{-\lambda t}) + (a\mu s - \lambda t)^2 - 2(1-a)\lambda t\big] \\
p_{2,1}(s,t) &= \frac{e^{-\mu s}}{a^2}\big[(1-a)(a-4-(1-a)\lambda t - a\mu s)(1-e^{-\lambda t}) \\
&\qquad\qquad + \lambda t(a^2 - 5a + 4 + a\mu s - \lambda t)\big] \\
p_{2,2}(s,t) &= \frac{e^{-\mu s}}{2a^2}\big[2(1-a)(3-a+(1-a)\lambda t)(1-e^{-\lambda t}) \\
&\qquad\qquad + \lambda t(\lambda t - 2(1-a)(3-a))\big]
\end{aligned}$$

6 Cross-correlations and relative timing

In this section we will briefly discuss the main correlation properties of the two processes. We first of all look at the point processes and we remark that, by recasting the self-decomposability equation (9) in the form

$$X_k = \frac{a\mu}{\lambda} W_k + B_k(1) Z_k \qquad\qquad W_k = \frac{\lambda}{\mu} Y_k \sim \mathfrak{E}_1(\mu)$$

the point processes appear as

$$T_n = \sum_{k=0}^{n} X_k \qquad\qquad S_n = \sum_{k=0}^{n} W_k$$

where now $X_k \sim \mathfrak{E}_1(\lambda)$ and $W_k \sim \mathfrak{E}_1(\mu)$ play at once the role of the correlated renewals. It is interesting to point out then that, at variance with other models [6],

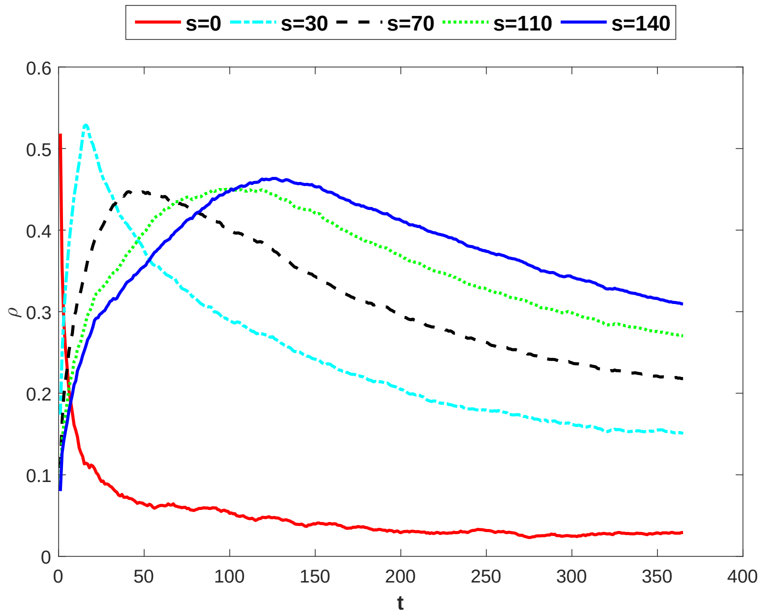


Figure 6: Cross-correlation $\rho(s,t)$ of the two Poisson processes $M(s)$ and $N(t)$ estimated by Monte Carlo simulations. Here $a = 0.5$, and $\lambda = \mu = 20$, namely ${}^{a\mu}/_{\lambda} = 0.5 < 1$

we are no longer tied to take truly *coincident* shocks: we will show indeed that with non-zero probabilities the values of the paired, and correlated renewals X_k, W_k (waiting times) can be in an order whatsoever, and they would almost never coincide. As a consequence the propagation of the shocks from a process to the other will quite plausibly happen with delays whose random sizes (and directions) could also be modeled by suitably choosing our parameters a, λ and μ. And moreover the random times T_n and S_n will be correlated by the summing up of the renewals, but will never fall at the same instant. This *relative timing* apparently allows for an enhanced flexibility of the model in the practical applications because we no longer have to rely on common shocks, but rather on correlated and randomly delayed ones

More precisely we can now single out two possible regimes for our processes: ${}^{a\mu}/_{\lambda} \leq 1$ and ${}^{a\mu}/_{\lambda} > 1$. It is then easy to see that for every $k = 1, 2, \ldots$

$$X_k = \frac{a\mu}{\lambda} W_k + B_k(1)Z_k \geq \frac{a\mu}{\lambda} W_k > W_k \qquad\qquad \frac{a\mu}{\lambda} > 1$$

and hence we first of all have

$$\boldsymbol{P}\{X_k > W_k\} = 1 \qquad\qquad \frac{a\mu}{\lambda} > 1$$

On the other hand for ${}^{a\mu}/_{\lambda} \leq 1$ the probability $\boldsymbol{P}\{X_k > W_k\}$ can still be explicitly calculated by taking into account the laws specified in the Section 4, and in this case it is possible to show that

$$\boldsymbol{P}\{X_k > W_k\} = \frac{(1-a)\mu}{\lambda + (1-a)\mu} \qquad \frac{a\mu}{\lambda} \leq 1$$

a value ranging from 0 to 1 according to the different possible choices of the parameters a, λ and μ

As for the relative timings T_n, S_m of the shocks along the point processes themselves, an explicit calculation of $\boldsymbol{P}\{T_n \leq S_m\}$ is certainly possible, but its results would turn out to be rather cumbersome because it would involve two or three convolutions of (positive and negative) Erlang laws with different parameters. We will then confine ourselves here to produce just the cross-correlations between T_n, S_m: since it is easy to check that

$$\boldsymbol{cov}[X_k, W_\ell] = \frac{a}{\lambda\mu}\,\delta_{k\ell}$$

it is also apparent that

$$\boldsymbol{cov}[T_n, S_m] = \sum_{k=1}^{n}\sum_{\ell=1}^{m}\boldsymbol{cov}[X_k, W_\ell] = \frac{a}{\lambda\mu}\sum_{k=1}^{n}\sum_{\ell=1}^{m}\delta_{k\ell} = \frac{a}{\lambda\mu} m \wedge n$$

and hence the cross-correlation coefficient of T_n, S_m will simply be

$$r_{nm} = a\,\frac{n \wedge m}{\sqrt{nm}} = \begin{cases} a\sqrt{{}^{n}/_{m}} & \text{for } \; n \leq m \\ a\sqrt{{}^{m}/_{n}} & \text{for } \; n \geq m \end{cases}$$

Finally, even closed formulas for the cross-correlation coefficient $\rho(s,t)$ between the Poisson processes $M(s)$ and $N(t)$ are still derivable on the ground of our previous results about the joint distributions, but it would be too long to thoroughly elaborate them here. As an alternative we have chosen to show the plots of their estimates based on a sample of 10^5 Monte Carlo simulations of their trajectories as shown in the Figures 6 and 7. There the behavior is displayed of $\rho(s,t)$ as a function of t for different, fixed values of s. More precisely, in the Figure 6 we have taken $a = 0.5$ and $\lambda = \mu = 20$ as the values for the relevant parameters of our coupled processes (then we have ${}^{a\mu}/_{\lambda} < 1$), while in the Figure 7 the same parameters are $a = 0.8, \lambda = 20$ and $\mu = 40$ (and then ${}^{a\mu}/_{\lambda} > 1$). It is apparent from these pictures that the behavior of $\rho(s,t)$ is comparable to that of the self-correlation of a simple Poisson process, but for the fact that the cumulative effect of the correlate renewals produces a smoothing of the shape around the maximum values near $t = s$. At first sight this could look as a little difference, but in the domain, for instance, of the financial applications even small deviations among the models could produce huge differences in gains and losses

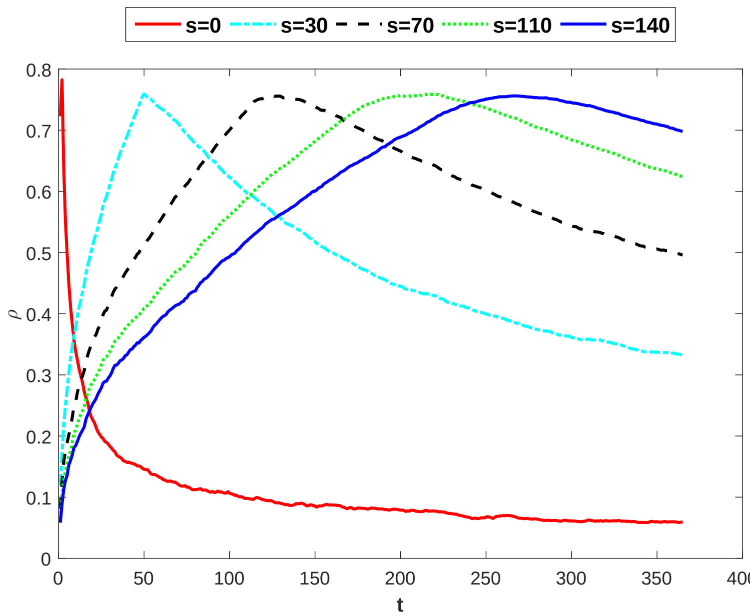


Figure 7: Cross-correlation $\rho(s,t)$ of the two Poisson processes $M(s)$ and $N(t)$ estimated by Monte Carlo simulations. Here $a = 0.8$, $\lambda = 20$ and $\mu = 40$, namely ${}^{a\mu}/_{\lambda} = 1.6 > 1$

7 Conclusions and further inquiries

It is apparent that, within the model discussed in the Section 2, from the self-decomposability alone we can only get pairs of *rv*'s X, Y with *positive correlations* $0 < r_{XY} < 1$ steered by the value of a parameter a. It would be interesting however to widen the scope of our models in order to achieve also Poisson processes whose correlation can span over all its possible values (both positive and negative) by changing the value of some numerical parameter. In this respect it is important to remark – as pointed out in the Section 3.3 – that while two *rv*'s X and Y which are, for instance, marginally exponentials can also be *totally correlated* ($r_{XY} = 1$), they can not instead be *totally anti-correlated* ($r_{XY} = -1$) because this would imply some *linear* dependence with a *negative* proportionality coefficient, and that would be at odds with the fact that both our *rv*'s take arbitrary large, but only positive values. Hence two exponential *rv*'s X and Y can always have a negative correlation, but only up to a minimal value which in any case must be larger than -1. We also showed in the Section 3.3 that this minimum is reached when between X and Y there is a peculiar kind of mutual functional, *decreasing* dependence, albeit clearly *not a linear* one. A model to produce pairs X, Y of *rv*'s which are marginally exponentials, and which – following the value of a numerical parameter a – show all the possible correlation values will be discussed in a subsequent paper

Our results in any case show that the self-decomposability, joined with the subordination techniques, can be a promising tool to study dependency beyond the Gaussian-Itō world. We have shown indeed how to obtain dependent exponential (gamma) *rv*'s that can be used to create and simulate dependent Poisson processes without resorting to definitely coincident jumps, but the path is now open to produce more general dependent gamma (Erlang at first) *rv*'s to simulate dependent variance gamma processes. A further extension could then be to study the self-decomposability of density functions that have a strictly proper rational characteristic function (Exponential Polynomial Trigonometric, *EPT* laws) in order to construct 2-dimensional correlated *EPT rv*'s (see for instance [16]). Finally it would be expedient to explore the Markov properties of the 2-component Poisson processes $(M,(t),N(t))$ with dependent marginals that we have introduced in this paper and the Master equations ruling them: this too will be the subject of future inquiries

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A Notations

All along this paper we will adopt the following notations: for a Poisson law $\mathfrak{P}(\alpha)$ we will introduce the symbols

$$\pi_n(\alpha)=\frac{\alpha^n}{n!}\,e^{-\alpha}\qquad\qquad \Pi_n(\alpha)=\sum_{k=n}^{\infty}\pi_k(\alpha)\qquad\qquad \alpha>0\qquad\quad n=0,1,2\ldots$$

and for a binomial law $\mathfrak{B}(n,1-a)$ the notation

$$\beta_k(n)=\left\{\begin{array}{ccc}1 & n=0, & k=0\\ \binom{n}{k}a^{n-k}(1-a)^k & n=1,2,\ldots, & k=0,\ldots,n\end{array}\right.\qquad\qquad 0\le a\le 1$$

It will be understood moreover that

$$\pi_n(0^+)=\delta_{n,0}$$

We will also use bot the Heaviside function ϑ, and the Heaviside symbol Θ

$$\vartheta(x)=\left\{\begin{array}{ll}1 & x\ge 0\\ 0 & x<0\end{array}\right.\qquad\qquad \Theta_j=\left\{\begin{array}{ll}1 & j\ge 1\\ 0 & j\le 0\end{array}\right.\qquad\qquad j=0,\pm1,\pm2,\ldots$$

The *pdf* and *chf* of a standard Erlang law $\mathfrak{E}_n(1)$ moreover will be denoted as

$$f_n(x)=\left\{\begin{array}{c}\delta(x)\\ \frac{x^{n-1}}{(n-1)!}e^{-x}\vartheta(x)\end{array}\right.\qquad\qquad \varphi_n(u)=\left\{\begin{array}{cc}1 & n=0\\ \left(\frac{1}{1-iu}\right)^n & n=1,2,\ldots\end{array}\right.$$

where it is understood for the Dirac delta $\delta(x)$ that for every $b > 0$

$$\int_0^b \delta(x)\,dx = 1 \qquad\qquad \lim_{z\downarrow 0^+}\int_z^b \delta(x)\,dx = 0$$

Remark also that apparently

$$f_n(x) = \pi_{n-1}(x)\vartheta(x) \qquad n = 1, 2, \dots$$

We will finally define for later convenience the functions

$$h_n(x) = \sum_{k=0}^{n} \beta_k(n) f_k(x) \qquad n = 0, 1, 2, \dots$$

which are the *pdf*'s of the mixtures of Erlang laws $\mathfrak{E}_k(1)$ with binomial $\mathfrak{B}(n, 1-a)$ weights for their indices k

B A proof of Proposition 5.1

To evaluate $\boldsymbol{P}\{S_m \le \rho,\, S_n \le \tau\}$ we first remark that

$$\boldsymbol{P}\{S_m \le \rho,\, S_n \le \tau\} = \boldsymbol{P}\{M(\rho) \ge m,\, M(\tau) \ge n\}$$

and then that, being a Poisson process, $M(t)$ is non-decreasing: as a consequence

$$\begin{array}{lcll} m \le n \text{ and } \tau \le \rho & \Longrightarrow & M(\tau) \le M(\rho) & \text{hence } \{M(\tau) \ge n\} \subseteq \{M(\rho) \ge m\} \\ n \le m \text{ and } \rho \le \tau & \Longrightarrow & M(\rho) \le M(\tau) & \text{hence } \{M(\rho) \ge m\} \subseteq \{M(\tau) \ge n\} \end{array}$$

In the case $m \le n$ we then have for $\tau \le \rho$

$$\boldsymbol{P}\{M(\rho) \ge m,\, M(\tau) \ge n\} = \boldsymbol{P}\{M(\tau) \ge n\}$$

while for $\rho \le \tau$ from the general properties of a Poisson process we get

$$\begin{aligned} &\boldsymbol{P}\{M(\rho) \ge m,\, M(\tau) \ge n\} \\ &\qquad = \sum_{k=m}^{\infty} \boldsymbol{P}\{M(\rho) \ge m,\, M(\tau) \ge n \,|\, M(\rho) = k\}\, \boldsymbol{P}\{M(\rho) = k\} \\ &\qquad = \sum_{k=m}^{n} \boldsymbol{P}\{M(\tau) \ge n \,|\, M(\rho) = k\}\, \boldsymbol{P}\{M(\rho) = k\} + \boldsymbol{P}\{M(\rho) > n\} \\ &\qquad = \sum_{k=m}^{n} \boldsymbol{P}\{M(\tau - \rho) \ge n - k\}\, \boldsymbol{P}\{M(\rho) = k\} + \boldsymbol{P}\{M(\rho) > n\} \end{aligned}$$

In the same vein, when $n \le m$ we have for $\rho \le \tau$

$$\boldsymbol{P}\{M(\rho) \ge m,\, M(\tau) \ge n\} = \boldsymbol{P}\{M(\rho) \ge n\}$$

while for $\tau \leq \rho$ we get

$$\begin{aligned}
&\boldsymbol{P}\{M(\rho) \geq m,\, M(\tau) \geq n\} \\
&\qquad = \sum_{k=m}^{\infty} \boldsymbol{P}\{M(\rho) \geq m,\, M(\tau) \geq n \,|\, M(\tau) = k\}\, \boldsymbol{P}\{M(\tau) = k\} \\
&\qquad = \sum_{k=n}^{m} \boldsymbol{P}\{M(\rho) \geq m \,|\, M(\tau) = k\}\, \boldsymbol{P}\{M(\tau) = k\} + \boldsymbol{P}\{M(\tau) > m\} \\
&\qquad = \sum_{k=n}^{m} \boldsymbol{P}\{M(\rho - \tau) \geq m - k\}\, \boldsymbol{P}\{M(\tau) = k\} + \boldsymbol{P}\{M(\tau) > m\}
\end{aligned}$$

Remark that for $m = n$ both the cases lead to the same result, namely

$$\boldsymbol{P}\{M(\rho) \geq n,\, M(\tau) \geq n\} = \begin{cases} \boldsymbol{P}\{M(\tau) \geq n\} & \text{when } \tau \leq \rho \\ \boldsymbol{P}\{M(\rho) \geq n\} & \text{when } \rho \leq \tau \end{cases}$$

that can also be conveniently summarized as

$$\boldsymbol{P}\{M(\rho) \geq n,\, M(\tau) \geq n\} = \boldsymbol{P}\{M(\rho \wedge \tau) \geq n\}$$

On the other hand for $m < n$ we have

$$\begin{aligned}
&\text{for } \tau \leq \rho \qquad \boldsymbol{P}\{M(\tau) \geq n\} \\
&\text{for } \rho \leq \tau \qquad \boldsymbol{P}\{M(\rho) \geq n\} + \sum_{k=m}^{n-1} \boldsymbol{P}\{M(\tau - \rho) \geq n - k\}\, \boldsymbol{P}\{M(\rho) = k\}
\end{aligned}$$

that can also be put in the form

$$\boldsymbol{P}\{M(\rho \wedge \tau) \geq n\} + \vartheta(\tau - \rho) \sum_{k=m}^{n-1} \boldsymbol{P}\{M(\tau - \rho) \geq n - k\}\, \boldsymbol{P}\{M(\rho) = k\}$$

while for $m > n$ it is

$$\begin{aligned}
&\text{for } \tau \leq \rho \qquad \boldsymbol{P}\{M(\tau) \geq m\} + \sum_{k=n}^{m-1} \boldsymbol{P}\{M(\rho - \tau) \geq m - k\}\, \boldsymbol{P}\{M(\tau) = k\} \\
&\text{for } \rho \leq \tau \qquad \boldsymbol{P}\{M(\rho) \geq m\}
\end{aligned}$$

namely

$$\boldsymbol{P}\{M(\rho \wedge \tau) \geq m\} + \vartheta(\rho - \tau) \sum_{k=n}^{m-1} \boldsymbol{P}\{M(\rho - \tau) \geq m - k\}\, \boldsymbol{P}\{M(\tau) = k\}$$

In both cases the first terms can expressed as $\boldsymbol{P}\{M(\rho \wedge \tau) \geq m \vee n\}$, and in this form they also coincide with the previous result for $m = n$. On the other hand the extra term with the sum (which is absent for $m = n$) must be taken in consideration either when we have both $m < n$ and $\rho \leq \tau$, or when it is $m > n$ and $\tau \leq \rho$. All these provisions can then be comprehensively taken into account in the formula

$$\begin{aligned}
&\boldsymbol{P}\{S_m \leq \rho,\, S_n \leq \tau\} \\
&\quad = \boldsymbol{P}\{M(\rho \wedge \tau) \geq m \vee n\} + [\Theta_{n-m}\vartheta(\tau-\rho) + \Theta_{m-n}\vartheta(\rho-\tau)] \cdot \\
&\qquad\qquad \sum_{k=m\wedge n}^{(m\vee n)-1} \boldsymbol{P}\{M(|\rho-\tau|) \geq (m \vee n) - k\}\, \boldsymbol{P}\{M(\rho \wedge \tau) = k\}
\end{aligned}$$

which finally takes the form of Proposition 5.1 by using the notations adopted in the Appendix A for the Poisson distributions and the Heaviside symbols

C Proof details for Proposition 5.3

C.1 The case $a\mu s > \lambda t$, namely $z < 0$

We begin with the case $a\mu s > \lambda t$, namely $z < 0$, by recalling also that for the integration variable in (13) it is $0 \leq w \leq y$. As a consequence, when the Proposition 5.1 in used in (13), we always have

$$0 \leq \tau = \frac{y-w}{\mu} \leq \frac{y-z}{\mu} = s = \rho \tag{29}$$

On the other hand, since the conditions of the Lemma 5.2 are met, we can also restrict ourselves to evaluate $p_{m,n}(s,t)$ for $0 \leq n \leq m$

C.1.1 $m = n \geq 0$

In this case from (13), (29) and from the Proposition 5.1 we find

$$\begin{aligned}
q_{n,n}(y,z) &= a\int_0^y dw\, h_n(aw)\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right) \geq n\right\} \\
q_{n+1,n}(y,z) &= a\int_0^y dw\, h_n(aw)\left[\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right) \geq n+1\right\}\right. \\
&\qquad\left. + \boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right) \geq 1\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right) = n\right\}\right] \\
q_{n,n+1}(y,z) &= a\int_0^y dw\, h_{n+1}(aw)\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right) \geq n+1\right\} \\
q_{n+1,n+1}(y,z) &= a\int_0^y dw\, h_{n+1}(aw)\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right) \geq n+1\right\}
\end{aligned}$$

and then from (12)

$$\begin{aligned} p_{n,n}(y,z) &= a\int_0^y dw\, h_n(aw)\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right)=0\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)=n\right\} \\ &= a\int_0^y dw\, h_n(aw)\,\pi_0(w-z)\,\pi_n(y-w) \end{aligned} \tag{30}$$

C.1.2 $m > n \geq 0$

In order to make use of Proposition 5.1 we remark that now

$$m \geq n+1 \qquad m+1 > n \qquad m+1 > n+1$$

so that from (13) and (29)

$$\begin{aligned}
q_{m,n}(y,z) = a\int_0^y dw\, h_n(aw)\Bigg[&\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right) \geq m\right\}\\
&+\sum_{k=n}^{m-1}\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right) \geq m-k\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right) = k\right\}\Bigg]\\
q_{m+1,n}(y,z) = a\int_0^y dw\, h_n(aw)\Bigg[&\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right) \geq m+1\right\}\\
&+\sum_{k=n}^{m}\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right) \geq m+1-k\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right) = k\right\}\Bigg]\\
q_{m,n+1}(y,z) = a\int_0^y dw\, h_{n+1}(aw)\Bigg[&\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right) \geq m\right\}\\
&+\Theta_{m-n-1}\sum_{k=n+1}^{m-1}\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right) \geq m-k\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right) = k\right\}\Bigg]\\
q_{m+1,n+1}(y,z) = a\int_0^y dw\, h_{n+1}(aw)\Bigg[&\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right) \geq m+1\right\}\\
&+\sum_{k=n+1}^{m}\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right) \geq m+1-k\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right) = k\right\}\Bigg]
\end{aligned}$$

and hence from (12) (in a slightly simplified notation)

$$\begin{aligned}
p_{m,n} \;=\; & a\int_0^y dw\, h_n\Bigg[\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)=m\right\} \\
&\qquad -\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right)\geq 1\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)=m\right\} \\
&\qquad +\sum_{k=n}^{m-1}\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right)=m-k\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)=k\right\}\Bigg] \\
& -a\int_0^y dw\, h_{n+1}\Bigg[\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)=m\right\} \\
&\qquad -\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right)\geq 1\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)=m\right\} \\
&\qquad +\Theta_{m-n-1}\sum_{k=n+1}^{m-1}\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right)=m-k\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)=k\right\}\Bigg] \\
=\; & a\int_0^y dw\Bigg[(h_n-h_{n+1})\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right)=0\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)=m\right\} \\
&\qquad +h_n\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right)=m-n\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)=n\right\} \\
&\qquad +(h_n-h_{n+1})\Theta_{m-n-1}\sum_{k=n+1}^{m-1}\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right)=m-k\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)=k\right\}\Bigg] \\
=\; & a\int_0^y dw\Bigg[h_{n+1}\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right)=m-n\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)=n\right\} \\
&\qquad +(h_n-h_{n+1})\sum_{k=n}^{m}\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right)=m-k\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)=k\right\}\Bigg]
\end{aligned}$$

so that we finally have in full notation

$$\begin{aligned}
p_{m,n}(y,z) \;=\; & a\int_0^y dw\Bigg\{h_{n+1}(aw)\pi_{m-n}(w-z)\pi_n(y-w) \\
&\qquad +\big[h_n(aw)-h_{n+1}(aw)\big]\sum_{k=n}^{m}\pi_{m-k}(w-z)\pi_k(y-w)\Bigg\}
\end{aligned} \tag{31}$$

Remark that this formula correctly encompasses also the case $m=n\geq 0$. As stated in the Proposition 5.3, we can also concisely write the overall result in the form

$$p_{m,n}(y,z)=\begin{cases} Q_{n,n}(y,z) & m=n\geq 0 \\ Q_{m,n}(y,z)-Q_{m,n+1}(y,z) & m>n\geq 0\end{cases} \tag{32}$$

where for $m \geq n \geq 0$ and $z < 0$ we define

$$Q_{m,n}(y,z) = a\int_0^y dw\, h_n(aw)\sum_{k=n}^{m}\pi_{m-k}(w-z)\pi_k(y-w)$$

C.1.3 A normalization check

We provide here a quick check of the correct normalization of the joint probabilities $p_{m,n}$ calculated in the previous section for $a\mu s > \lambda t$, namely $z < 0 < y$. First remark that with $x = aw$

$$\begin{aligned}
Q_{m,0}(y,z) &= a\int_0^y dw\, h_0(aw)\sum_{k=0}^{m}\pi_{m-k}(w-z)\pi_k(y-w)\\
&= \int_0^{ay} dx\,\delta(x)\sum_{k=0}^{m}\pi_{m-k}\left(\frac{x}{a}-z\right)\pi_k\left(y-\frac{x}{a}\right) = \sum_{k=0}^{m}\pi_{m-k}(-z)\pi_k(y)\\
&= e^{-(y-z)}\sum_{k=0}^{m}\frac{y^k(-z)^{m-k}}{k!(m-k)!} = \frac{e^{-(y-z)}}{m!}\sum_{k=0}^{m}\binom{m}{k}y^k(-z)^{m-k}\\
&= \frac{e^{-(y-z)}}{m!}(y-z)^m = \pi_m(y-z)
\end{aligned}$$

Then, since here only the $p_{m,n}$ with $m \geq n \geq 0$ do not vanish, we have (by neglecting the arguments y, z)

$$\begin{aligned}
\sum_{n,m} p_{m,n} &= \sum_{m=0}^{\infty}\sum_{n=0}^{m} p_{m,n} = Q_{0,0} + \sum_{m=1}^{\infty}\left[Q_{m,m} + \sum_{n=0}^{m-1}(Q_{m,n} - Q_{m,n+1})\right]\\
&= Q_{0,0} + \sum_{m=1}^{\infty}\left(\sum_{n=0}^{m} Q_{m,n} - \sum_{n=0}^{m-1} Q_{m,n+1}\right)\\
&= Q_{0,0} + \sum_{m=1}^{\infty}\left(\sum_{n=0}^{m} Q_{m,n} - \sum_{n=1}^{m} Q_{m,n}\right) = Q_{0,0} + \sum_{m=1}^{\infty} Q_{m,0}\\
&= \sum_{m=0}^{\infty} Q_{m,0} = \sum_{m=0}^{\infty}\pi_m = 1
\end{aligned}$$

which confirms the normalization

C.2 The case $a\mu s < \lambda t$, namely $y > z > 0$

In this case, since $0 < z < y$ and $0 < w < y$, when we use Proposition 5.1 in (13) instead of (29) we find only

$$0 \leq \tau = \frac{y-w}{\mu} \qquad 0 \leq \rho = s = \frac{y-z}{\mu} \tag{33}$$

so that ρ and τ can now happen to be in an order whatsoever. As a consequence Lemma 5.2 does not hold, and we must consider all the possible choices of m, n

C.2.1 $n > m \geq 0$

In order to make use of Proposition 5.1 we remark that now

$$m < n+1 \qquad m+1 \leq n \qquad m+1 < n+1$$

so that from (13) and (33)

$$\begin{aligned}
q_{m,n} \;&=\; a\int_0^y dw\, h_n(aw)\boldsymbol{P}\left\{M\left(\frac{y-z}{\mu}\wedge\frac{y-w}{\mu}\right)\geq n\right\}\\
&\quad +a\int_0^z dw\, h_n(aw)\\
&\qquad \sum_{k=m}^{n-1}\boldsymbol{P}\left\{M\left(\frac{z-w}{\mu}\right)\geq n-k\right\}\boldsymbol{P}\left\{M\left(\frac{y-z}{\mu}\right)=k\right\}\\
q_{m+1,n} \;&=\; a\int_0^y dw\, h_n(aw)\boldsymbol{P}\left\{M\left(\frac{y-z}{\mu}\wedge\frac{y-w}{\mu}\right)\geq n\right\}\\
&\quad +a\,\Theta_{n-m-1}\int_0^z dw\, h_n(aw)\\
&\qquad \sum_{k=m+1}^{n-1}\boldsymbol{P}\left\{M\left(\frac{z-w}{\mu}\right)\geq n-k\right\}\boldsymbol{P}\left\{M\left(\frac{y-z}{\mu}\right)=k\right\}\\
q_{m,n+1} \;&= a\int_0^y dw\, h_{n+1}(aw)\boldsymbol{P}\left\{M\left(\frac{y-z}{\mu}\wedge\frac{y-w}{\mu}\right)\geq n+1\right\}\\
&\quad +a\int_0^z dw\, h_{n+1}(aw)\\
&\qquad \sum_{k=m}^{n}\boldsymbol{P}\left\{M\left(\frac{z-w}{\mu}\right)\geq n+1-k\right\}\boldsymbol{P}\left\{M\left(\frac{y-z}{\mu}\right)=k\right\}\\
q_{m+1,n+1} \;&=\; a\int_0^y dw\, h_{n+1}(aw)\boldsymbol{P}\left\{M\left(\frac{y-z}{\mu}\wedge\frac{y-w}{\mu}\right)\geq n+1\right\}\\
&\quad +a\int_0^z dw\, h_{n+1}(aw)\\
&\qquad \sum_{k=m+1}^{n}\boldsymbol{P}\left\{M\left(\frac{z-w}{\mu}\right)\geq n+1-k\right\}\boldsymbol{P}\left\{M\left(\frac{y-z}{\mu}\right)=k\right\}
\end{aligned}$$

and hence from (12)

$$\begin{aligned}
p_{m,n} &= a\int_0^z dw\, h_n(aw)\boldsymbol{P}\left\{M\left(\frac{z-w}{\mu}\right)\geq n-m\right\}\boldsymbol{P}\left\{M\left(\frac{y-z}{\mu}\right)=m\right\}\\
&\quad -a\int_0^z dw\, h_{n+1}(aw)\boldsymbol{P}\left\{M\left(\frac{z-w}{\mu}\right)\geq n+1-m\right\}\boldsymbol{P}\left\{M\left(\frac{y-z}{\mu}\right)=m\right\}\\
&= \boldsymbol{P}\left\{M\left(\frac{y-z}{\mu}\right)=m\right\} a\int_0^z dw\left[h_n(aw)\left(1-\boldsymbol{P}\left\{M\left(\frac{z-w}{\mu}\right)<n-m\right\}\right)\right.\\
&\qquad\left. -h_{n+1}(aw)\left(1-\boldsymbol{P}\left\{M\left(\frac{z-w}{\mu}\right)\leq n-m\right\}\right)\right]\\
&= \boldsymbol{P}\left\{M\left(\frac{y-z}{\mu}\right)=m\right\} a\int_0^z dw\left\{h_n(aw)\boldsymbol{P}\left\{M\left(\frac{z-w}{\mu}\right)=n-m\right\}\right.\\
&\qquad\left. +[h_n(aw)-h_{n+1}(aw)]\left(1-\boldsymbol{P}\left\{M\left(\frac{z-w}{\mu}\right)\leq n-m\right\}\right)\right\}
\end{aligned}$$

which, by plugging in the explicit Poisson probabilities, can be written as

$$\begin{aligned}
p_{m,n}(y,z) &= \pi_m(y-z)\, a\int_0^z dw\,[h_n(aw)-h_{n+1}(aw)]\\
&\quad +\pi_m(y-z)\, a\int_0^z dw\Big\{h_n(aw)\pi_{n-m}(z-w)\\
&\qquad -[h_n(aw)-h_{n+1}(aw)]\sum_{k=0}^{n-m}\pi_k(z-w)\Big\} \qquad (34)\\
&= A_{m,n}(y,z)-A_{m,n+1}(y,z)+B_{m,n}(y,z)-B_{m,n-1}(y,z)
\end{aligned}$$

where, with the notations (17) and (18) adopted in the Proposition 5.3, we have defined

$$\begin{aligned}
A_{m,n}(y,z) &= \pi_m(y-z)\, a\int_0^z dw\, h_n(aw)\\
B_{m,n}(y,z) &= \pi_m(y-z)\, a\int_0^z dw\, h_{n+1}(aw)\sum_{k=0}^{n-m}\pi_k(z-w)
\end{aligned}$$

C.2.2 $\boldsymbol{n = m \geq 0}$

In this case from (13), (33) and from the Proposition 5.1 we find

$$\begin{aligned}
q_{n,n} &= a\int_0^y dw\, h_n(aw)\boldsymbol{P}\left\{M\left(\frac{y-z}{\mu}\wedge\frac{y-w}{\mu}\right)\geq n\right\}\\
q_{n+1,n} &= a\int_0^y dw\, h_n(aw)\boldsymbol{P}\left\{M\left(\frac{y-z}{\mu}\wedge\frac{y-w}{\mu}\right)\geq n+1\right\}\\
&\qquad +a\int_z^y dw\, h_n(aw)\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right)\geq 1\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)= n\right\}\\
q_{n,n+1} &= a\int_0^y dw\, h_{n+1}(aw)\boldsymbol{P}\left\{M\left(\frac{y-z}{\mu}\wedge\frac{y-w}{\mu}\right)\geq n+1\right\}\\
&\qquad +a\int_0^z dw\, h_{n+1}(aw)\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right)\geq 1\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)= n\right\}\\
q_{n+1,n+1} &= a\int_0^y dw\, h_{n+1}(aw)\boldsymbol{P}\left\{M\left(\frac{y-z}{\mu}\wedge\frac{y-w}{\mu}\right)\geq n+1\right\}
\end{aligned}$$

and then from (12)

$$\begin{aligned}
p_{m,n} &= a\int_0^y dw\, h_n(aw)\boldsymbol{P}\left\{M\left(\frac{y-z}{\mu}\wedge\frac{y-w}{\mu}\right)= n\right\}\\
&\quad -a\int_z^y dw\, h_n(aw)\left(1-\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right)=0\right\}\right)\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)= n\right\}\\
&\quad -a\int_0^z dw\, h_{n+1}(aw)\left(1-\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right)=0\right\}\right)\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)= n\right\}\\
&= a\int_0^z dw\left[h_n(aw)-h_{n+1}(aw)\right]\boldsymbol{P}\left\{M\left(\frac{y-z}{\mu}\right)= n\right\}\\
&\qquad -a\int_z^y dw\, h_n(aw)\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right)=0\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)= n\right\}\\
&\qquad -a\int_0^z dw\, h_{n+1}(aw)\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right)=0\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)= n\right\}
\end{aligned}$$

which with the Poisson probabilities become

$$\begin{aligned}
p_{n,n}(y,z) &= \pi_n(y-z)\,a\int_0^z dw\left[h_n(aw)-h_{n+1}(aw)\right]\\
&\qquad +\pi_n(y-z)\,a\int_0^z dw\, h_{n+1}(aw)\pi_0(z-w)\\
&\qquad\qquad +a\int_z^y dw\, h_n(aw)\pi_0(w-z)\pi_n(y-w)\\
&= A_{n,n}(y,z)-A_{n,n+1}(y,z)+B_{n,n}(y,z)+C_{n,n}(y,z)
\end{aligned} \tag{35}$$

where, as in the Proposition 5.3, along with (17) and (18) we also introduced (19)

$$C_{m,n}(y,z) = a\int_z^y dw\, h_n(aw)\sum_{k=n}^{m}\pi_{m-k}(w-z)\pi_k(y-w)$$

C.2.3 $m > n \geq 0$

In this case it is

$$m \geq n+1 \qquad m+1 > n \qquad m+1 > n+1$$

so that from (13) and (33)

$$\begin{aligned}
q_{m,n} &= a\int_0^y dw\, h_n(aw)\boldsymbol{P}\left\{M\left(\frac{y-z}{\mu}\wedge\frac{y-w}{\mu}\right)\geq m\right\}\\
&\quad + a\int_z^y dw\, h_n(aw)\\
&\qquad \sum_{k=n}^{m-1}\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right)\geq m-k\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)=k\right\}\\
q_{m+1,n} &= a\int_0^y dw\, h_n(aw)\boldsymbol{P}\left\{M\left(\frac{y-z}{\mu}\wedge\frac{y-w}{\mu}\right)\geq m+1\right\}\\
&\quad + a\int_z^y dw\, h_n(aw)\\
&\qquad \sum_{k=n}^{m}\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right)\geq m+1-k\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)=k\right\}\\
q_{m,n+1} &= a\int_0^y dw\, h_{n+1}(aw)\boldsymbol{P}\left\{M\left(\frac{y-z}{\mu}\wedge\frac{y-w}{\mu}\right)\geq m\right\}\\
&\quad + \Theta_{m-n-1}a\int_z^y dw\, h_{n+1}(aw)\\
&\qquad \sum_{k=n+1}^{m-1}\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right)\geq m-k\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)=k\right\}\\
q_{m+1,n+1} &= a\int_0^y dw\, h_{n+1}(aw)\boldsymbol{P}\left\{M\left(\frac{y-z}{\mu}\wedge\frac{y-w}{\mu}\right)\geq m+1\right\}\\
&\quad + a\int_z^y dw\, h_{n+1}(aw)\\
&\qquad \sum_{k=n+1}^{m}\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right)\geq m+1-k\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)=k\right\}
\end{aligned}$$

and hence from (12)

$$
\begin{aligned}
p_{m,n} \;=\;& a\int_0^y dw\,[h_n(aw)-h_{n+1}(aw)]\,\boldsymbol{P}\left\{M\left(\frac{y-z}{\mu}\wedge\frac{y-w}{\mu}\right)=m\right\}\\
&+a\int_z^y dw\,h_n(aw)\left[\sum_{k=n}^{m-1}\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right)=m-k\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)=k\right\}\right.\\
&\qquad\left.-\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right)\geq 1\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)=m\right\}\right]\\
&-a\int_z^y dw\,h_{n+1}(aw)\\
&\qquad\left[\Theta_{m-n-1}\sum_{k=n+1}^{m-1}\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right)=m-k\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)=k\right\}\right.\\
&\qquad\left.-\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right)\geq 1\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)=m\right\}\right]
\end{aligned}
$$

On the other hand, since for every $t>0$

$$
\boldsymbol{P}\{M(t)\geq 1\}=1-\boldsymbol{P}\{M(t)=0\}
$$

we can also write

$$
\begin{aligned}
p_{m,n} \;=\;& a\int_0^y dw\,[h_n(aw)-h_{n+1}(aw)]\,\boldsymbol{P}\left\{M\left(\frac{y-z}{\mu}\wedge\frac{y-w}{\mu}\right)=m\right\}\\
&-a\int_z^y dw\,[h_n(aw)-h_{n+1}(aw)]\,\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)=m\right\}\\
&+a\int_z^y dw\,h_n(aw)\sum_{k=n}^{m}\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right)=m-k\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)=k\right\}\\
&-a\int_z^y dw\,h_{n+1}(aw)\sum_{k=n+1}^{m}\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right)=m-k\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)=k\right\}\\
=\;& a\int_0^z dw\,[h_n(aw)-h_{n+1}(aw)]\,\boldsymbol{P}\left\{M\left(\frac{y-z}{\mu}\right)=m\right\}\\
&+a\int_z^y dw\left\{h_{n+1}(aw)\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right)=m-n\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)=n\right\}\right.\\
&\qquad+[h_n(aw)-h_{n+1}(aw)]\\
&\qquad\left.\sum_{k=n}^{m}\boldsymbol{P}\left\{M\left(\frac{w-z}{\mu}\right)=m-k\right\}\boldsymbol{P}\left\{M\left(\frac{y-w}{\mu}\right)=k\right\}\right\}
\end{aligned}
$$

which when the Poisson probabilities are introduced goes into

$$
\begin{aligned}
p_{m,n}(y,z) &= \pi_m(y-z)a\int_0^z dw\,[h_n(aw)-h_{n+1}(aw)] \\
&\qquad +a\int_z^y dw\,h_{n+1}(aw)\pi_{m-n}(w-z)\pi_n(y-w) \qquad (36)\\
&\qquad\qquad +a\int_z^y dw\,[h_n(aw)-h_{n+1}(aw)]\sum_{k=n}^m \pi_{m-k}(w-z)\pi_k(y-w) \\
&= A_{m,n}(y,z)-A_{m,n+1}(y,z)+C_{m,n}(y,z)-C_{m,n+1}(y,z)
\end{aligned}
$$

namely the result of Proposition 5.3 once the definitions (17) and (19) are taken into account

C.3 Continuity in $z = 0$

We will now compare the values of $p_{m,n}(y,z)$ separately listed in the Proposition 5.3 for $z < 0$ and $z > 0$, and we will show that they connect with continuity in $z = 0$, in the sense that for every $y > 0$

$$p_{m,n}(y,0^-) = p_{m,n}(y,0^+)$$

For $z < 0$ (namely $a\mu s > \lambda t$) the results are given in the Lemma 5.2 and in formulas (30) and (31), so that for $z \uparrow 0^-$ we simply have

$$
\begin{aligned}
p_{m,n}(y,0^-) &= 0 && n > m \geq 0 \qquad (37)\\
p_{n,n}(y,0^-) &= a\int_0^y dw\,h_n(aw)\,\pi_0(w)\,\pi_n(y-w) && n = m \geq 0 \qquad (38)\\
p_{m,n}(y,0^-) &= a\int_0^y dw\Big\{h_{n+1}(aw)\pi_{m-n}(w)\pi_n(y-w) && m > n \geq 0 \qquad (39)\\
&\qquad +\big[h_n(aw)-h_{n+1}(aw)\big]\sum_{k=n}^m \pi_{m-k}(w)\pi_k(y-w)\Big\}
\end{aligned}
$$

On the other hand, when $z > 0$ (namely $a\mu s < \lambda t$) the results are given in formulas (34), (35) and (36), where z appears also as an integration limit so that it is advisable to premise a few formal remarks. If the integrand is a regular function $\xi(x)$ we of course have with $y > z > 0$

$$\lim_{z\downarrow 0^+}\int_0^z \xi(x)\,dx = \int_0^{0^+}\xi(x)\,dx = 0 \qquad\qquad \lim_{z\downarrow 0^+}\int_z^y \xi(x)\,dx = \int_0^y \xi(x)\,dx$$

When instead the integrand contains the distribution $\delta(x) = \delta(x-0^+)$ concentrated on $x = 0^+$, as indeed happens in every first term of $h_n(x)$ which is $\beta_0(n)\delta(x) = a^n\delta(x)$

(see also Appendix A), we have

$$\lim_{z\downarrow 0^+}\int_0^z \xi(x)\delta(x)\,dx = \int_0^{0^+}\xi(0)\delta(x)\,dx = \xi(0) \qquad\qquad \lim_{z\downarrow 0^+}\int_z^y \xi(x)\delta(x)\,dx = 0$$

As a consequence in the limit $z\downarrow 0^+$, by neglecting the vanishing terms, the following integrals on $[0,z]$ only retain the contribution of the term $k=0$ in h_n

$$\begin{aligned}\lim_{z\downarrow 0^+}\int_0^z dx\,\xi(x)h_n(x) &= \int_0^{0^+} dx\,\xi(x)\sum_{k=0}^n \beta_k(n)f_k(x) = \int_0^{0^+} dx\,\xi(x)\beta_0(n)f_0(x)\\ &= a^n\int_0^{0^+} dx\,\xi(0)\delta(x) = a^n\xi(0)\end{aligned}$$

while on the other hand for the integrals on $[z,y]$ we have

$$\lim_{z\downarrow 0^+}\int_z^y dx\,\xi(x)h_n(x) = \int_{0^+}^y dx\,\xi(x)\sum_{k=0}^n \beta_k(n)f_k(x) = \int_0^y dx\,\xi(x)\sum_{k=1}^n \beta_k(n)f_k(x)$$

where now the term $k=0$ is apparently missing

Take first $p_{m,n}(y,z)$ for $z>0$ and $n>m\geq 0$ as given in (34): from the previous remarks, with $x=aw$ and recalling also that $\pi_k(0)=\delta_{k,0}$ (so that $\pi_{n-m}(0)=0$ because $n>m$), in the limit $z\downarrow 0^+$ we immediately have

$$\begin{aligned}p_{m,n}(y,0^+) &= \pi_m(y)\int_0^{0^+} dx\,\left[a^n-a^{n+1}+a^n\pi_{n-m}(0)-(a^n-a^{n+1})\sum_{k=0}^{n-m}\pi_k(0)\right]\delta(x)\\ &= \pi_m(y)\left[a^n-a^{n+1}-(a^n-a^{n+1})\sum_{k=0}^{n-m}\delta_{k,0}\right]=0\end{aligned}$$

which coincides with $p_{m,n}(y,0^-)$ in (37). Then, always with $z>0$, consider the case $n=m\geq 0$ given in (35): now (since $\pi_0(0)=1$) in the limit $z\downarrow 0^+$ we have that the first integral exactly compensates the term $k=0$ missing in the sum of the second integral, so that

$$\begin{aligned}p_{n,n}(y,0^+) &= \pi_n(y)\int_0^{0^+} dx\,\left[a^n-a^{n+1}+a^{n+1}\pi_0(0)\right]\delta(x)\\ &\qquad\qquad +a\int_0^y dw\,\pi_0(w)\pi_n(y-w)\sum_{k=1}^n\beta_k(n)f_k(aw)\\ &= a^n\pi_n(y)+a\int_0^y dw\,\pi_0(w)\pi_n(y-w)\sum_{k=1}^n\beta_k(n)f_k(aw)\\ &= a\int_0^y dw\,\pi_0(w)\pi_n(y-w)\sum_{k=0}^n\beta_k(n)f_k(aw)\\ &= a\int_0^y dw\,\pi_0(w)\pi_n(y-w)h_n(aw)\end{aligned}$$

which again coincides with $p_{n,n}(y,0^-)$ in (38). Finally consider the case $z>0$ and $m>n\geq 0$ which is given in (36): here with the same remarks as before in the limit $z\downarrow 0^+$ we again have $p_{m,n}(y,0^-)$ of (39)

$$\begin{aligned}
p_{m,n}(y,0^+) &= \pi_m(y)\int_0^{0^+}dx\,(a^n-a^{n+1})\,\delta(x)\\
&\qquad +a\int_{0^+}^{y}dw\Bigg\{h_{n+1}(aw)\pi_{m-n}(w)\pi_n(y-w)\\
&\qquad\qquad +[h_n(aw)-h_{n+1}(aw)]\sum_{k=n}^{m}\pi_{m-k}(w)\pi_k(y-w)\Bigg\}\\
&= a^n(1-a)\pi_m(y)+a\int_{0^+}^{y}dw\Bigg\{h_{n+1}(aw)\pi_{m-n}(w)\pi_n(y-w)\\
&\qquad\qquad +[h_n(aw)-h_{n+1}(aw)]\sum_{k=n}^{m}\pi_{m-k}(w)\pi_k(y-w)\Bigg\}\\
&= a\int_0^{y}dw\Bigg\{h_{n+1}(aw)\pi_{m-n}(w)\pi_n(y-w)\\
&\qquad\qquad +[h_n(aw)-h_{n+1}(aw)]\sum_{k=n}^{m}\pi_{m-k}(w)\pi_k(y-w)\Bigg\}
\end{aligned}$$

because it is easy to see that the first term exactly compensates the missing term in the second integral:

$$\begin{aligned}
&\int_0^{ay}dx\left[a^{n+1}\pi_{m-n}(0)\pi_n(y)+(a^n-a^{n+1})\sum_{k=n}^{m}\pi_{m-k}(0)\pi_k(y)\right]\delta(x)\\
&\qquad\qquad =\int_0^{ay}dx\,a^n(1-a)\sum_{k=n}^{m}\delta_{m-k,0}\pi_k(y)=a^n(1-a)\pi_m(y)
\end{aligned}$$

D A proof of Proposition 5.4

To find an explicit elementary formula for $Q_{m,n}(y,z)$ (here $z<0$, namely $a\mu s>\lambda t$) let us begin by supposing $m>n\geq 1$: in this case – within our usual notations –

from (15) we have

$$\begin{aligned}
Q_{m,n}(y,z) &= \frac{e^{-(y-z)}}{m!}a\int_0^y dw\left[\beta_0(n)f_0(aw)+\sum_{\ell=1}^n\beta_\ell(n)f_\ell(aw)\right]\\
&\qquad\cdot\sum_{k=n}^m\binom{m}{k}(w-z)^{m-k}(y-w)^k\\
&= \frac{e^{-(y-z)}}{m!\,a^m}\int_0^{ay}dx\left[a^n\delta(x)+\sum_{\ell=1}^n\beta_\ell(n)f_\ell(x)\right]\\
&\qquad\cdot\sum_{k=n}^m\binom{m}{k}(x-az)^{m-k}(ay-x)^k\\
&= \frac{e^{-(y-z)}}{m!\,a^m}\sum_{k=n}^m\binom{m}{k}\Bigg[a^n\,(-az)^{m-k}\,(ay)^k\\
&\qquad+\sum_{\ell=1}^n\frac{\beta_\ell(n)}{(\ell-1)!}\int_0^{ay}dx\,e^{-x}x^{\ell-1}(x-az)^{m-k}(ay-x)^k\Bigg]
\end{aligned}$$

On the other hand (see [17] 3.383.1) with $v=ay-x$ it is

$$\begin{aligned}
\int_0^{ay}dx\,e^{-x}x^{\ell-1}(x-az)^{m-k}(ay-x)^k &= e^{-ay}\int_0^{ay}dv\,e^{v}(ay-v)^{\ell-1}[a(y-z)-v]^{m-k}v^k\\
&= e^{-ay}\int_0^{ay}dv\,e^{v}(ay-v)^{\ell-1}v^k\sum_{i=0}^{m-k}\binom{m-k}{i}[a(y-z)]^i(-v)^{m-k-i}\\
&= e^{-ay}\sum_{i=0}^{m-k}\binom{m-k}{i}(-1)^{m-k-i}[a(y-z)]^i\int_0^{ay}dv\,e^{v}(ay-v)^{\ell-1}v^{m-i}\\
&= e^{-ay}\sum_{i=0}^{m-k}\binom{m-k}{i}(-1)^{m-k-i}[a(y-z)]^i(ay)^{\ell+m-i}\\
&\qquad\mathrm{B}(\ell,m-i+1)\,\Phi(m-i+1;m-i+\ell+1;ay)
\end{aligned}$$

where $\Phi(\alpha;\beta;x)$ is a confluent hypergeometric function (see [17] 9.2) and $\mathrm{B}(\alpha,\beta)$ is the beta function (see [17] 8.38) with

$$\mathrm{B}(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}=\frac{(\alpha-1)!(\beta-1)!}{(\alpha+\beta-1)!}$$

As a consequence we have

$$
\begin{aligned}
Q_{m,n}(y,z) &= \frac{e^{-(y-z)}}{m!\,a^m}\sum_{k=n}^{m}\binom{m}{k}\Bigg[a^n\,(-az)^{m-k}\,(ay)^k \\
&\qquad + \sum_{\ell=1}^{n}\beta_\ell(n)e^{-ay}\sum_{i=0}^{m-k}\binom{m-k}{i}(-1)^{m-k-i}[a(y-z)]^i(ay)^{\ell+m-i} \\
&\qquad\qquad \frac{(m-i)!}{(m-i+\ell)!}\,\Phi(m-i+1;m-i+\ell+1;ay)\Bigg] \\
&= \frac{e^{-(y-z)}}{m!\,a^m}\sum_{k=n}^{m}\binom{m}{k}\sum_{\ell=0}^{n}\beta_\ell(n)e^{-ay}\sum_{i=0}^{m-k}\binom{m-k}{i}[a(y-z)]^i(ay)^{\ell+m-i} \\
&\qquad (-1)^{m-k-i}\frac{(m-i)!}{(m-i+\ell)!}\,\Phi(m-i+1;m-i+\ell+1;ay)\Bigg] \\
&= \sum_{k=n}^{m}\sum_{i=0}^{m-k}\frac{(-1)^{m-k-i}}{a^{m-i}}\binom{m-i}{k}\sum_{\ell=0}^{n}\beta_\ell(n) \\
&\qquad \pi_i(y-z)\pi_{m-i+\ell}(ay)\Phi(m-i+1;m-i+\ell+1;ay)
\end{aligned}
$$

and with the change of the summation index $j=m-i$

$$
\begin{aligned}
Q_{m,n}(y,z) &= \sum_{k=n}^{m}\sum_{j=k}^{m}\frac{(-1)^{j-k}}{a^j}\binom{j}{k}\sum_{\ell=0}^{n}\beta_\ell(n) \\
&\qquad \pi_{m-j}(y-z)\pi_{j+\ell}(ay)\Phi(j+1;j+\ell+1;ay)
\end{aligned}
$$

as stated in the Proposition 5.4. We have proved this result by supposing $m>n\geq 1$, but it is possible to check now by direct calculation that also its extension to $m\geq n\geq 0$ gives the right results for $Q_{m,0}$ with $m\geq 0$, and for $Q_{n,n}$ with $n\geq 1$. We have in fact from the definition (15) that for $m\geq n=0$ it is

$$
\begin{aligned}
Q_{0,0}(y,z) &= e^{-(y-z)}a\int_0^y dw\,\delta(aw) = e^{-(y-z)} = \pi_0(y-z) \\
Q_{m,0}(y,z) &= \frac{e^{-(y-z)}}{m!}\,a\int_0^y dw\,h_0(aw)\sum_{k=0}^{m}\binom{m}{k}(w-z)^{m-k}(y-w)^k \\
&= \frac{e^{-(y-z)}(y-z)^m}{m!}\,a\int_0^y dw\,\delta(aw) = \pi_m(y-z)
\end{aligned}
$$

while for $m=n\geq 1$ we have with $v=a(y-w)$

$$
\begin{aligned}
Q_{n,n}(y,z) &= \frac{e^{-(y-z)}}{n!}\,a\int_0^y dw\left[a^n\delta(aw)+\sum_{k=1}^{n}\beta_k(n)f_k(aw)\right](y-w)^n \\
&= \frac{e^{-(y-z)}}{n!}\left[(ay)^n+\frac{e^{-ay}}{a^n}\sum_{k=1}^{n}\frac{\beta_k(n)}{(k-1)!}\int_0^{ay}e^v v^n(ay-v)^{k-1}dv\right]
\end{aligned}
$$

and since (see [17] 3.383.1)

$$\int_0^{ay} e^v v^n (ay-v)^{k-1} dv = (ay)^{k+n} \mathrm{B}(k, n+1)\Phi(n+1; n+k+1; ay)$$

we finally get

$$\begin{aligned} Q_{n,n}(y,z) &= \frac{e^{-(y-z)}}{n!}(ay)^n \left[1 + \frac{e^{-ay}}{a^n}\sum_{k=1}^n \beta_k(n)\frac{n!(ay)^k}{(n+k)!}\Phi(n+1; n+k+1; ay)\right] \\ &= e^{-(y-z)}\pi_n(ay)\sum_{k=0}^n \frac{\beta_k(n)}{a^n}\frac{n!}{(n+k)!}(ay)^k\Phi(n+1; n+k+1; ay) \end{aligned}$$

It would be easy to see now that all these results can also be derived as particular cases from our explicit expression (15) that can hence be considered as completely general

We proceed now to calculate for $z > 0$ (namely for $\lambda t > a\mu s$) the explicit elementary form of the terms A, B and C defined in (17), (18) and (19). With $x = aw$, and by taking into account [17] 8.350.1 and 8.352.1, from (17) we first have for $m \geq 0$ and $n \geq 1$

$$\begin{aligned} A_{m,n}(y,z) &= \pi_m(y-z)\, a\int_0^z dw \left[\beta_0(n)f_0(aw) + \sum_{k=1}^n \beta_k(n)f_k(aw)\right] \\ &= \pi_m(y-z)\left[a^n + \sum_{k=1}^n \frac{\beta_k(n)}{(k-1)!}\int_0^{az} x^{k-1}e^{-x}dx\right] \\ &= \pi_m(y-z)\left\{a^n + \sum_{k=1}^n \beta_k(n)\left[1 - \sum_{j=0}^{k-1}\pi_j(az)\right]\right\} \\ &= \pi_m(y-z)\sum_{k=0}^n \beta_k(n)\left[1 + \pi_k(az) - \sum_{j=0}^k \pi_j(az)\right] \end{aligned}$$

Since however this result can be extended also to $n = 0$ giving the correct result

$$A_{m,0}(y,z) = \pi_m(y-z)\int_0^{az}\delta(x)\,dx = \pi_m(y-z)$$

it can be definitely used for every possible value of $n, m \geq 0$ as recalled in (26) of the Proposition 5.4

Then, for $n \geq m \geq 0$ and $x = aw$, by using [17] 3.383.1 we have from (18)

$$
\begin{aligned}
B_{m,n}(y,z) &= \pi_m(y-z)\, a\int_0^z dw\, h_{n+1}(aw)\sum_{k=0}^{n-m}\pi_k(z-w)\\
&= \pi_m(y-z)\sum_{k=0}^{n-m}\int_0^{az} dx\, h_{n+1}(x)\pi_k\left(z-\frac{x}{a}\right)\\
&= \pi_m(y-z)\sum_{k=0}^{n-m}\sum_{\ell=0}^{n+1}\beta_\ell(n+1)\int_0^{az} dx\, f_\ell(x)\pi_k\left(z-\frac{x}{a}\right)\\
&= \pi_m(y-z)\sum_{k=0}^{n-m}\Bigg[a^{n+1}\pi_k\left(z\right)\\
&\qquad +\frac{e^{-z}}{a^k k!}\sum_{\ell=1}^{n+1}\frac{\beta_\ell(n+1)}{(\ell-1)!}\int_0^{az} dx\, x^{\ell-1}(az-x)^k e^{\frac{1-a}{a}x}\Bigg]\\
&= \pi_m(y-z)\sum_{k=0}^{n-m}\pi_k\left(z\right)\Bigg[a^{n+1}\\
&\qquad +\sum_{\ell=1}^{n+1}\beta_\ell(n+1)\frac{(az)^\ell k!}{(k+\ell)!}\,\Phi\left(\ell,k+\ell+1,(1-a)z\right)\Bigg]\\
&= \pi_m(y-z)\sum_{k=0}^{n-m}\pi_k\left(z\right)\sum_{\ell=0}^{n+1}\beta_\ell(n+1)\frac{(az)^\ell k!}{(k+\ell)!}\,\Phi\left(\ell,k+\ell+1,(1-a)z\right)
\end{aligned}
$$

because $\Phi(0,k+1,x) = 1$. This result also coincides with (27) in the Proposition 5.4

Finally, for $m \geq n \geq 0$ and $0 < z < y$, from (19) we first remark that $C_{m,0}(y,z) = 0$ because $h_0(aw) = \delta(aw)$ is now peaked outside the integration interval $[z, y]$. Then for $m \geq n \geq 1$ (the term of the sum with $\ell = 0$ again vanishes for the same reason as before) with $v = a(y-w)$, and $u = a(y-z)$ for short, from (19) we have

$$
\begin{aligned}
C_{m,n}(y,z) &= \int_0^u dv\, h_n(ay-v)\sum_{k=n}^{m}\pi_{m-k}\left(\frac{u-v}{a}\right)\pi_k\left(\frac{v}{a}\right)\\
&= \sum_{\ell=1}^{n}\beta_\ell(n)\int_0^u dv\, f_\ell(ay-v)\sum_{k=n}^{m}\pi_{m-k}\left(\frac{u-v}{a}\right)\pi_k\left(\frac{v}{a}\right)\\
&= \frac{e^{-(y-z)}}{a^m}\sum_{\ell=1}^{n}\beta_\ell(n)\int_0^u dv\, f_\ell(ay-v)\sum_{k=n}^{m}\frac{(u-v)^{m-k}v^k}{(m-k)!k!}\\
&= \frac{e^{-(y-z)}}{a^m m!}\sum_{\ell=1}^{n}\beta_\ell(n)\sum_{k=n}^{m}\binom{m}{k}\int_0^u dv\,\frac{(ay-v)^{\ell-1}e^{-ay+v}}{(\ell-1)!}(u-v)^{m-k}v^k\\
&= \frac{e^{-(y-z)}e^{-ay}}{a^m m!}\sum_{\ell=1}^{n}\frac{\beta_\ell(n)}{(\ell-1)!}\sum_{k=n}^{m}\binom{m}{k}\int_0^u dv\, e^v(ay-v)^{\ell-1}(u-v)^{m-k}v^k
\end{aligned}
$$

and since it is (see [17] 3.383.1)

$$\begin{aligned}
&\int_0^u dv\, e^v (ay-v)^{\ell-1}(u-v)^{m-k}v^k \\
&\quad = \sum_{j=0}^{\ell-1}\binom{\ell-1}{j}(ay)^j(-1)^{\ell-1-j}\int_0^u dv\, e^v v^{k+\ell-1-j}(u-v)^{m-k} \\
&\quad = \sum_{j=0}^{\ell-1}\binom{\ell-1}{j}(ay)^j(-1)^{\ell-1-j} \\
&\qquad\qquad B(m-k+1,k+\ell-j)u^{m+\ell-j}\Phi(k+\ell-j,m+\ell-j+1,u) \\
&\quad = \sum_{j=0}^{\ell-1}\frac{(\ell-1)!(m-k)!(k+\ell-1-j)!}{j!(\ell-1-j)!(m+\ell-j)!}(-1)^{\ell-1-j} \\
&\qquad\qquad\qquad (ay)^j u^{m+\ell-j}\Phi(k+\ell-j,m+\ell-j+1,u)
\end{aligned}$$

we can write

$$\begin{aligned}
C_{m,n}(y,z) &= \frac{e^{-(y-z)}e^{-ay}}{a^m}\sum_{\ell=1}^{n}\beta_\ell(n)\sum_{k=n}^{m}\sum_{j=0}^{\ell-1}\binom{k+\ell-j-1}{k} \\
&\qquad \frac{(-1)^{\ell-1-j}(ay)^j u^{m+\ell-j}}{j!(m+\ell-j)!}\,\Phi(k+\ell-j,m+\ell-j+1,u) \\
&= \frac{e^{-(1-a)(y-z)}}{a^m}\sum_{\ell=1}^{n}\beta_\ell(n)\sum_{k=n}^{m}\sum_{j=0}^{\ell-1}(-1)^{\ell-1-j}\binom{k+\ell-j-1}{k} \\
&\qquad \pi_j(ay)\pi_{m+\ell-j}(a(y-z))\Phi(k+\ell-j,m+\ell-j+1,a(y-z))
\end{aligned}$$

as stated in (28) of the Proposition 5.4

As for the confluent hypergeometric functions appearing in all these results they in fact boil down to finite combinations of elementary functions as stated in the final formulas of the Proposition 5.4. The case $n = 0$ immediately follows indeed from the definitions (see [17] 9.210.1). On the other hand, when $1 \le \alpha \le \beta$, with the change of variables $y = x - z$ and by taking sequentially into account [17] 3.383.1, 8.350.1 and 8.352.1 we have

$$
\begin{aligned}
\Phi(\alpha,\beta+1,x) &= \frac{1}{B(\beta-\alpha+1,\alpha)x^\beta}\int_0^x e^z z^{\alpha-1}(x-z)^{\beta-\alpha}dz \\
&= \frac{\beta!\,e^x}{(\beta-\alpha)!(\alpha-1)!\,x^\beta}\int_0^x e^{-y}(x-y)^{\alpha-1}y^{\beta-\alpha}dy \\
&= \frac{\beta!\,e^x}{(m-\alpha)!(\alpha-1)!\,x^\beta}\sum_{\gamma=0}^{\alpha-1}\binom{\alpha-1}{\gamma}x^\gamma(-1)^{\alpha-\gamma-1}\int_0^x y^{\beta-\gamma-1}e^{-y}dy \\
&= e^x\sum_{\gamma=0}^{\alpha-1}\frac{(-1)^{\alpha-\gamma-1}(\beta-\gamma-1)!}{(\beta-\alpha)!(\alpha-\gamma-1)!}\,\frac{\pi_\gamma(x)}{\pi_\beta(x)}\left(1-\sum_{\eta=0}^{\alpha-\gamma-1}\pi_\eta(x)\right) \\
&= e^x\sum_{\gamma=0}^{\alpha-1}(-1)^{\alpha-\gamma-1}\binom{\beta-\gamma-1}{\beta-\alpha}\frac{\pi_\gamma(x)}{\pi_\beta(x)}\left(1-\sum_{\eta=0}^{\alpha-\gamma-1}\pi_\eta(x)\right)
\end{aligned}
$$

as stated in the proposition

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