

Gompertz and logistic stochastic dynamics: Advances in an ongoing quest

Nicola CUFARO PETRONI*

Department of *Mathematics* and *TIRES*, University of Bari
INFN Sezione di Bari
via E. Orabona 4, 70125 Bari, Italy

Salvatore DE MARTINO†

Full prof. of General Physics (Ret), University of Salerno
personal address: via Luigi Settembrini 61, 80139 Napoli, Italy

Silvio DE SIENA‡

Full prof. of Theoretical Physics (Ret), University of Salerno
personal address: via Bastioni 15, 84122 Salerno, Italy

Abstract

In this report we summarize a few methods for solving the stochastic differential equations (*SDE*) and the corresponding Fokker-Planck equations describing the Gompertz and logistic random dynamics. It is shown that the solutions of the Gompertz *SDE* are completely known, while for the logistic *SDE*'s we can provide the solution as an explicit process, but we find much harder to write down its distribution in closed form. Many details of possible ways out of this maze are listed in the paper and its appendices. We also briefly discuss the prospects of performing a suitable averaging, or a deterministic limit. The possibility is also suggested of associating these equations to the stochastic mechanics of a quantum harmonic oscillator adopted as a tool serviceable also in the field of stochastic control: in particular we propose to investigate the equations associated to the quantum stationary states

Key words: Gompertz and logistic equations; Averaging and deterministic limit; Stochastic mechanics and control

*cufaro@ba.infn.it

†demartino@sa.infn.it

‡silvio.desiena@gmail.com

Contents

1	Introduction	3
2	Gompertz stochastic equations	6
2.1	Smoluchowsky <i>SDE</i> : stationary <i>pdf</i>	6
2.2	Linearized <i>SDE</i>	6
2.3	Parametric equations	8
3	Logistic stochastic equations	11
3.1	Smoluchowsky <i>SDE</i> : stationary <i>pdf</i>	11
3.2	Linearized <i>SDE</i>	12
3.2.1	Semi-explicit transition <i>pdf</i>	14
3.2.2	Fokker-Planck equation	17
3.3	θ -logistic <i>SDE</i>	22
3.4	Integrals of a geometric Wiener process	25
3.4.1	Moments	25
3.4.2	Characteristic function	27
3.4.3	Finite sums of a geometric Wiener process	28
3.4.4	Differential equations	31
4	Conclusions and outlooks	37
A	Deterministic Gompertz and logistic equations	40
A.1	Gompertz equation	40
A.2	Logistic and θ -logistic equations	40
B	Solving stochastic differential equations	41
B.1	An epitome of Itô calculus	41
B.2	<i>SDE</i> transformations	43
B.2.1	Elementary solvable <i>SDE</i> 's	44
B.2.2	Transformations to constant coefficients	48
B.2.3	Transformations to process-independent coefficients	50
B.2.4	Transformations to process-linear coefficients	52
C	Solving Fokker-Planck equations	53
C.1	Semi-explicit transition <i>pdf</i> 's	54
C.2	Random bridges	55
C.2.1	Non-random interpolation	55
C.2.2	Brownian bridges	58
C.3	Eigenfunction expansion	59
C.4	Invariant Boltzmann distributions	61

D An *anamnesis* of stochastic mechanics 64

D.1 The *FPE* for stationary states 66

D.2 The Itô *SDE* for stationary states 67

D.3 *QHO* stationary states 68

D.4 The processes for the *QHO* stationary states 69

 D.4.1 The ground state $n = 0$ 70

 D.4.2 The the first excited state $n = 1$ 71

 D.4.3 The second excited state $n = 2$ 74

D.5 The macroscopic limit 75

D.6 Looking deeper into the $n = 1$ case 77

E Quantiles and medians: a reminder 79

E.1 Definitions 79

E.2 Properties 81

E.3 Expectations and medians for symmetric laws 85

1 Introduction

In the present paper we will mainly deal with two classes of one-dimensional *SDE*'s (*stochastic differential equations*, see Appendix B) with a non linear drift that have been recently discussed in a number of papers (see for instance [1, 2, 3] and references quoted therein) and that are the noisy version of their deterministic counterparts (see Appendix A): the Gompertz and the logistic equations. The general form of a ***Gompertz SDE*** for the process $\xi(s)$ is

$$d\xi(s) = [a_1\xi(s) - a_2 \xi(s) \ln(b_1\xi(s))] ds + b_1\xi(s) d\beta(s) \tag{1}$$

where $\beta(s)$ is a Wiener process with diffusion coefficient 2δ , namely $\mathbf{E}[\beta^2(s)] = 2\delta s$. Remark that a possible difference between the two coefficients b_1 – that in the logarithm argument, and the other in front of the Wiener process – can be easily reabsorbed by redefining a_1 and a_2 . The physical dimensions (by supposing for instance that ξ and β are lengths L , while s is a time T) are then

$$[a_1] = \frac{1}{T} \quad [a_2] = \frac{1}{T} \quad [b_1] = \frac{1}{L} \quad [\delta] = \frac{L^2}{T}$$

Within a similar notation the ***logistic SDE*** is

$$d\xi(s) = \xi(s) [a_1 - a_2 \xi(s)] ds + b_1\xi(s)d\beta(s) \tag{2}$$

with the following physical dimensions

$$[a_1] = \frac{1}{T} \quad [a_2] = \frac{1}{LT} \quad [b_1] = \frac{1}{L} \quad [\delta] = \frac{L^2}{T}$$

while the generalized **θ -logistic SDE** ($\theta > 0$) is

$$d\xi(s) = \xi(s) [a_1 - a_2 \xi^\theta(s)] ds + b_1 \xi(s) d\beta(s) \quad (3)$$

with the physical dimensions

$$[a_1] = \frac{1}{T} \quad [a_2] = \frac{1}{L^\theta T} \quad [b_1] = \frac{1}{L} \quad [\delta] = \frac{L^2}{T}$$

The logistic (2) is apparently recovered for $\theta = 1$. Therefore – within the notations of the Appendix B – the time independent coefficients of the Gompertz SDE are

$$a(x) = a_1 x - a_2 x \ln(b_1 x) \quad b(x) = b_1 x \quad (4)$$

those of the logistic SDE are

$$a(x) = a_1 x - a_2 x^2 \quad b(x) = b_1 x \quad (5)$$

while for the θ -logistic we finally have

$$a(x) = a_1 x - a_2 x^{1+\theta} \quad b(x) = b_1 x \quad \theta > 0 \quad (6)$$

In order to simplify their look, however, it is expedient to recast these equation in a dimensionless form so that only the essential parameters will remain in evidence. As for the equation (1), a transformation to the dimensionless quantities

$$t = a_1 s \quad X(t) = b_1 \xi \left(\frac{t}{a_1} \right) \quad W(t) = b_1 \beta \left(\frac{t}{a_1} \right) \quad D = \frac{b_1^2 \delta}{a_1} \quad \alpha = \frac{a_2}{a_1}$$

would give rise to the ***dimensionless Gompertz SDE***

$$dX(t) = X(t) [1 - \alpha \ln X(t)] dt + X(t) dW(t) \quad (7)$$

with coefficients

$$a(x) = x(1 - \alpha \ln x) \quad b(x) = x \quad \mathbf{E} [W^2(t)] = 2Dt \quad (8)$$

while with the similar transformations

$$t = a_1 s \quad X(t) = \frac{a_2}{a_1} \xi \left(\frac{t}{a_1} \right) \quad W(t) = b_1 \beta \left(\frac{t}{a_1} \right) \quad D = \frac{b_1^2 \delta}{a_1}$$

the equation (2) becomes the ***dimensionless logistic SDE***

$$dX(t) = X(t) [1 - X(t)] dt + X(t) dW(t) \quad (9)$$

with coefficients

$$a(x) = x(1 - x) \quad b(x) = x \quad \mathbf{E} [W^2(t)] = 2Dt \quad (10)$$

Finally the transformations

$$t = a_1 s \quad X(t) = \left(\frac{a_2}{a_1}\right)^{1/\theta} \xi\left(\frac{t}{a_1}\right) \quad W(t) = b_1 \beta \left(\frac{t}{a_1}\right) \quad D = \frac{b_1^2 \delta}{a_1}$$

will give rise to the ***dimensionless θ -logistic SDE***

$$dX(t) = X(t)[1 - X(t)^\theta]dt + X(t) dW(t) \tag{11}$$

with coefficients

$$a(x) = x(1 - x^\theta) \quad b(x) = x \quad \mathbf{E} [W^2(t)] = 2Dt \tag{12}$$

We will adopt these dimensionless notations all along the present paper: in the Section 2 we will give a virtually complete solution of the Gompertz *SDE* (7), while in the subsequent Section 2.3 these solutions will be extended to the parametric Gompertz equation with a time-dependent drift coefficient. As for the logistic and θ -logistic *SDE*'s (9) and (11) the Section 3 will list several partial results, in particular the explicit solutions (32) and (55) in the guise of processes whose distributions however – albeit derivable from the existing literature [4] – can not be easily presented in a manageable closed form. The same problem is addressed again in the Section 3.4 in the reduced form of the distribution of the integrals of geometric Wiener processes, or even of their finite sums at different times, but here too the results are only preliminary while a discussion of the exact results is postponed to a forthcoming paper

The Gompertz and logistic random dynamics will also be looked at from the standpoint of the *FPE*'s (*Fokker-Planck equations*, see Appendix C) for their *pdf* (*probability density functions*) and this will give in the future the opportunity of exploring a further perspective. In a few previous papers [5, 6] we indeed analyzed the solutions of the *FPE*'s associated by the stochastic mechanics to the quantum wave functions (see Appendix D for the particular case of the stationary states of a *QHO*, *quantum harmonic oscillator*), and we looked into the possibility of controlling the stochastic evolution by means of suitable potentials. We plan therefore to extend in the near future this analysis to the Gompertz and logistic random dynamics. In particular we will focus our attention on the relationship between these equations for the stationary states of a *QHO* with frequency ω , and the Gompertz and logistic equations: this will be listed among other suggestions for future research in the conclusive Section 4. In a last Appendix E about quantiles and medians, definitions and results are finally collected to serve in scattered discussions about a suitable coarse-graining of our stochastic equations: this last step is proposed here in order to recover the deterministic equations of the Appendix A, and therefore to show the global consistency of these models

2 Gompertz stochastic equations

2.1 Smoluchowsky *SDE*: stationary *pdf*

We begin implementing first the transformations presented in the Appendix B.2.2 and leading to a Smoluchowsky *SDE*: since our coefficients (8) are time-independent, the transformation (132) which is now

$$y = h(x) = \ln x \quad x = g(y) = e^y \quad Y(t) = \ln X(t) \quad X(t) = e^{Y(t)}$$

leads to $\widehat{b}(y, t) = 1$, while (133) gives the drift coefficient

$$\widehat{a}(y) = 1 - D - \alpha y \tag{13}$$

Therefore, provided that $\alpha > 0$, the Gompertz *SDE* (7) becomes a Smoluchowsky *SDE* that essentially turns out to be an Ornstein-Uhlenbeck (*OU*) *SDE* with an additional constant drift

$$dY(t) = (1 - D - \alpha Y(t)) dt + dW(t) \tag{14}$$

This *OU SDE* for $Y(t)$ can be completely solved and its Gaussian transition *pdf* is well known: as a consequence we can also easily find the log-normal transition *pdf* for the process $X(t)$, but we will postpone to the next section a discussion of these details taking a look for the time being only at the stationary solution of the transformed $Y(t)$ process: with a dimensionless potential $\chi(y)$, from (108) and (13) we first have

$$-D\chi'(y) = \widehat{a}(y) = 1 - D - \alpha y \quad \chi(y) = \frac{\phi(y)}{kT} \tag{15}$$

and therefore

$$\chi(y) = \frac{\alpha}{2D} y^2 - \frac{1-D}{D} y + c$$

so that, if $\alpha > 0$ and if the integration constant c is suitably chosen, the stationary Boltzmann distribution comes out to be a Gaussian law $\mathfrak{N}\left(\frac{1-D}{\alpha}, \frac{D}{\alpha}\right)$ as it is for every *OU* process; the original process $X(t) = e^{Y(t)}$ then is a *geometric OU* and its stationary solutions are log-normal

2.2 Linearized *SDE*

We consider next the transformation presented in the Appendix B.2.3 and leading to process independent coefficients, and we check first whether the compatibility condition (142) holds for our Gompertz *SDE*: an answer in the affirmative follows from a direct calculation since

$$b(x) \left[D b''(x) - \frac{d}{dx} \left(\frac{a(x)}{b(x)} \right) \right] = \alpha$$

As a consequence, with $c = \alpha$, we first have from (143) that $\widehat{b}(t) = e^{\alpha t}$, then from (144) it is

$$h(x, t) = e^{\alpha t} \int \frac{1}{x} dx = e^{\alpha t} \ln x$$

and finally from (145) we find that

$$\widehat{a}(t) = e^{\alpha t} \left(\alpha \ln x + \frac{x - \alpha x \ln x}{x} - D \right) = (1 - D) e^{\alpha t}$$

The transformed Gompertz *SDE* for the process

$$Z(t) = e^{\alpha t} \ln X(t)$$

is then

$$dZ(t) = (1 - D) e^{\alpha t} dt + e^{\alpha t} dW(t) \tag{16}$$

whose solution, with $Z(0) = Z_0 = \ln X_0$, from (111) is

$$Z(t) = Z_0 + (1 - D) \frac{e^{\alpha t} - 1}{\alpha} + \int_0^t e^{\alpha u} dW(u)$$

so that the solution of the *SDE* (7) is

$$X(t) = e^{e^{-\alpha t} Z(t)} = X_0 e^{-\alpha t} e^{(1-D)(1-e^{-\alpha t})/\alpha} e^{\int_0^t e^{-\alpha(t-u)} dW(u)} \tag{17}$$

It is interesting to remark moreover that, with a degenerate initial condition $X_0 = x_0$, **P**-*a.s.* and by switching off the Wiener noise ($D = 0$) this solution exactly coincides with the solution (82) of the deterministic Gompertz *ODE* discussed in the Appendix A.1. It is easy to see on the other hand that for the process $Y(t) = e^{-\alpha t} Z(t) = \ln X(t)$ we also have

$$dZ(t) = \alpha e^{\alpha t} Y(t) dt + e^{\alpha t} dY(t)$$

so that by comparing it with (16) we get

$$dY(t) = (1 - D - \alpha Y(t)) dt + dW(t)$$

apparently coincident with the Smoluchowsky equation (14) already discussed in the Section 2.1. Since moreover it is easy to see that

$$\int_s^t \widehat{a}(u) du = (1 - D) e^{\alpha s} \frac{e^{\alpha(t-s)} - 1}{\alpha} \quad \int_s^t \widehat{b}^2(u) du = e^{2\alpha s} \frac{e^{2\alpha(t-s)} - 1}{2\alpha}$$

from (111) and (112) we have for the solution of (16) with $Z(s) = ze^{\alpha s}$, **P**-*a.s.* at $t = s$

$$\begin{aligned} Z(t) &= ze^{\alpha s} + (1 - D) e^{\alpha s} \frac{e^{\alpha(t-s)} - 1}{\alpha} + \int_s^t e^{\alpha u} dW(u) \\ &\sim \mathfrak{N} \left(ze^{\alpha s} + (1 - D) e^{\alpha s} \frac{e^{\alpha(t-s)} - 1}{\alpha}, D e^{2\alpha s} \frac{e^{2\alpha(t-s)} - 1}{\alpha} \right) \end{aligned}$$

and for the solution of the Gompertz *SDE* (7) with $\ln X(s) = e^{\alpha s} \ln y = ze^{\alpha s}$, **P**-a.s. at $t = s$

$$\begin{aligned} \ln X(t) &= e^{-\alpha t} Z(t) \\ &= e^{-\alpha(t-s)} \ln y + \frac{1-D}{\alpha} (1 - e^{-\alpha(t-s)}) + e^{-\alpha t} \int_s^t e^{\alpha u} dW(u) \\ &\sim \mathfrak{N} \left(e^{-\alpha(t-s)} \ln y + \frac{1-D}{\alpha} (1 - e^{-\alpha(t-s)}), D \frac{1 - e^{-2\alpha(t-s)}}{\alpha} \right) \end{aligned}$$

Therefore the **transition pdf** $f(x, t|y, s)$ of the Gompertz process $X(t)$ is the log-normal law associated to the previous Gaussian distribution which also asymptotically goes to the stationary log-normal distribution

$$\ln \mathfrak{N} \left(\frac{1-D}{\alpha}, \frac{D}{\alpha} \right)$$

already found in the section 2.1, with expectation

$$\mathbf{E} [X(+\infty)] = e^{\frac{2-D}{2\alpha}}$$

corresponding to the asymptotic value $e^{1/\alpha}$ of the deterministic Gompertz equation when $D = 0$ (see Appendix A.1)

2.3 Parametric equations

In a generalization of the previous investigations, let us consider next the so called **parametric OU SDE** (with a time dependent drift velocity) for a process $\eta(s)$

$$d\eta(s) = -\omega a(s)\eta(s) ds + b d\beta(s)$$

where $\beta(s)$ again is a Wiener process with diffusion coefficient 2δ so that $\mathbf{E} [\beta^2(s)] = 2\delta s$. Here ω is a frequency, b a dimensionless constant and $a(s)$ a dimensionless function of the time s . Since $\sigma = \sqrt{\delta/\omega}$ has the same physical dimensions of η and β (a length, for instance), we can now switch to a dimensionless formulation by taking $t = \omega s$ and

$$Y(t) = \frac{1}{\sigma} \eta \left(\frac{t}{\omega} \right) \quad W(t) = \frac{b}{\sigma} \beta \left(\frac{t}{\omega} \right) \quad D = \frac{\delta b^2}{\omega \sigma^2} = b^2 \quad \alpha(t) = a \left(\frac{t}{\omega} \right)$$

a transformation leading to the dimensionless parametric *OU SDE*

$$dY(t) = -\alpha(t)Y(t)dt + dW(t) \quad Y(0) = Y_0 \quad (18)$$

where $W(t)$ is a Wiener process with diffusion coefficient $2D$, namely $\mathbf{E} [W^2(t)] = 2Dt$. For further purposes $\alpha(t)$ could also become a stochastic process, but for the

time being we will take it just as a suitable deterministic function. If we now define the new process $X(t)$ as

$$X(t) = e^{Y(t)} \quad Y(t) = \ln X(t)$$

taking $a(y, t) = -\alpha(t)y$ and $b(y, t) = 1$, with $x = h(y) = e^y$ and $y = g(x) = \ln x$, from (95) and (96) – swapping the symbols x, y – we obtain the coefficients for the transformed equation

$$\begin{aligned} \widehat{a}(x, t) &= [e^y(1 - \alpha(t)y)]_{y=\ln x} = x(1 - \alpha(t) \ln x) \\ \widehat{b}(x, t) &= [e^y]_{y=\ln x} = x \end{aligned}$$

finally leading to the ***parametric Gompertz SDE***

$$dX(t) = X(t)(1 - \alpha(t) \ln X(t)) dt + X(t) dW(t) \quad (19)$$

Therefore this new equation (19), that apparently generalizes (7), can be solved by looking first at a solution of (18): to this end we remark that, since $a(y, t) = -\alpha(t)y$ and $b(y, t) = 1$, using again (95) and (96), the transformation

$$Z(t) = Y(t)e^{\int_0^t \alpha(u)du} = h(Y(t), t) \quad h(y, t) = ye^{\int_0^t \alpha(u)du}$$

leads to the coefficients

$$\widehat{a}(z, t) = 0 \quad \widehat{b}(z, t) = e^{\int \alpha(t)dt}$$

namely to the *SDE*

$$dZ(t) = e^{\int_0^t \alpha(u)du} dW(t) \quad Z(0) = Y_0$$

whose solution is

$$Z(t) = Y_0 + \int_0^t e^{\int_0^r \alpha(u)du} dW(r)$$

so that we finally have

$$Y(t) = Y_0 e^{-\int_0^t \alpha(u)du} + \int_0^t e^{-\int_r^t \alpha(u)du} dW(r) \quad (20)$$

that with a condition at a time s becomes

$$Y(t) = Y_0 e^{-\int_s^t \alpha(u)du} + \int_s^t e^{-\int_r^t \alpha(u)du} dW(r) \quad (21)$$

As long as the *rv* $Y_0 \sim \mathfrak{N}(y_0, \sigma_0^2)$ is Gaussian, the process (21) is Gaussian too, and if Y_0 is degenerate the *pdf* of $Y(t)$ also is the *transition pdf*. On the other hand

when (21) is Gaussian their laws are completely determined by the expectations and the covariance functions that can be explicitly calculated. In fact we first have

$$\mathbf{E}[Y(t)] = y_0 e^{-\int_s^t \alpha(u) du} \quad (22)$$

then, by taking $\tilde{Y}_0 = Y_0 - y_0$ and

$$\tilde{Y}(t) = Y(t) - \mathbf{E}[Y(t)] = \tilde{Y}_0 e^{-\int_s^t \alpha(u) du} + \int_s^t e^{-\int_r^t \alpha(u) du} dW(r)$$

from the independence of X_0 and $W(t)$ and from the usual properties (85) of the increments $dW(t)$, we get for $t < t'$

$$\begin{aligned} \mathbf{cov}[Y(t), Y(t')] &= \mathbf{E}[\tilde{Y}(t)\tilde{Y}(t')] \\ &= \sigma_0^2 e^{-\int_s^t \alpha(u) du - \int_s^{t'} \alpha(u) du} \\ &\quad + \mathbf{E}\left[\int_s^t e^{-\int_r^t \alpha(u) du} dW(r) \int_s^{t'} e^{-\int_{r'}^{t'} \alpha(u) du} dW(r')\right] \\ &= \sigma_0^2 e^{-\int_s^t \alpha(u) du - \int_s^{t'} \alpha(u) du} \\ &\quad + \mathbf{E}\left[\int_s^t e^{-\int_r^t \alpha(u) du} dW(r) \int_s^t e^{-\int_{r'}^t \alpha(u) du} dW(r')\right] \\ &= \sigma_0^2 e^{-\int_s^t \alpha(u) du - \int_s^{t'} \alpha(u) du} + 2D \int_s^t e^{-\int_r^t \alpha(u) du - \int_r^{t'} \alpha(u) du} dr \end{aligned}$$

and hence in any case

$$\mathbf{cov}[Y(t), Y(t')] = \sigma_0^2 e^{-\int_s^t \alpha(u) du - \int_s^{t'} \alpha(u) du} + 2D \int_s^{t \wedge t'} e^{-\int_r^t \alpha(u) du - \int_r^{t'} \alpha(u) du} dr \quad (23)$$

By the way remark that the results (22) and (23) hold even if $Y(t)$ is not Gaussian, but when $Y_0 \sim \mathfrak{N}(y_0, \sigma_0^2)$ they also completely determines the distribution of the process (21) and in particular

$$Y(t) \sim \mathfrak{N}\left(y_0 e^{-\int_s^t \alpha(u) du}, \sigma_0^2 e^{-\int_s^t \alpha(u) du} + 2D \int_s^t e^{-2\int_r^t \alpha(u) du} dr\right) \quad (24)$$

With $\sigma_0 = 0$ (degenerate initial condition) we moreover have

$$Y(t) \sim \mathfrak{N}\left(y_0 e^{-\int_s^t \alpha(u) du}, 2D \int_s^t e^{-2\int_r^t \alpha(u) du} dr\right) \quad (25)$$

which plays the role of the transition *pdf* for the processes solution of (18).

Going back now to the parametric Gompertz *SDE* (19), we find that when the parametric *OU* process $Y(t)$ is Gaussian, then the process $X(t)$ is log-normal and

we can explicitly give all the details of its distribution, in particular from (25) and with $X(s) = y = e^{y_0}$ its transition *pdf* $f(x, t|y, s)$ is

$$X(t) \sim \text{ln}\mathfrak{N} \left(e^{-\int_s^t \alpha(u) du} \ln y, 2D \int_s^t e^{-2\int_r^t \alpha(u) du} dr \right) \quad (26)$$

It is possible then to calculate also its expectation and variance according to (228)

$$\mathbf{E} [X(t)] = e^{\mathbf{E}[Y(t)] + \mathbf{V}[Y(t)]/2} \quad \mathbf{V} [X(t)] = e^{2\mathbf{E}[Y(t)] + \mathbf{V}[Y(t)]} (e^{\mathbf{V}[Y(t)]} - 1)$$

that in the case (26) of degenerate initial conditions become

$$\mathbf{E} [X(t)] = y e^{-\int_s^t \alpha(u) du} e^{D \int_s^t e^{-2\int_r^t \alpha(u) du} dr} \quad (27)$$

$$\mathbf{V} [X(t)] = y^{2e^{-\int_s^t \alpha(u) du}} e^{2D \int_s^t e^{-2\int_r^t \alpha(u) du} dr} \left(e^{2D \int_s^t e^{-2\int_r^t \alpha(u) du} dr} - 1 \right) \quad (28)$$

As for the median, when $Y(t)$ is Gaussian, from (229) we simply get

$$\mathbf{M} [X(t)] = e^{\mathbf{E}[Y(t)]} = y e^{-\int_s^t \alpha(u) du} \quad (29)$$

that can be used along with (27) and (28) to analyze the oscillations of the system. However, as hinted also in the Appendix E.3, we must keep into account that these results about means and medians are not completely general since they hold only in so far as the parametric *OU* process $Y(t)$ is Gaussian: this is not always the case because it requires that the initial condition itself should be Gaussian. We could alternatively take advantage of the more general relation (232), but also in this case there is a snag because we should calculate the median $\mathbf{M} [Y(t)]$ which is not always a straightforward job for arbitrary initial conditions. The adoption of medians as a way to retrieve the deterministic evolution has been extensively discussed in a few previous papers [1]

3 Logistic stochastic equations

3.1 Smoluchowsky *SDE*: stationary *pdf*

The transformation (132) for the coefficients (10), namely

$$y = h(x) = \ln x \quad x = g(y) = e^y \quad Y(t) = \ln X(t) \quad X(t) = e^{Y(t)}$$

applied to the logistic *SDE* (9) leads to $\widehat{b}(y, t) = 1$, and from (133) to the drift coefficient

$$\widehat{a}(y) = 1 - D - e^y$$

namely to the Smoluchowsky *SDE*

$$dY(t) = (1 - D - e^{Y(t)}) dt + dW(t)$$

and from (108) to a dimensionless potential

$$\chi(y) = \frac{\phi(y)}{kT} = \frac{e^y}{D} - \frac{1-D}{D}y + c$$

that – provided now that $1 > D$ – gives rise to the following stationary log-gamma Boltzmann distribution (see [7] 3.328 for the normalization integral)

$$\frac{e^{-\frac{e^y}{D} + \frac{1-D}{D}y}}{D^{\frac{1-D}{D}} \Gamma\left(\frac{1-D}{D}\right)} \quad 1 > D$$

We will not elaborate further about this stationary distribution for the process $Y(t)$, and we will rather confine ourselves to remark that by transforming back to the original process $X(t) = e^{Y(t)}$ we find that its stationary density is

$$\frac{1}{D \Gamma\left(\frac{1-D}{D}\right)} \left(\frac{x}{D}\right)^{\frac{1-D}{D}-1} e^{-\frac{x}{D}} \quad 1 > D \quad x > 0$$

namely that the stationary distribution of $X(t)$ is the gamma law $\mathfrak{G}\left(\frac{1-D}{D}, \frac{1}{D}\right)$. There is no easy way instead at this stage to find the form of the transition *pdf* for both the processes $Y(T)$ and $X(t)$. Remark finally that the condition $1 > D$, required to have a normalizable stationary solution, amounts to the explicit condition

$$a_1 > b_1^2 \delta$$

among the coefficients of the original logistic *SDE* (2) laden with its physical dimensions, and hence it represent an equilibrium condition between the dynamical and the diffusive components of the process

3.2 Linearized *SDE*

Considering first to the transformation to process-independent new coefficients discussed in the Appendix B.2.3, since the coefficients (10) are time-independent, we must preliminarily check the compatibility condition (142), but we find

$$b(x) \left[D b''(x) - \frac{d}{dx} \left(\frac{a(x)}{b(x)} \right) \right] = x$$

that is not constant, and hence the said compatibility condition (142) is not satisfied by the logistic coefficients

As next step we then explore the possibility of linearizing the *SDE* (9) in the sense discussed in the Appendix B.2.4: in order to check the condition (150) we find from the coefficients (10) that

$$q(x) = 1 - D - x \quad b(x)q'(x) = -x \quad \frac{1}{q'(x)} \frac{d}{dx} [b(x)q'(x)] = 1$$

so that the compatibility condition (150) is satisfied and from (148) and (149) we also find

$$\widehat{b}_1 = -1 \quad p(x) = \int \frac{dx}{b(x)} = \ln x \quad h(x) = c e^{-p(x)} = c e^{-\ln x} = \frac{c}{x}$$

The reciprocal transformation relations then are

$$Y(t) = \frac{c}{X(t)} \quad X(t) = \frac{c}{Y(t)}$$

and if we choose $c = 1$ as integration constant we get the transformation

$$y = h(x) = \frac{1}{x} \quad x = g(y) = \frac{1}{y} \quad Y(t) = \frac{1}{X(t)} \quad X(t) = \frac{1}{Y(t)}$$

so that with

$$h'(x) = -\frac{1}{x^2} \quad h''(x) = \frac{2}{x^3}$$

from (95) and (96) we have

$$\begin{aligned} \widehat{a}(y) &= h'(g(y))a(g(y)) + D h''(g(y))b^2(g(y)) = (2D - 1)y + 1 \\ \widehat{b}(y) &= h'(g(y))b(g(y)) = -y \end{aligned}$$

namely from (121)

$$\widehat{a}_0 = 1 \quad \widehat{a}_1 = 2D - 1 \quad \widehat{b}_0 = 0 \quad \widehat{b}_1 = -1$$

and hence the new *SDE* is

$$dY(t) = [(2D - 1)Y(t) + 1] dt - Y(t) dW(t) \quad (30)$$

As a consequence, by taking as in (123)

$$\overline{Z}(t) = (D - 1)t - W(t) \sim \mathfrak{N}((D - 1)t, 2Dt)$$

the general solution (124) of the linearized *SDE* (30) for $Y(0) = Y_0$ is

$$Y(t) = e^{\overline{Z}(t)} \left(Y_0 + \int_0^t e^{-\overline{Z}(u)} du \right)$$

while the solution $X(t) = \frac{1}{Y(t)}$ of the logistic *SDE* (9) for $X(0) = X_0 = \frac{1}{Y_0}$ is

$$X(t) = \frac{X_0 e^{-\overline{Z}(t)}}{1 + X_0 \int_0^t e^{-\overline{Z}(u)} du} \quad (31)$$

Remark that with a degenerate initial condition $X_0 = x_0$, **P**-a.s. and by switching off the Wiener noise ($D = 0$) we get $\overline{Z}(t) = -t$, and the solution (31) exactly

coincides with the solution (83) of the deterministic logistic *ODE* discussed in the Appendix A.2. By taking on the other hand $X_0 = y$, \mathbf{P} -a.s. at a time $0 \leq s \leq t$ we have the solution

$$X(t) = \frac{y e^{-\bar{Z}(t-s)}}{1 + y \int_s^t e^{-\bar{Z}(u)} du} \quad (32)$$

whose *pdf* $f(x, t|y, s)$ will be the **transition pdf** of our logistic process. If moreover we define the derivable process

$$A(t) = X_0 \int_0^t e^{-\bar{Z}(u)} du \quad \dot{A}(t) = X_0 e^{-\bar{Z}(t)} \quad (33)$$

the solution (31) takes the equivalent forms (see also [7] 3.434.2)

$$\begin{aligned} X(t) &= \frac{\dot{A}(t)}{1 + A(t)} = \frac{d}{dt} \ln(1 + A(t)) = \frac{d}{dt} \int_0^\infty e^{-u} \frac{1 - e^{-uA(t)}}{u} du \\ &= - \int_0^\infty \frac{e^{-u}}{u} \frac{d}{dt} e^{-uA(t)} du = \dot{A}(t) \int_0^\infty e^{-u(1+A(t))} du \end{aligned} \quad (34)$$

hinting additionally to a possible direct connection between the moments of $X(t)$ and the generating function of $A(t)$, namely $\mathbf{E} [e^{-uA(t)}]$

At first sight these results seem to be coherent with those elaborated in other papers [2], where also a few procedures leading to the calculation of expectations and variances of $X(t)$ are discussed. For the time being however we will neglect a detailed analysis of these claims, noting instead that an explicit, exact form of the *pdf* of (32) could be retrieved only by taking advantage of a few rather intricate results available in the literature [4]. While indeed $\bar{Z}(t)$ is a Gaussian process (it is just a re-scaled Wiener process plus a uniform drift) so that $e^{-\bar{Z}(t)}$ is a geometric Gaussian process with a log-normal law, it is totally another matter to find the law of the integral process

$$\int_s^t e^{-\bar{Z}(u)} du$$

Extensive research [4, 8, 9, 10] has been devoted to this problem, but the available answers are far from being easy to handle (see also the subsequent discussion in the Section 3.4). We will therefore present in the following, for the time being, only a few *elementary* approaches with their associated partial results, looking forward instead to scrutinize the question in further detail in a forthcoming paper within the framework of a more general setting

3.2.1 Semi-explicit transition *pdf*

We will present first a semi-explicit form of the transition *pdf* by recalling the notations and the results referred to in the Appendix C.1. Since the coefficients (10)

are time-independent we begin by defining the functions

$$y = h(x) = \int \frac{dx}{b(x)} = \ln x \quad x = g(y) = e^y$$

$$\widehat{a}(y) = \frac{a(g(y))}{b(g(y))} - D b'(g(y)) = 1 - D - e^y$$

so that we also find

$$\begin{aligned} \beta(y) &= -\frac{\widehat{a}^2(y)}{4D} - \frac{\widehat{a}'(y)}{2} = -\frac{1}{4D} (1 - D - e^y)^2 + \frac{e^y}{2} \\ &= -\frac{1}{4D} (1 - e^y)^2 + \frac{1}{2} - \frac{D}{4} \end{aligned}$$

As a consequence we have the following expressions

$$\begin{aligned} \bar{h}(r) &= r \ln x + (1 - r) \ln y & e^{\bar{h}(r)} &= x^r y^{1-r} \\ \bar{W}_{st}(r) &= W(s + (t - s)r) - (rW(t) + (1 - r)W(s)) \\ \beta(\bar{W}_{st}(r) + \bar{h}(r)) &= \frac{2 - D}{4} - \frac{1}{4D} \left(1 - x^r y^{1-r} e^{\bar{W}_{st}(r)}\right)^2 \end{aligned}$$

and hence we get

$$\begin{aligned} Z(s, t) &= \int_0^1 \beta(\bar{W}_{st}(r) + \bar{h}(r)) dr = \frac{2 - D}{4} - \frac{1}{4D} \int_0^1 \left(1 - x^r y^{1-r} e^{\bar{W}_{st}(r)}\right)^2 dr \\ &= \frac{2 - D}{4} - \frac{1}{4D} + \frac{y}{2D} \int_0^1 \left(\frac{x}{y}\right)^r e^{\bar{W}_{st}(r)} dr - \frac{y^2}{4D} \int_0^1 \left(\frac{x}{y}\right)^{2r} e^{2\bar{W}_{st}(r)} dr \\ &= -\frac{(1 - D)^2}{4D} + \frac{y}{2D} \int_0^1 \left(\frac{x}{y}\right)^r e^{\bar{W}_{st}(r)} dr - \frac{y^2}{4D} \int_0^1 \left(\frac{x}{y}\right)^{2r} e^{2\bar{W}_{st}(r)} dr \end{aligned}$$

We thus find

$$\begin{aligned} e^{(t-s)Z(s,t)} &= e^{\frac{2-D}{4}(t-s)} \exp \left\{ -\frac{t-s}{4D} \int_0^1 \left(1 - x^r y^{1-r} e^{\bar{W}_{st}(r)}\right)^2 dr \right\} \\ &= e^{-\frac{(1-D)^2}{4D}(t-s)} \\ &\quad \exp \left\{ -\frac{t-s}{4D} \left(y^2 \int_0^1 \left(\frac{x}{y}\right)^{2r} e^{2\bar{W}_{st}(r)} dr - 2y \int_0^1 \left(\frac{x}{y}\right)^r e^{\bar{W}_{st}(r)} dr \right) \right\} \end{aligned}$$

so that finally for the expectation factor in our transition *pdf* (155) we have

$$\begin{aligned} \mathbf{E} [e^{(t-s)Z(s,t)}] &= e^{-\frac{(1-D)^2}{4D}(t-s)} \mu(x, t; y, s) \\ \mu(x, t; y, s) &= \mathbf{E} \left[\exp \left\{ -\frac{t-s}{4D} \left(y^2 \int_0^1 \left(\frac{x}{y}\right)^{2r} e^{2\bar{W}_{st}(r)} dr - 2y \int_0^1 \left(\frac{x}{y}\right)^r e^{\bar{W}_{st}(r)} dr \right) \right\} \right] \end{aligned}$$

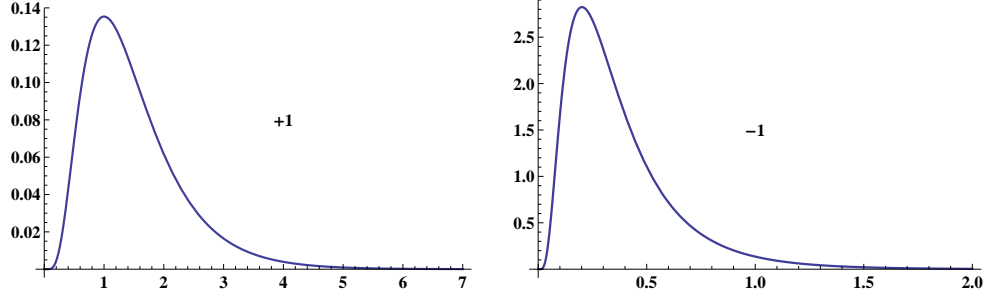


Figure 1: Behavior of the function (36) for both the possible signs (beware: the plot scales in the two pictures are rather different)

As for the other factors in (155) we first remark that

$$\begin{aligned} \frac{1}{2D} \int_y^x \frac{a(z)}{b^2(z)} dz &= \frac{1}{2D} \int_y^x \frac{1-z}{z} dz = \frac{1}{2D} \left(\ln \frac{x}{y} - x + y \right) \\ -\frac{1}{4D(t-s)} \left(\int_y^x \frac{dz}{b(z)} \right)^2 &= -\frac{1}{4D(t-s)} \left(\int_y^x \frac{dz}{z} \right)^2 = -\frac{1}{4D(t-s)} \ln^2 \frac{x}{y} \end{aligned}$$

and then we find out for the transition *pdf*

$$\begin{aligned} f(x, t|y, s) &= \frac{e^{-\frac{(1-D)^2}{4D}(t-s)}}{x\sqrt{4\pi D(t-s)}} \sqrt{\frac{y}{x}} e^{\frac{1}{2D}(\ln \frac{x}{y} - x + y)} e^{-\frac{1}{4D(t-s)} \ln^2 \frac{x}{y}} \mu(x, t; y, s) \\ &= \frac{e^{-\frac{(1-D)^2}{4D}(t-s) - \frac{1}{2} \ln \frac{x}{y} + \frac{1}{2D}(\ln \frac{x}{y} - x + y) - \frac{1}{4D(t-s)}(\ln \frac{x}{y})^2}}{x\sqrt{4\pi D(t-s)}} \mu(x, t; y, s) \\ &= \frac{e^{-\frac{x-y}{2D} - \frac{1}{4D(t-s)}((1-D)(t-s) - \ln \frac{x}{y})^2}}{x\sqrt{4\pi D(t-s)}} \mu(x, t; y, s) \end{aligned} \quad (35)$$

Here too, however, despite the fact that the transition *pdf* is given in closed form, the calculation of the expectation in (35) depends on the knowledge of the law of the integral of a geometric Gaussian process similar to that of the solution (32). Therefore – at variance with the case of the Gompertz *SDE* – the expression (35) seems to represent the farthest point we can reach at present along this path in our quest for an explicit formula of the transition *pdf* for the logistic *SDE*. The main hurdle apparently is the computation of the expectation term $\mu(x, t; y, s)$ which contains integrals of geometric Wiener processes with non elementary distributions [4]. All that we can easily assess for the time being is the behavior of the explicit term in front of μ that is of the type

$$\frac{e^{-x - (\pm 1 - \ln x)^2}}{x} \quad (36)$$

This function turns out to be regular in the origin $x = 0$ for both the possible signs and, coherently with the behavior of the stationary distribution, it displays a gamma-like shape for $x > 0$ as can be seen from the Figure 1

3.2.2 Fokker-Planck equation

We will next turn our attention to the possible solutions of the corresponding *FPE* along the lines presented in the Appendix C: for a process $X(t)$ solution of the logistic *SDE* (9) the *FPE* (165) is

$$\begin{aligned} \partial_t f_X(x, t) &= -\partial_x \left[\vec{v}(x) f_X(x, t) \right] + \partial_x^2 [B(x) f_X(x, t)] \\ &= -\partial_x [x(1-x) f_X(x, t)] + D \partial_x^2 [x^2 f_X(x, t)] \\ &= D x^2 \partial_x^2 f_X(x, t) + x(4D - 1 + x) \partial_x f_X(x, t) + (2D - 1 + 2x) f_X(x, t) \end{aligned} \quad (37)$$

where

$$\vec{v}(x) = a(x) = x(1-x) \quad B(x) = D b^2(x) = D x^2$$

while for the transformed process $Y(t) = \frac{1}{X(t)}$ solution of the *SDE* (30) we have

$$\begin{aligned} \partial_t f_Y(y, t) &= -\partial_y \left[\vec{v}(y) f_Y(y, t) \right] + \partial_y^2 [B(y) f_Y(y, t)] \\ &= -\partial_y [(2D - 1)y + 1] f_Y(y, t) + D \partial_y^2 [y^2 f_Y(y, t)] \\ &= D y^2 \partial_y^2 f_Y(y, t) + [(2D + 1)y - 1] \partial_y f_Y(y, t) + f_Y(y, t) \end{aligned} \quad (38)$$

with

$$\vec{v}(y) = \hat{a}(y) = (2D - 1)y + 1 \quad B(y) = D \hat{b}^2(y) = D y^2$$

Remark by the way that the *pdf* of $Y(t) = \frac{1}{X(t)}$ can always be derived from that of $X(t)$ as

$$f_Y(y, t) = \frac{1}{y^2} f_X\left(\frac{1}{y}, t\right)$$

so that the corresponding equations could be deduced one from the other by means of this transformation

We can then look for the solutions with eigenfunction expansions starting with (37): since we already know that the *pdf* of the gamma law $\mathfrak{G}\left(\frac{1-D}{D}, \frac{1}{D}\right)$

$$\tilde{f}_X(x) = \frac{\left(\frac{1}{D}\right)^{\frac{1-D}{D}}}{\Gamma\left(\frac{1-D}{D}\right)} x^{\frac{1-D}{D}-1} e^{-\frac{x}{D}} = \frac{1}{D} \frac{\left(\frac{x}{D}\right)^{\frac{1-D}{D}-1}}{\Gamma\left(\frac{1-D}{D}\right)} e^{-\frac{x}{D}} \quad 1 > D \quad (39)$$

is a stationary solution of (37) (this can be also checked by direct calculation), from the Section C.3 stems that by taking

$$f_X(x, t) = \sqrt{\tilde{f}_X(x)} g_X(x, t)$$

we get for g_X the new equation

$$\partial_t g_X(x, t) = \mathcal{L} [g_X] (x, t) \quad (40)$$

where \mathcal{L} is an operator of the Sturm-Liouville form

$$\mathcal{L} [\varphi] (x) = \frac{d}{dx} \left[p(x) \frac{d\varphi(x)}{dx} \right] - q(x) \varphi(x) \quad (41)$$

that is is self-adjoint for functions satisfying suitable boundary conditions in $x = 0$ and $x = +\infty$. It can be shown (and cross-checked by direct calculation) that for our equation (37) we have in particular

$$\begin{aligned} p(x) &= B(x) = Dx^2 \\ q(x) &= \frac{[B'(x) - \vec{v}(x)]^2}{4B(x)} - \frac{[B'(x) - \vec{v}(x)]'}{2} = \frac{(x-1)^2 - 2D}{4D} \end{aligned}$$

so that we finally get

$$\partial_t g_X(x, t) = Dx^2 \partial_x^2 g_X(x, t) + 2Dx \partial_x g_X(x, t) + \frac{2D - (x-1)^2}{4D} g_X(x, t)$$

We then separate the variables by taking

$$g_X(x, t) = e^{-\lambda t} G_X(x)$$

obtaining the eigenvalue equation

$$\mathcal{L} [G_X] (x) + \lambda G_X(x) = 0$$

that can be explicitly written as

$$x^2 G_X''(x) + 2x G_X'(x) + \left[\frac{2D - (x-1)^2}{4D^2} + \frac{\lambda}{D} \right] G_X(x) = 0 \quad (42)$$

Now this is a totally Fuchsian equation with two singularities in $x = 0$ and $x = +\infty$ and consequently can be treated with the usual methods: first of all it is possible to check by direct calculation that $\lambda_0 = 0$ is an eigenvalue for the eigenfunction $G_0(x) = \sqrt{\tilde{f}_X(x)}$. Then to simplify the notation we change the variable according to

$$z = \frac{x}{D} \quad G_X(x) = G_X(Dz) = \psi(z)$$

and we get

$$z^2 \psi''(z) + 2z \psi'(z) + \left[\frac{1 + 2\lambda}{2D} - \left(\frac{Dz - 1}{2D} \right)^2 \right] \psi(z) = 0 \quad (43)$$

Now we take

$$\begin{aligned}\psi(z) &= \frac{e^{z/2}}{z} u(z) \\ \psi'(z) &= \frac{e^{z/2}}{z} \left[u'(z) + \frac{z-2}{2z} u(z) \right] \\ \psi''(z) &= \frac{e^{z/2}}{z} \left[u''(z) + \frac{z-2}{z} u'(z) + \frac{4+(z-2)^2}{4z^2} u(z) \right]\end{aligned}$$

and we find

$$u''(z) + u'(z) + \left[\frac{1}{2Dz} + \frac{(1+2\lambda)2D-1}{4D^2z^2} \right] u(z) = 0$$

that can be put in the form of a confluent hypergeometric equation (see [7] formula 9.202.1)

$$u''(z) + u'(z) + \left[\frac{1}{z} + \frac{\frac{1}{2D} + \frac{1}{4} - \frac{(1-D)^2 - 4\lambda D}{4D^2}}{z^2} \right] u(z) = 0 \quad (44)$$

where moreover the term

$$\mu^2 = \frac{(1-D)^2 - 4\lambda D}{4D^2} = \left(\frac{1-D}{2D} \right)^2 - \frac{\lambda}{D}$$

is required to be positive, which would happen only if

$$\lambda < D \left(\frac{1-D}{2D} \right)^2$$

When this happens two linearly independent solutions are ([7] formula 9.202.2 – 3)

$$\begin{aligned}u_1(z) &= z^{\frac{1}{2}+\mu} e^{-z} \Phi \left(\frac{D-1}{2D} + \mu, 1 + 2\mu; z \right) \\ u_2(z) &= z^{\frac{1}{2}-\mu} e^{-z} \Phi \left(\frac{D-1}{2D} - \mu, 1 - 2\mu; z \right)\end{aligned}$$

where $\Phi(\alpha, \gamma; z)$ is the confluent hypergeometric function (see [7] formula 9.210.1). It is well known that the eigenvalues are found by requiring that $\Phi(\alpha, \gamma; z)$ degenerates in a Laguerre polynomial (see [7] formula 8.970.1) and that this happens when $\alpha = -n$ is a negative integer (see [7] formula 8.972.1). As a consequence our eigenvalues are selected by the requirement

$$\frac{D-1}{2D} \pm \mu = \frac{D-1}{2D} \pm \sqrt{\left(\frac{1-D}{2D} \right)^2 - \frac{\lambda}{D}} = -n$$

namely

$$\lambda_n = D \left(\frac{1-D}{D} n - n^2 \right) \quad (45)$$

However, while $\lambda_0 = 0$ is confirmed as an eigenvalue, we find that just a finite number of eigenvalues are possibly positive, and that they turn negative as soon as

$$n > \frac{1 - D}{D}$$

For small D this limit can be quite large, but that notwithstanding it remains a finite number, and hence it seems apparent that we can not have the infinite sequence of (increasing) positive eigenvalues that we would have supposed to have: this is a puzzling point, and these results can be also checked with a shortcut through MATHEMATICA asking for the solutions of (42). Remark that the positivity of the eigenvalues is directly linked to the ergodicity of the system, because it would entail that all the eigenfunctions other than the stationary solution are wiped out in time exponentially fast: failing to have positive eigenvalues would instead present the case of exploding terms in the eigenfunction expansion

If on the other hand we try to look as an alternative for the eigenfunction expansion of the solution of the Fokker-Planck equation (38) for the transformed process $Y(t) = \frac{1}{X(t)}$ we would meet again seeming insurmountable problems: we indeed immediately find by direct calculation that the stationary solution is now, quite understandably, the inverse gamma law *inv-G* $(\frac{1-D}{D}, \frac{1}{D})$ with *pdf*

$$\tilde{f}_Y(y) = \frac{\left(\frac{1}{D}\right)^{\frac{1-D}{D}}}{\Gamma\left(\frac{1-D}{D}\right)} y^{-\frac{1-D}{D}-1} e^{-\frac{1}{Dy}} = D \frac{\left(\frac{1}{Dy}\right)^{\frac{1}{D}}}{\Gamma\left(\frac{1-D}{D}\right)} e^{-\frac{1}{Dy}} \quad 1 > D \quad (46)$$

then by taking

$$f_Y(y, t) = \sqrt{\tilde{f}_Y(y)} g_Y(y, t)$$

we get the equation

$$\partial_t g_Y(y, t) = \mathcal{L}[g_Y](y, t) \quad (47)$$

where $\mathcal{L}[\cdot]$ is now a Sturm-Liouville operator (41) with

$$\begin{aligned} p(y) &= B(y) = Dy^2 \\ q(y) &= \frac{\left[B'(y) - \vec{v}(y)\right]^2}{4B(y)} - \frac{\left[B'(y) - \vec{v}(y)\right]'}{2} = \frac{(y-1)^2 - 2Dy^2}{4Dy^2} \end{aligned}$$

so that we finally get

$$\partial_t g_Y(y, t) = Dy^2 \partial_y^2 g_Y(y, t) + 2Dy \partial_y g_Y(y, t) + \frac{2Dy^2 - (y-1)^2}{4Dy^2} g_Y(y, t)$$

We next separate the variables by taking

$$g_Y(y, t) = e^{-\lambda t} G_Y(y)$$

to have the eigenvalue equation

$$\mathcal{L}[G_Y](y) + \lambda G_Y(y) = 0$$

which can be explicitly written as

$$y^2 G_Y''(y) + 2y G_Y'(y) + \left[\frac{2Dy^2 - (y-1)^2}{4D^2y^2} + \frac{\lambda}{D} \right] G_Y(y) = 0 \quad (48)$$

or equivalently as

$$G_Y''(y) + \frac{2}{y} G_Y'(y) + \left[\frac{2Dy^2 - (y-1)^2}{4D^2y^4} + \frac{\lambda}{Dy^2} \right] G_Y(y) = 0$$

Now it is apparent that this equation has a Fuchsian singularity at $y = +\infty$ because its coefficients asymptotically vanish quickly enough, but also has non Fuchsian singularity in $y = 0$ where its second coefficient displays a 4th order pole. As a consequence there is no standard procedure to solve it

The two Fokker-Planck equations (37) and (38) could finally also be recast in the form of *ODE*'s by means of a Laplace transform in t

$$\begin{aligned} \phi_X(x) &= \int_0^{+\infty} e^{-pt} f_X(x, t) dt & \phi_Y(y) &= \int_0^{+\infty} e^{-pt} f_Y(y, t) dt \\ \int_0^{+\infty} e^{-pt} \partial_t f_X(x, t) dt &= p \phi_X(x) - f_0(x) & \int_0^{+\infty} e^{-pt} \partial_t f_Y(y, t) dt &= p \phi_Y(y) - g_0(y) \end{aligned}$$

where, to keep the notation to a minimum, we hid the explicit dependence on p : on the other hand the variable p becomes an external parameter in the subsequent *ODE*'s which respectively are

$$Dx^2 \phi_X''(x) + [x^2 + (4D - 1)x] \phi_X'(x) + (2x + 2D - 1 - p) \phi_X(x) = -f_0(x) \quad (49)$$

$$Dy^2 \phi_Y''(y) + [(2D + 1)y - 1] \phi_Y'(y) + (1 - p) \phi_Y(y) = -g_0(y) \quad (50)$$

where $f_0(x)$ and $g_0(y)$ are the initial *pdf*'s. The associated, homogeneous equations

$$Dx^2 \phi_X''(x) + [x^2 + (4D - 1)x] \phi_X'(x) + (2x + 2D - 1 - p) \phi_X(x) = 0 \quad (51)$$

$$Dy^2 \phi_Y''(y) + [(2D + 1)y - 1] \phi_Y'(y) + (1 - p) \phi_Y(y) = 0 \quad (52)$$

are exactly solved by MATHEMATICA in terms of confluent hypergeometric functions. In particular, taking

$$\beta = \frac{1 - D}{D} \quad \eta(p) = \sqrt{\beta^2 + \frac{p}{D}}$$

the general solution of (51) for instance is

$$\begin{aligned} \phi_X(x) &= e^{-\frac{x}{D}} x^{\beta-1+\eta(p)} \left[C_1 \Psi \left(-\beta + \eta(p), 1 + 2\eta(p), \frac{x}{D} \right) \right. \\ &\quad \left. + \frac{C_2}{2\eta(p)B(1 + \beta - \eta(p), 2\eta(p))} \Phi \left(-\beta + \eta(p), 1 + 2\eta(p), \frac{x}{D} \right) \right] \end{aligned}$$

where Φ and Ψ are confluent hypergeometric functions (see [7] 9.201.1, 9.210.2). However, even if we can manage to find the complete solution of the non homogeneous equation, the problem of inverting such Laplace transforms would still stay with us making these results rather incomplete, at least for the time being

3.3 θ -logistic *SDE*

We can now generalize the previous results to the θ -logistic *SDE* (11) and we start by looking for a stationary Boltzmann distribution. The transformation (132) for the coefficients (12), namely

$$y = h(x) = \ln x \quad x = g(y) = e^y \quad Y(t) = \ln X(t) \quad X(t) = e^{Y(t)}$$

applied to the θ -logistic *SDE* (11) leads to $\widehat{b}(y, t) = 1$, and from (133) to the drift coefficient

$$\widehat{a}(y) = 1 - D - e^{\theta y}$$

namely to the Smoluchowsky *SDE*

$$dY(t) = (1 - D - e^{\theta Y(t)}) dt + dW(t)$$

and from (108) to a dimensionless potential

$$\chi(y) = \frac{\phi(y)}{kT} = \frac{e^{\theta y}}{\theta D} - \frac{1 - D}{D} y + c$$

that – provided now that $1 > D$ – gives rise to the following stationary generalized log-gamma Boltzmann distribution (see [7] 3.328 for the normalization integral)

$$\frac{e^{-\frac{\theta y}{\theta D} + \frac{1-D}{D} y}}{\frac{1}{\theta}(\theta D)^{\frac{1-D}{\theta D}} \Gamma\left(\frac{1-D}{\theta D}\right)} \quad 1 > D$$

This can finally be transformed back to the original process $X(t) = e^{Y(t)}$ giving as stationary density

$$\frac{\theta x^{\frac{1-D}{D}-1} e^{-\frac{x^\theta}{\theta D}}}{(\theta D)^{\frac{1-D}{\theta D}} \Gamma\left(\frac{1-D}{\theta D}\right)} \quad 1 > D$$

which is the *pdf* of the generalized gamma law $\mathfrak{G}_\theta\left(\frac{1-D}{D}, \frac{1}{(\theta D)^{1/\theta}}\right)$

We then go on to linearize the *SDE* (11): from the coefficients (12) we find

$$q(x) = 1 - D - x^\theta \quad b(x)q'(x) = -\theta x^\theta \quad \frac{1}{q'(x)} \frac{d}{dx} [b(x)q'(x)] = \theta$$

so that the compatibility condition (150) is satisfied and from (148), (149) we have

$$\widehat{b}_1 = -\theta \quad p(x) = \int \frac{dx}{b(x)} = \ln x \quad h(x) = c e^{\widehat{b}_1 p(x)} = c e^{-\theta \ln x} = \frac{c}{x^\theta}$$

If we then choose $c = 1$ as integration constant the reciprocal transformation relations are

$$y = h(x) = \frac{1}{x^\theta} \quad x = g(y) = \frac{1}{y^{1/\theta}} \quad Y(t) = \frac{1}{X(t)^\theta} \quad X(t) = \frac{1}{Y(t)^{1/\theta}}$$

so that with

$$h'(x) = -\frac{\theta}{x^{1+\theta}} \quad h''(x) = \frac{\theta(1+\theta)}{x^{2+\theta}}$$

from (95) and (96) we have

$$\begin{aligned} \widehat{a}(y) &= h'(g(y))a(g(y)) + D h''(g(y))b^2(g(y)) = ((1+\theta)D - 1)\theta y + \theta \\ \widehat{b}(y) &= h'(g(y))b(g(y)) = -\theta y \end{aligned}$$

namely from (121)

$$\widehat{a}_0 = \theta \quad \widehat{a}_1 = ((1+\theta)D - 1)\theta \quad \widehat{b}_0 = 0 \quad \widehat{b}_1 = -\theta$$

and hence the new *SDE* is

$$dY(t) = [((1+\theta)D - 1)Y(t) + 1]\theta dt - \theta Y(t) dW(t) \quad (53)$$

Taking now as in (123)

$$\overline{Z}(t) = \theta(D - 1)t - \theta W(t) \sim \mathfrak{N}((D - 1)\theta t, 2D\theta^2 t)$$

the general solution (124) of the linearized *SDE* (53) for $Y(0) = Y_0$ is

$$Y(t) = e^{\overline{Z}(t)} \left(Y_0 + \theta \int_0^t e^{-\overline{Z}(u)} du \right)$$

while the solution $X(t)$ of the θ -logistic *SDE* (11) for $Y_0 = X_0^{-\theta}$ is

$$X(t) = \left(\frac{X_0^\theta e^{-\overline{Z}(t)}}{1 + \theta X_0^\theta \int_0^t e^{-\overline{Z}(u)} du} \right)^{1/\theta} \quad (54)$$

Remark that here too, with a degenerate initial condition $X_0 = x_0$, *P*-*a.s.* and by switching off the Wiener noise ($D = 0$) we get $\overline{Z}(t) = -\theta t$, and the solution (54) exactly coincides with the solution (84) of the deterministic θ -logistic *ODE* discussed in the Appendix A.2. Then again, taking $X_0 = y$, *P*-*a.s.* at a time $0 \leq s \leq t$ we have the solution

$$X(t) = \left(\frac{y^\theta e^{-\overline{Z}(t-s)}}{1 + \theta y^\theta \int_s^t e^{-\overline{Z}(u)} du} \right)^{1/\theta} \quad (55)$$

whose *pdf* $f(x, t|y, s)$ will be the **transition pdf** of our θ -logistic process. If moreover we define the derivable process

$$A(t) = X_0^\theta \int_0^t e^{-\overline{Z}(u)} du \quad \dot{A}(t) = X_0^\theta e^{-\overline{Z}(t)} \quad (56)$$

the solution (54) takes the equivalent forms (see also [7] 3.434.2)

$$\begin{aligned} X(t) &= \left(\frac{\dot{A}(t)}{1 + \theta A(t)} \right)^{\frac{1}{\theta}} = \left(\frac{1}{\theta} \frac{d}{dt} \ln [1 + \theta A(t)] \right)^{\frac{1}{\theta}} = \left(\frac{d}{dt} \int_0^\infty e^{-u} \frac{1 - e^{-\theta u A(t)}}{\theta u} du \right)^{\frac{1}{\theta}} \\ &= \left(- \int_0^\infty \frac{e^{-u}}{\theta u} \frac{d}{dt} e^{-\theta u A(t)} du \right)^{\frac{1}{\theta}} = \left(\dot{A}(t) \int_0^\infty e^{-u(1+\theta A(t))} du \right)^{\frac{1}{\theta}} \end{aligned} \quad (57)$$

Retracing finally the procedure of the Appendix C.1 leading to the semi-explicit transition *pdf* (35), from the coefficients (12) we begin by defining the functions

$$y = h(x) = \ln x \quad x = g(y) = e^y \quad \hat{a}(y) = 1 - D - e^{\theta y}$$

so that we also find

$$\beta(y) = -\frac{1}{4D} (1 - e^{\theta y})^2 + \frac{1 + (\theta - 1)e^{\theta y}}{2} - \frac{D}{4}$$

Keeping then for $\bar{h}(r)$ and $\bar{W}_{st}(r)$ the same definitions of the Section 3.2.1, we have now

$$\begin{aligned} \beta(\bar{W}_{st}(r) + \bar{h}(r)) &= \frac{2 - D + 2(\theta - 1)x^{r\theta}y^{(1-r)\theta}e^{\theta\bar{W}_{st}(r)}}{4} - \frac{1}{4D} \left(1 - x^{r\theta}y^{(1-r)\theta}e^{\theta\bar{W}_{st}(r)} \right)^2 \end{aligned}$$

and hence we get

$$\begin{aligned} Z(s, t) &= -\frac{(1 - D)^2}{4D} + \frac{1 + (\theta - 1)D}{2D} y^\theta \int_0^1 \left(\frac{x}{y} \right)^{r\theta} e^{\theta\bar{W}_{st}(r)} dr - \frac{y^{2\theta}}{4D} \int_0^1 \left(\frac{x}{y} \right)^{2r\theta} e^{2\theta\bar{W}_{st}(r)} dr \end{aligned}$$

We thus find for the expectation factor in the transition *pdf* (155)

$$\begin{aligned} \mathbf{E} [e^{(t-s)Z(s,t)}] &= e^{-\frac{(1-D)^2}{4D}(t-s)} \mu_\theta(x, t; y, s) \\ \mu_\theta(x, t; y, s) &= \mathbf{E} \left[\exp \left\{ -\frac{t-s}{4D} \left(y^{2\theta} \int_0^1 \left(\frac{x}{y} \right)^{2r\theta} e^{2\theta\bar{W}_{st}(r)} dr \right. \right. \right. \\ &\quad \left. \left. \left. - 2(1 + (\theta - 1)D) y^\theta \int_0^1 \left(\frac{x}{y} \right)^{r\theta} e^{\theta\bar{W}_{st}(r)} dr \right) \right\} \right] \end{aligned}$$

As for the other factors in (155) we now have

$$\begin{aligned} \frac{1}{2D} \int_y^x \frac{a(z)}{b^2(z)} dz &= \frac{1}{2D} \left(\ln \frac{x}{y} - \frac{x^\theta - y^\theta}{\theta} \right) \\ -\frac{1}{4D(t-s)} \left(\int_y^x \frac{dz}{b(z)} \right)^2 &= -\frac{1}{4D(t-s)} \ln^2 \frac{x}{y} \end{aligned}$$

and therefore with a little algebra we find out for the transition *pdf*

$$f(x, t|y, s) = \frac{e^{-\frac{x^\theta - y^\theta}{2D\theta} - \frac{1}{4D(t-s)} \left((1-D)(t-s) - \ln \frac{x}{y} \right)^2}}{x\sqrt{4\pi D(t-s)}} \mu_\theta(x, t; y, s) \quad (58)$$

3.4 Integrals of a geometric Wiener process

In the Section 3.2 we have discussed the solutions of the *SDE* ruled by a logistic dynamics and we have found them contingent on processes basically of the type

$$X(t) = \int_0^t e^{W(s)} ds \quad (59)$$

where $W(t) \sim \mathfrak{N}(0, 2Dt)$ is a Wiener process with diffusion coefficient $2D$. The processes (59) are also known as *Exponential Functionals of Brownian Motion* and have been extensively studied in the financial context in a rather mathematical setting [4]. By postponing a more accurate analysis to a forthcoming paper, in the present section we will instead scrutinize the distribution of $X(t)$ with more elementary tools leading of course just to partial results.

3.4.1 Moments

We will begin by looking at the moments of (59)

$$\begin{aligned} M_n(t) = \mathbf{E} [X^n(t)] &= \mathbf{E} \left[\int_0^t ds_1 \dots \int_0^t ds_n e^{W(s_1) + \dots + W(s_n)} \right] \\ &= \int_0^t ds_1 \dots \int_0^t ds_n \mathbf{E} \left[e^{\sum_{k=1}^n W(s_k)} \right] \end{aligned} \quad (60)$$

Since it is

$$\mathbf{cov} [W(s), W(t)] = 2D (s \wedge t)$$

we have for every choice of t_1, \dots, t_n that

$$(W(t_1), \dots, W(t_n)) \sim \mathfrak{N}(\mathbf{0}, 2D \mathbb{A})$$

where $\mathbf{0} = (0, \dots, 0)$ and

$$\mathbb{A} = \begin{pmatrix} t_1 & t_1 \wedge t_2 & t_1 \wedge t_3 & \dots & t_1 \wedge t_n \\ t_2 \wedge t_1 & t_2 & t_2 \wedge t_3 & & t_2 \wedge t_n \\ t_3 \wedge t_1 & t_3 \wedge t_2 & t_3 & & t_3 \wedge t_n \\ \vdots & & & \ddots & \vdots \\ t_n \wedge t_1 & t_n \wedge t_2 & t_n \wedge t_3 & \dots & t_n \end{pmatrix}$$

As a consequence

$$\sum_{k=1}^n W(s_k) = (1, \dots, 1) \begin{pmatrix} W(s_1) \\ \vdots \\ W(s_n) \end{pmatrix} \sim \mathfrak{N} \left(0, 2D \sum_{j,k=1}^n (s_j \wedge s_k) \right)$$

and $e^{\sum_{k=1}^n W(s_k)}$ is the corresponding log-normal rv , so that

$$\mathbf{E} \left[e^{\sum_{k=1}^n W(s_k)} \right] = e^{D \sum_{j,k=1}^n (s_j \wedge s_k)} \quad (61)$$

which is apparently invariant under every permutation of the variables, while the moments are

$$M_n(t) = \int_0^t ds_1 \dots \int_0^t ds_n e^{D \sum_{j,k=1}^n (s_j \wedge s_k)} = \int_{[0,t]^n} ds_1 \dots ds_n e^{D \sum_{j,k=1}^n (s_j \wedge s_k)}$$

In this integral the variables s_k are not ordered, but we can go around this problem in the following way: consider the subset of $[0, t]^n$

$$B = \{(s_1, \dots, s_n) \in [0, t]^n : 0 \leq s_1 \leq \dots \leq s_n \leq t\}$$

where the variables are ordered according to their indices, and let \mathcal{P} be the family of the $n!$ permutations Π of s_1, \dots, s_n . The $n!$ subsets $\Pi(B)$ obtained by permutations of the variables in B are then such that

$$\bigcup_{\Pi \in \mathcal{P}} \Pi(B) = [0, t]^n$$

while on the other hand – because of the symmetry of (61) under permutations – all the integrals of (61) on every $\Pi(B)$ take the same value. We have then that

$$M_n(t) = n! \int_B ds_1 \dots ds_n e^{D \sum_{j,k=1}^n (s_j \wedge s_k)} = n! \int_0^t ds_n \dots \int_0^{s_3} ds_2 \int_0^{s_2} ds_1 e^{D \sum_{j,k=1}^n (s_j \wedge s_k)}$$

and since for $s_1 \leq s_2 \leq \dots \leq s_n$ it is

$$\begin{aligned} \sum_{j,k=1}^n (s_j \wedge s_k) &= \sum_{k=1}^n s_k + 2 \sum_{j < k} (s_j \wedge s_k) = \sum_{k=1}^n s_k + 2 \sum_{j=1}^{n-1} \sum_{k=j+1}^n (s_j \wedge s_k) \\ &= \sum_{k=1}^n s_k + 2 \sum_{j=1}^{n-1} \sum_{k=j+1}^n s_j = \sum_{k=1}^n s_k + 2 \sum_{j=1}^{n-1} (n-j) s_j \\ &= \sum_{k=1}^n s_k + 2 \sum_{j=1}^n (n-j) s_j = \sum_{k=1}^n [2(n-k) + 1] s_k \end{aligned}$$

we finally get

$$M_n(t) = n! \int_0^t ds_n \dots \int_0^{s_3} ds_2 \int_0^{s_2} ds_1 e^{D \sum_{k=1}^n [2(n-k)+1]s_k}$$

and with the changes of variables $v_k = Ds_k$

$$\begin{aligned} M_n(t) &= \frac{n!}{D^n} \int_0^t dv_n \dots \int_0^{v_3} dv_2 \int_0^{v_2} dv_1 e^{\sum_{k=1}^n [2(n-k)+1]v_k} \\ &= \frac{n!}{D^n} \int_0^t dv_n e^{v_n} \int_0^{v_n} dv_{n-1} e^{3v_{n-1}} \dots \int_0^{v_3} dv_2 e^{(2n-3)v_2} \int_0^{v_2} dv_1 e^{(2n-1)v_1} \end{aligned} \quad (62)$$

By looking now at the explicit calculations for the first few values of n

$$\begin{aligned} M_1(t) &= \frac{e^{Dt} - 1}{D} \\ M_2(t) &= \frac{e^{4Dt} - 4e^{Dt} + 3}{6D^2} \\ M_3(t) &= \frac{e^{9Dt} - 6e^{4Dt} + 15e^{Dt} - 10}{60D^3} \\ M_4(t) &= \frac{e^{16Dt} - 8e^{9Dt} + 28e^{4Dt} - 56e^{Dt} + 35}{840D^4} \dots \end{aligned}$$

we can conjecture the following general form for the moments of (59)

$$M_n(t) = (-1)^n \frac{n!}{D^n} \sum_{k=0}^n (-1)^k \frac{2 - \delta_{k0}}{(n-k)!(n+k)!} e^{k^2 Dt} \quad (63)$$

but we have yet no proof by recurrence and induction, so that (62) remains for the time being our last validated result

3.4.2 Characteristic function

Leaving aside for now every convergence question¹, starting from (63) we could surmise that the characteristic function of $X(t)$ takes the form

$$\begin{aligned} \varphi(u, t) &= \sum_{n=0}^{\infty} \frac{(iu)^n}{n!} M_n(t) = \sum_{n=0}^{\infty} \left(\frac{-iu}{D} \right)^n \sum_{k=0}^n (-1)^k \frac{2 - \delta_{k0}}{(n-k)!(n+k)!} e^{k^2 Dt} \\ &= \sum_{k=0}^{\infty} (-1)^k (2 - \delta_{k0}) e^{k^2 Dt} \sum_{n=k}^{\infty} \left(\frac{-iu}{D} \right)^n \frac{1}{(n-k)!(n+k)!} \end{aligned}$$

¹This is not at all a small detail, as it will be clear at the end of the present section. On the other hand it is known that this problem already exists for the lognormal distributions: while all the moments exist and are finite the generating function does not exist, and the characteristic function can not be represented as a convergent series [11]. This is related indeed to the fact that the lognormal distribution is not uniquely determined by its moments, and it would not be surprising then to find that this behavior extends also to the integrals of lognormal processes

and by changing the index n into $\ell = n - k$

$$\varphi(u, t) = \sum_{k=0}^{\infty} \left(\frac{iu}{D}\right)^k (2 - \delta_{k0}) e^{k^2 Dt} \sum_{\ell=0}^{\infty} \left(\frac{-iu}{D}\right)^\ell \frac{1}{\ell! (\ell + 2k)!}$$

It is known on the other hand that (see [7] 8.402)

$$\sum_{\ell=0}^{\infty} \frac{z^\ell}{\ell! (\ell + 2k)!} = (-z)^{-k} J_{2k}(2i\sqrt{z})$$

where $J_n(x)$ are the Bessel functions of the first kind, and hence

$$\varphi(u, t) = \sum_{k=0}^{\infty} (2 - \delta_{k0}) e^{k^2 Dt} J_{2k} \left(2i\sqrt{\frac{-iu}{D}} \right)$$

Since in general $\varphi(-u) = \overline{\varphi(u)}$, we can restrict ourselves to $u > 0$ and in this case we have

$$2i\sqrt{\frac{-iu}{D}} = \pm 2e^{i\frac{\pi}{4}} \sqrt{\frac{u}{D}} = \pm(1+i)\sqrt{\frac{2u}{D}} \quad u > 0$$

On the other hand we also know that $J_{2k}(-z) = J_{2k}(z)$ so that we finally have

$$\begin{aligned} \varphi(u, t) &= \sum_{k=0}^{\infty} (2 - \delta_{k0}) e^{k^2 Dt} J_{2k} \left((1+i)\sqrt{\frac{2u}{D}} \right) \\ &= 2 \sum_{k=1}^{\infty} e^{k^2 Dt} J_{2k} \left((1+i)\sqrt{\frac{2u}{D}} \right) + J_0 \left((1+i)\sqrt{\frac{2u}{D}} \right) \quad u > 0 \end{aligned} \quad (64)$$

This result looks however only formal because the presence of terms $e^{k^2 Dt}$ in the sums lends no hope for a convergence whatsoever. This was moreover a foregone conclusion since the (absolute) moments (63) utterly fail the convergence test for the Taylor expansion of the characteristic function: the moments (63) coincide indeed with the absolute moments because our rv 's are always positive (as for every exponential function), and hence the convergence of the Taylor expansion would require

$$\overline{\lim}_n \frac{\sqrt[n]{M_n(t)}}{n} < +\infty$$

while a few numerical trials show that the limit diverges for every choice of t and D

3.4.3 Finite sums of a geometric Wiener process

Since the process $X(t)$ in (59) is the integral of a geometric Wiener process, we could first of all investigate the laws of sums of a geometric Wiener process at different times. Let us begin with the simplest case

$$Z = e^{W(s)} + e^{W(t)} \quad s \leq t \quad (65)$$

by remarking first that the two log-normal rv 's $e^{W(s)}$ and $e^{W(t)}$ are not independent. The rv Z can however be put in the form of the product of two independent rv 's

$$Z = e^{W(s)} (1 + e^{W(t)-W(s)}) = XY$$

where we know that $X > 0$ is log-normal, while $Y > 1$ is a 1-shifted log-normal:

$$X = e^{W(s)} \sim \ln\mathfrak{N}(0, 2Ds) \quad Y - 1 = e^{W(t)-W(s)} \sim \ln\mathfrak{N}(0, 2D(t-s))$$

namely the pdf 's respectively are

$$f_X(x) = \frac{e^{-\frac{\ln^2 x}{4Ds}}}{x\sqrt{4\pi Ds}} \vartheta(x) \quad f_Y(y) = \frac{e^{-\frac{\ln^2(y-1)}{4D(t-s)}}}{(y-1)\sqrt{4\pi D(t-s)}} \vartheta(y-1) \quad (66)$$

$\vartheta(x)$ being the Heaviside function. To find the pdf $f_Z(z)$ of Z we could then remark that $\ln Z = \ln X + \ln Y$ is the sum of two independent rv 's where in particular $\ln X = W(s) \sim \mathfrak{N}(0, 2Ds)$. We could hence first calculate the pdf of $\ln Z$ as the convolution of the pdf 's of $W(s)$ and $\ln Y$, and then transform it back to the pdf of Z . It is important to remark however that $\ln Y$, as the logarithm of a 1-shifted log-normal, by no means is a normal rv as can be apparently argued from the simple remark that, being $Y > 1$, we always get $\ln Y > 0$. The pdf of $\ln Y$ can of course be explicitly calculated with the usual procedure, but it turns out to have an involuted form which makes hard to calculate the required convolution, and still harder to find back $f_Z(z)$. Alternatively we could try to directly calculate the pdf of the product of two non-negative, independent rv 's according to the following result

Proposition 3.1. *If $Z = XY$ is the product of two ac rv 's with joint pdf $f(x, y)$, then its pdf is*

$$f_Z(z) = \int_0^\infty \frac{dx}{x} \left[f\left(x, \frac{z}{x}\right) + f\left(-x, -\frac{z}{x}\right) \right] \quad (67)$$

and when in particular we take $X \geq 0, Y \geq 0$ it becomes

$$f_Z(z) = \vartheta(z) \int_0^\infty \frac{dx}{x} f\left(x, \frac{z}{x}\right) \quad (68)$$

Finally, if we also suppose that X, Y are independent with marginals $f_X(x)$ and $f_Y(y)$ the pdf is

$$f_Z(z) = \vartheta(z) \int_0^\infty \frac{dx}{x} f_X(x) f_Y\left(\frac{z}{x}\right) \quad (69)$$

Proof: Starting from the cdf of Z we have

$$F_Z(z) = \mathbf{P}\{Z \leq z\} = \mathbf{P}\{XY \leq z\} = \iint_{D_z} f(x, y) dx dy$$

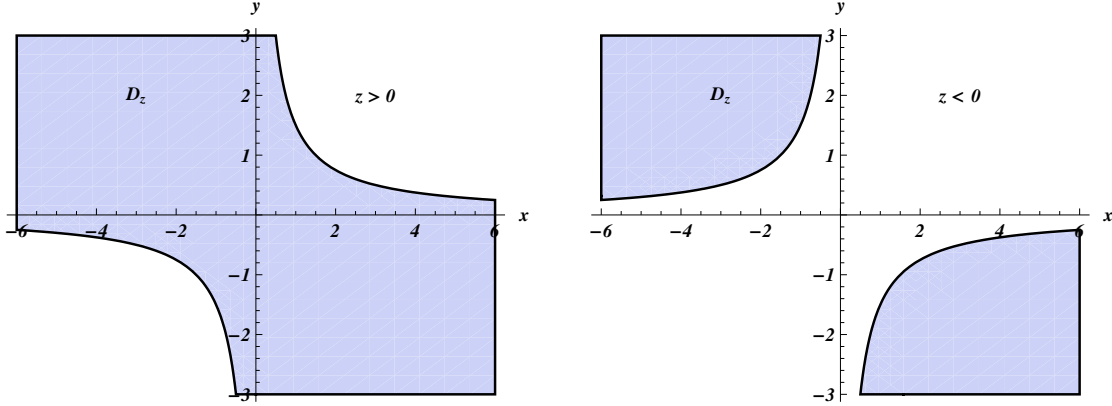


Figure 2: The shadowed regions show the integration domain D_z

where $D_z = \{(x, y) \in \mathbf{R}^2 : xy \leq z\}$. Looking at the Figure 2 we see then that

$$\begin{aligned} F_Z(z) &= \int_{-\infty}^0 dx \int_{z/x}^{\infty} dy f(x, y) + \int_0^{\infty} dx \int_{-\infty}^{z/x} dy f(x, y) \\ &= \int_0^{\infty} dx \int_{-z/x}^{\infty} dy f(-x, y) + \int_0^{\infty} dx \int_{-\infty}^{z/x} dy f(x, y) \\ &= \int_0^{\infty} dx \int_{-\infty}^{z/x} dy f(-x, -y) + \int_0^{\infty} dx \int_{-\infty}^{z/x} dy f(x, y) \end{aligned}$$

and by introducing a new variable $u = xy$

$$\begin{aligned} F_Z(z) &= \int_0^{\infty} dx \int_{-\infty}^z \frac{du}{x} f\left(-x, -\frac{u}{x}\right) + \int_0^{\infty} dx \int_{-\infty}^z \frac{du}{x} f\left(x, \frac{u}{x}\right) \\ &= \int_{-\infty}^z du \int_0^{\infty} \frac{dx}{x} \left[f\left(x, \frac{u}{x}\right) + f\left(-x, -\frac{u}{x}\right) \right] \end{aligned}$$

so that at once we get the *pdf* (70), while the other two formulas (68) and (69) immediately follow ■

From (69) and (66) we then have the following *pdf* for Z

$$f_Z(z) = \vartheta(z) \int_0^z \frac{e^{-\frac{\ln^2 x}{4Ds}}}{x\sqrt{4\pi Ds}} \frac{e^{-\frac{[\ln(z-x)-\ln x]^2}{4D(t-s)}}}{(z-x)\sqrt{4\pi D(t-s)}} dx \quad (70)$$

which again is not an easy calculation to perform, even if it looks tantalizingly near to an explicit answer. Numerical integration shows that (70) is correctly normalized, and numerical plots in Figure 3 display a very reasonable behavior confirming that our calculation is so far acceptable: that notwithstanding, the unavailability of a complete result for such a simple case as the *rv* Z in (65) also uphold the view that finding the law of $X(t)$ in (59) is a problem hard to crack

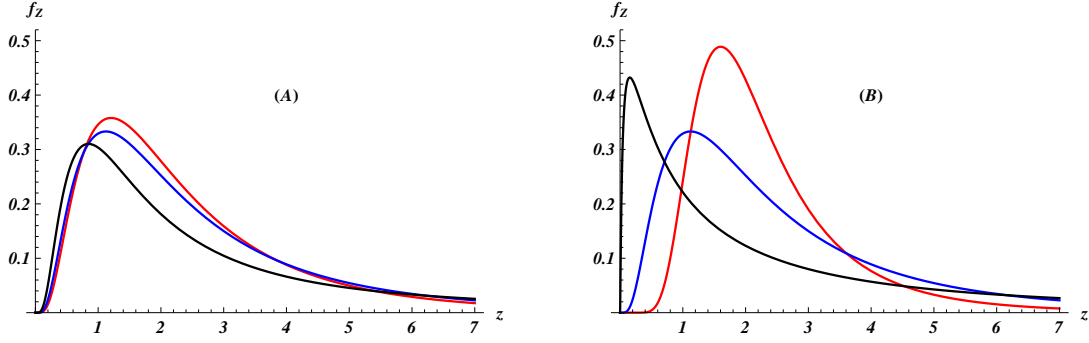


Figure 3: Numerical instances of the *pdf* $f_Z(z)$ in (70): **(A)** $4Ds = 1.0$, while $4D(t - s) = 0.1$ (red) 1.0 (blue) 8.0 (black); **(B)** $4D(t - s) = 1$, while $4Ds = 0.2$ (red) 1.0 (blue) 5.0 (black)

3.4.4 Differential equations

When an *ac* process $X(t)$ is solution of a *SDE*

$$dX(t) = a(X(t), t) dt + b(X(t), t) dW(t) \tag{71}$$

then $X(t)$ is Markovian, with almost every trajectory everywhere continuous, and its *pdf* $f_X(x, t)$ is solution of a Fokker-Planck equation

$$\partial_t f_X(x, t) = -\partial_x [A(x, t) f_X(x, t)] + \frac{1}{2} \partial_x^2 [B(x, t) f_X(x, t)] \tag{72}$$

where

$$a(x, t) = A(x, t) \quad B(x, t) = 2Db^2(x, t)$$

In this case it also satisfies a *continuity equation* which represents a requirement of probability conservation: we can indeed immediately recast (72) into the form

$$\partial_t f_X(x, t) + \partial_x [v(x, t) f_X(x, t)] = 0 \tag{73}$$

provided that

$$v(x, t) = A(x, t) - \frac{\partial_x [B(x, t) f_X(x, t)]}{2f_X(x, t)} \tag{74}$$

It is apparent however that in the present context the continuity equation (73) is not a new equation really different from the Fokker-Planck equation (72), and this is made clear in particular by the fact that the velocity field (74) is contingent on the solution $f_X(x, t)$ of (72): in other words here $v(x, t)$ does not represent an external, given field but depends on the solution $f_X(x, t)$ so that (if $A(x, t)$ and $B(x, t)$ are given) we can directly calculate v from f_X , and conversely f_X from v

Not every stochastic process, however, is Markovian, and in particular $X(t)$ defined in (59) is not. In this case neither the process trajectories will satisfy a *SDE* of the type (71), nor its *pdf* will be a solution of a *PDE* of the type (72). This is quite understandable, and in fact every quest for some other kind (for instance) of *PDE*, even if possible, is doomed to futility: since $X(t)$ is not Markovian, in order to find the law of the process it would be far from enough to know its one time *pdf* $f_X(x, t)$ together with its transition *pdf* $f_X(x, t|y, s)$. We would need instead the knowledge of every joint *pdf* $f_X(x_1, t; \dots; x_n, t_n)$ that in any case could not be extracted from a single *PDE*.

On the other hand the conservation of the probability should be guaranteed in any case, and hence the *pdf* $f_X(x, t)$ is supposed to satisfy some kind of continuity equation of the type (73), but for the fact that now this continuity equation can no longer be derived from a corresponding *FPE*. We must at once remark, however, that (73) in no way can surrogate the role of a *FPE*: first of all its possible solutions will not constitute the basis to build the process laws; and furthermore – as we already have remarked – $v(x, t)$ is not a given function independent from the solution. In general the connection between $v(x, t)$ and $f_X(x, t)$ will not be as simple as (74), but in any case every possible solution will be associated to its own velocity field.

As a matter of fact, however, when the process is not Markovian we should give up our old habit of thinking to the *different processes* selected by different initial conditions for a single transition *pdf* as to a *unified process*: now every global law (represented by the said joint *pdf*'s $f_X(x_1, t; \dots; x_n, t_n)$) defines a different process and we do not see in general a way to detect homogeneous classes among them. This means, among others, that for every continuity equation with a given $v(x, t)$ there will be just one possible solution of interest for us, and that a family of processes could be located only through their mating with the velocity fields

Two remarks are in order here: first, we could revert to our initial, more narrow aim of finding just the law of the *rv*'s $X(t)$ in (59) and not the global law of the geometric Wiener process that these *rv*'s represent for $t > 0$. We have found however that even restricting the scope of our enquiry to this carefully circumscribed problem will not make easy to pick up a meaningful solution. Second, we could try to circumvent the non Markovianity of $X(t)$ in a way reminiscent of an Ornstein-Uhlenbeck procedure: the process $X(t)$ is apparently derivable with $\dot{X}(t) = e^{W(t)} = Y(t)$ so that its stochastic differential is

$$dX(t) = e^{W(t)} dt = Y(t) dt$$

This however does not constitute a *SDE*, and hence the *pdf* of $X(t)$ (which arguably is non Markovian, as in the Ornstein-Uhlenbeck case) does not satisfy a Fokker-Planck equation. The process $Y(t) = e^{W(t)}$ on the other hand – from the Itô formula – is a solution of the *SDE* for a geometric Wiener process

$$dY(t) = DY(t) dt + Y(t) dW(t)$$

and hence it is Markovian so that its *pdf* $f_Y(y, t)$ obeys the corresponding Fokker-Planck equation

$$\begin{aligned} \partial_t f_Y(y, t) &= -D\partial_x [y f_Y(y, t)] + D\partial_y^2 [y^2 f_Y(y, t)] \\ &= Dy^2 \partial_Y^2 f_Y(y, t) + 3D\partial_y f_Y(y, t) + Df_Y(y, t) \end{aligned} \quad (75)$$

The pair $X(t), Y(t)$ will thus satisfy the system

$$\begin{cases} dX(t) &= Y(t) dt \\ dY(t) &= DY(t) dt + Y(t) dW(t) \end{cases}$$

If then we define the vector process

$$\mathbf{Z}(t) = \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}$$

it will be a solution of the two-components, vector *SDE*

$$d\mathbf{Z}(t) = \mathbf{a}(\mathbf{Z}(t)) dt + \mathbb{C}(\mathbf{Z}(t)) d\mathbf{W}(t) \quad (76)$$

where

$$\mathbf{a}(\mathbf{z}) = \mathbf{a}(x, y) = \begin{pmatrix} y \\ Dy \end{pmatrix} \quad \mathbb{C}(\mathbf{z}) = \mathbb{C}(x, y) = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}$$

while the vector Wiener process can be taken as

$$\mathbf{W}(t) = \begin{pmatrix} W_X(t) \\ W(t) \end{pmatrix}$$

$W_X(t)$ being any auxiliary Wiener process apparently playing no role in the discussion of the *SDE* (76). The vector process $\mathbf{Z}(t)$ is then Markovian and to its vector *SDE* (76) it is possible to associate a (2+1)-dimensional Fokker-Planck equation for the *pdf* $f(\mathbf{z}, t) = f(x, y, t)$: solving this equation would lead in principle to the complete law of the process $\mathbf{Z}(t)$ and hence, by marginalization, to the much sought-after law of its component $X(t)$

Proposition 3.2. *The Fokker-Planck equation of the process $\mathbf{Z}(t)$ is*

$$\partial_t f(x, y, t) = D\partial_y^2 [y^2 f(x, y, t)] - y\partial_x f(x, y, t) - D\partial_y [yf(x, y, t)] \quad (77)$$

while the marginal *pdf* $f_X(x, t)$ of the process $X(t)$ obeys the continuity equation

$$\partial_t f_X(x, t) + \partial_x [v(x, t)f_X(x, t)] = 0 \quad (78)$$

where the velocity field is

$$v(x, t) = \mathbf{E} \left[\dot{X}(t) \mid X(t) = x \right] = \mathbf{E} \left[e^{W(t)} \left| \int_0^t e^{W(s)} ds = x \right. \right] \quad (79)$$

Proof: The vector process $\mathbf{Z}(t)$ is Markovian and to its vector *SDE* (76) it is possible to associate a (2+1)-dimensional Fokker-Planck equation for the *pdf* $f(\mathbf{z}, t) = f(x, y, t)$ with the coefficients

$$\mathbf{A}(\mathbf{z}) = \mathbf{a}(\mathbf{z}) = \begin{pmatrix} y \\ Dy \end{pmatrix} \quad \mathbb{B}(\mathbf{z}) = 2D \mathbb{C}(\mathbf{z})\mathbb{C}^T(\mathbf{z}) = 2D \begin{pmatrix} 0 & 0 \\ 0 & y^2 \end{pmatrix}$$

giving rise to

$$\begin{aligned} \partial_t f(x, y, t) &= D\partial_y^2 [y^2 f(x, y, t)] - \partial_x [yf(x, y, t)] - \partial_y [Dyf(x, y, t)] \\ &= Dy^2\partial_y^2 f(x, y, t) - y[\partial_x f(x, y, t) - 3D\partial_y f(x, y, t)] + Df(x, y, t) \end{aligned}$$

namely (180). This *PDE* essentially is confined to the quadrant $x > 0, y > 0$ because the processes $X(t)$ and $Y(t)$ are positive and never vanish. When the *pdf* $f(x, y, t)$ has been found, we can calculate the marginals $f_X(x, t)$ and $f_Y(y, t)$ as

$$f_X(x, t) = \int_0^{+\infty} f(x, y, t) dy \quad f_Y(y, t) = \int_0^{+\infty} f(x, y, t) dx$$

As a matter of fact, however, from the laws of a Wiener process $W(t) \sim \mathfrak{N}(0, 2Dt)$ we already know that the log-normal *pdf* of $Y(t)$ which is

$$f_Y(y, t) = \frac{e^{-\frac{\ln^2 y}{4Dt}}}{y\sqrt{4\pi Dt}} \quad (80)$$

and hence we can also perform a first check of the coherence of our joint equation (180): by x -marginalization of (180) we indeed have

$$\begin{aligned} \partial_t f_Y(y, t) &= Dy^2\partial_y^2 f_Y(y, t) + 3Dy\partial_y f_Y(y, t) + Df_Y(y, t) - y \int_0^{+\infty} \partial_x f(x, y, t) dx \\ &= Dy^2\partial_y^2 f_Y(y, t) + 3Dy\partial_y f_Y(y, t) + Df_Y(y, t) - y [f(x, y, t)]_{x=0}^{x=+\infty} \\ &= Dy^2\partial_y^2 f_Y(y, t) + 3Dy\partial_y f_Y(y, t) + Df_Y(y, t) + yf(0, y, t) \end{aligned} \quad (81)$$

that coincides with (75) provided that $f(0, y, t) = 0$, as it is reasonable to require. On the other hand, by taking (80) into account, we also have by direct calculation that

$$\frac{\partial_t f_Y(y, t)}{f_Y(y, t)} = \frac{\ln^2 y - 2Dt}{4Dt^2} = \frac{Dy^2\partial_y^2 f_Y(y, t) + 3Dy\partial_y f_Y(y, t)}{f_Y(y, t)} + D$$

so that the *pdf* (80) of $Y(t)$ actually is a solution of the x -marginalized equation (81). In the same vein we can then study the y -marginalized equation of (180): let us first remark that

$$[yf(x, y, t)]_{y=0}^{y=\infty} = [y^2 f(x, y, t)]_{y=0}^{y=\infty} = 0$$

because all the moments of a log-normal distribution (which is the marginal of $f(x, y, t)$) are finite so that $y^n f(x, y, t)$ must be infinitesimal for $y \rightarrow +\infty$, while

$f(x, y, t)$ can diverge in $y \rightarrow 0$ to an order strictly lesser than 1. As a consequence by y -marginalizing (180) with integrations by part (the finite terms vanish) we have the continuity equation (78)

$$\begin{aligned} \partial_t f_X(x, t) &= D \int_0^\infty y^2 \partial_y^2 f(x, y, t) dy - \partial_x \int_0^\infty y f(x, y, t) dy \\ &\quad + 3D \int_0^\infty y \partial_y f(x, y, t) dy + D f_X(x, t) \\ &= 2D f_X(x, t) - \partial_x \left[f_X(x, t) \int_0^\infty y \frac{f(x, y, t)}{f_X(x, t)} dy \right] \\ &\quad - 3D f_X(x, t) + D f_X(x, t) \\ &= -\partial_x [v(x, t) f_X(x, t)] \end{aligned}$$

where $v(x, t)$ is defined in (79) ■

All this is hardly surprising: we would get the same continuity equation for the vector process constituted by the pair position/velocity of a Brownian motion in the Ornstein-Uhlenbeck dynamical model. The continuity equation (78) however is not very useful for us because the velocity field $v(x, t)$ is in some sense dependent from the form of the solution: the defining conditional expectation (79) is indeed calculated with a law also involving the marginal $f_X(x, t)$. In other words a well behaved *pdf* always satisfies a continuity equation when the velocity field is rightly defined as in (79), and hence it expresses a consistence requirement, rather than a true equation ... unless you already know the (well behaved) velocity field, and hence the solution. On the other hand we do not have here an explicit expression of $v(x, t)$, but it would be interesting to remark here that its definition (79) seems to hint to the need of some kind of *mean-conditioning* since after all the condition is expressed in terms of a sum (integral) of *rv*'s of the same kind of the averaged one: this is a point worth of a further enquiry

Alternatively, by postponing every marginalization, we could try first to solve the joint Fokker-Planck equation (180): separating the variables with $f(x, y, t) = g(x, y)h(t)$ we have

$$\frac{\dot{h}(t)}{h(t)} = \frac{Dy^2 \partial_y^2 g(x, y) - y [\partial_x g(x, y) - 3D \partial_y g(x, y)] + Dg(x, y)}{g(x, y)} = \lambda$$

and hence

$$\begin{aligned} \dot{h}(t) &= \lambda h(t) \\ Dy^2 \partial_y^2 g(x, y) - y [\partial_x g(x, y) - 3D \partial_y g(x, y)] + (D - \lambda)g(x, y) &= 0 \end{aligned}$$

Take then $g(x, y) = u(x)v(y)$ to have

$$\frac{u'(x)}{u(x)} = Dy \frac{v''(y)}{v(y)} + 3D \frac{v'(y)}{v(y)} + \frac{D - \lambda}{y} = -\mu$$

and finally with $f(x, y, t) = u(x)v(y)h(t)$

$$\begin{aligned}\dot{h}(t) &= \lambda h(t) \\ u'(x) &= -\mu u(x) \\ Dy^2 v''(y) + 3Dy v'(y) + (D - \lambda + \mu y)v(y) &= 0\end{aligned}$$

From the first two equations we simply have

$$h(t) = ae^{\lambda t} \quad u(x) = be^{-\mu x}$$

while the third can be written as

$$v''(y) + \frac{3}{y} v'(y) + \left(\frac{D - \lambda}{Dy^2} + \frac{\mu}{Dy} \right) v(y) = 0$$

Comparing now this equation with the Bessel equation 8.491.12 in [7], we see that within the notations adopted there we must require

$$2\alpha - 2\beta\nu + 1 = 3 \quad \beta = \frac{1}{2} \quad \beta^2\gamma^2 = \frac{\mu}{D} \quad \alpha(\alpha - 2\beta\nu) = \frac{D - \lambda}{D}$$

namely

$$\alpha = 1 \pm \sqrt{\frac{\lambda}{D}} \quad \beta = \frac{1}{2} \quad \gamma = \pm 2\sqrt{\frac{\mu}{D}} \quad \nu = \pm 2\sqrt{\frac{\lambda}{D}}$$

so that the solutions will have the form

$$v(y) = \frac{1}{y} Z_{\pm 2\sqrt{\frac{\lambda}{D}}} \left(\pm 2\sqrt{\frac{\mu y}{D}} \right)$$

where the symbol $Z_\nu(z)$ denotes one of the Bessel functions $J, N, H^{(1)}, H^{(2)}$, as any linear combination of them. This seems to confirm the relation with the Bessel functions found in the Section 3.4.2.

Several remarks, however, are in order: first, to keep α and ν real we must require $\lambda \geq 0$; on the other hand if $\lambda > 0$ the factor $h(t)$ will result in a time-exploding term, so that the most reasonable option seems to be $\lambda = 0$ (stationary solution). This choice would result in the $Z_0(z)$ Bessel functions, but empirical evidence (from *Mathematica*) seems to imply that either $y^{-1}Z_0(\sqrt{y})$ is not always non-negative, or it diverges for $z \rightarrow +\infty$, and in any case it diverges for $y \rightarrow 0$ in a non integrable way so that the normalization of these functions appears to be hopeless. For example, the unique Bessel function giving rise to non negative, asymptotically infinitesimal solutions is $K_0(z)$, but $y^{-1}K_0(\sqrt{y})$ diverges for $y \rightarrow 0$ at an order $1 + \epsilon$ with $\epsilon > 0$ arbitrarily small. This is in any case coherent with the remark that a stationary solution is hardly conceivable for processes based on the exponentials of a Wiener process

4 Conclusions and outlooks

The present paper summarizes both a rather conclusive discussion of the solutions of the Gompertz *SDE* (7), and several partial results related to the solutions of the logistic *SDE* (9). By postponing to future enquiries the completion of this program, the definition of a viable deterministic coarse-graining of these equations and a connection between their solutions and the Nelson stochastic mechanics recalled in the Appendix D, we will end our discussion by listing here a few among the many points that would deserve a further elaboration

1. **Random parametric Gomperts *SDE*'s:** The parametric Gompertz *SDE* (19) with a time-dependent frequency $\alpha(t)$ is associated to the parametric *OU SDE* (18): ask then what happens when $\alpha(t)$ is also random, for instance of the type $\alpha(t) = \alpha_0(1+U(t))$ where $U(t)$ is a suitable external process (for instance either another Wiener process independent from $W(t)$, or $W(t)$ itself). The case of random coefficients can be compared to the case of systems of *SDE* where the second *SDE* defines the new process $U(t)$. The two equations can be either coupled, or uncoupled
2. **Modified Gompertzian growth:** Can we devise some modified Gompertz equation giving rise – beside the usual growth – either to oscillations, or to some decrease to some other stable level? This could be done by means of two mechanisms: either an external forcing term inscribed in the time depending (but not necessarily random) coefficients; or some shrewd random term (as in the previous point 1) ruling, for instance, in an unpredictable way the sign of the exponentials of the Gompertz functions. This enquiry could lead to compare these systems with the random *Lotka-Volterra* systems where the oscillations are induced by a coupling in a system of equations describing populations: should we think, then, to coupled Gompertzian systems?
3. **Two kinds of deterministic correspondence:** Take the *SDE*

$$dX(t) = a(X(t), t)dt + b(X(t), t)dW(t)$$

we can recover a deterministic equation in two, non-equivalent ways: either we can just consider that the diffusion coefficient D vanishes, drop the random term and consider the *ODE*²

$$\dot{x}(t) = a(x(t), t)$$

or we can take the expectation of the *SDE*

$$\frac{d\mathbf{E}[X(t)]}{dt} = \mathbf{E}[a(X(t), t)]$$

²This in some sense reverses the usual procedures leading to the Langevin equation starting from a Newton equation

which is rather different from the previous one because $\mathbf{E}[a(X(t), t)]$ is not $a(\mathbf{E}[X(t)], t)$, but for the linear case. The previous ambiguity is not relevant indeed for linear *SDE* of the Smoluchowsky type

$$dX(t) = [q(t) - p(t)X(t)]dt + dW(t)$$

whose deterministic counterpart in both the cases is

$$\dot{x}(t) + p(t)x(t) = q(t), \quad x(t) = \mathbf{E}[X(t)]$$

namely the most general, first order, linear *ODE* whose solutions are completely known. On the other hand also the general solution of the corresponding *SDE* is completely known, so that it would be telling to compare first these two solutions. Contrariwise the problem for the non linear $a(x, t)$ still stay with us

4. **Equations for the medians $\mathbf{M}[X(t)]$:** In the quest for the deterministic counterpart of a *SDE* it could be useful to look to the medians (see Appendix E) rather than to the expectations. It is not easy however to find the equations for the medians of a process satisfying even the simple linear *SDE*

$$dY(t) = [b(t) - a(t)Y(t)]dt + dW(t)$$

because in general the medians of sums are not sums of the medians

5. **The $n = 1$ state of a quantum harmonic oscillator:** In the Appendix D we explicitly solved the *FPE* associated by the stochastic mechanics to the $n = 1$ eigenstate of a quantum harmonic oscillator, but at the same time we were not able to solve the corresponding *SDE*. As a matter of fact we were not able to find a process displaying as a transition *pdf* that derived from the harmonic oscillator solution. The problem is that this solution looks like a mixture (and even in this case: what about the *rv* with a mixture law in terms of the *rv*'s obeying the separate mixing laws?), but in fact it is not a true mixture because the coefficients sum up to 1, but are not in the range $[0, 1]$ as should be for every convex combination. It looks then more as an affine combination, but nothing seems to be known about this case. Alternatively we could consider our solution as a convex combination of *pdf* taking also negative values. Either way we seem to be obliged to take seriously the existence of (at least virtual) negative probabilities [19, 20, 21] and follow this path to its bitter end trying to guess what kind of *rv*'s – if any – can be distributed in this way
6. **Logistic systems:** Finally, even if we choose to focus our attention on the more manageable Gompertz systems, at least from a mathematical standpoint it would be relevant to complete the investigation about the solutions of the

Logistic equations that we gave in two different explicit ways (32) and (35): these, however, are for the time being more formal than substantial because, as we pointed out in the corresponding sections, we did not yet explicitly elaborate the laws of the involved rv 's that will be instead scrutinized in a forthcoming paper

A Deterministic Gompertz and logistic equations

In the present appendix we will briefly recall the deterministic variants of the *SDE*'s presented all along the present paper: we refer in particular to the dimensionless *SDE*'s (7), (9) and (11). These deterministic versions are attained simply by switching off the Wienerian noise $W(t)$ of the said *SDE*'s (namely by taking $D = 0$), and are first order *ODE*'s (*ordinary differential equations*) of the type

$$\dot{x}(t) = v[x(t), t]$$

Their solutions will be used in the present paper as a benchmark to check the consistency of the solutions of the corresponding *SDE*'s. The reason why in the said *SDE*'s the noise turns out to be also proportional to the process itself will be addressed in a forthcoming paper devoted to the deterministic and stochastic models of population dynamics

A.1 Gompertz equation

The deterministic version of the Gompertz *SDE* (7) apparently is

$$\dot{x}(t) = x(t)(1 - \alpha \ln x(t))$$

that, with an initial condition $x(0) = x_0$, is easily integrated by separating the variables providing the solution

$$x(t) = x_0^{e^{-\alpha t}} e^{(1-e^{-\alpha t})/\alpha} \quad (82)$$

that is a monotonic function starting from x_0 and asymptotically relaxing to the limiting value $e^{1/\alpha}$. This asymptotic value, however, is always larger than 1 because we require $\alpha > 0$: we could nevertheless outflank this limitation by adding a new parameter β into the equation

$$\dot{x}(t) = x(t)(\beta - \alpha \ln x(t))$$

which now has the solution

$$x(t) = x_0^{e^{-\alpha t}} e^{\beta(1-e^{-\alpha t})/\alpha}$$

with an asymptotic value $e^{\beta/\alpha}$ that can now be both larger and smaller than 1 according to the sign of β

A.2 Logistic and θ -logistic equations

The logistic *ODE* associated to the *SDE* (9) is

$$\dot{x}(t) = x(t)(1 - x(t))$$

with a $v(x) = x(1 - x)$, and its solution with the initial condition $x(0) = x_0$ is again retrieved by separating the variables:

$$x(t) = \frac{x_0}{x_0 + (1 - x_0)e^{-t}} \tag{83}$$

Here too we are dealing with a function monotonically going, for $t \rightarrow +\infty$, from an arbitrary x_0 to 1. The θ -logistic *ODE* is finally its generalization with $\theta > 0$

$$\dot{x}(t) = x(t)(1 - x^\theta(t))$$

whose solution is easily seen to be now

$$x(t) = \left(\frac{x_0^\theta}{x_0^\theta + (1 - x_0^\theta)e^{-\theta t}} \right)^{1/\theta} \tag{84}$$

with the same qualitative behavior as before

B Solving stochastic differential equations

B.1 An epitome of Itô calculus

First of all let us recall in a simplified, mnemonic form a few results of the Itô calculus that can be found for example in [12], p. 11-27. Leaving aside a rigorous presentation, we will work here in the setting of the *stochastic differentials (sdif)* whose precise meaning must be retrieved from their role and use in the standard definition of the Itô integral

$$\int_a^b X(t) dW(t)$$

where $W(t)$ is a Wiener process with diffusion coefficient $2D$, while $X(t)$ is a well behaved processes non-anticipative w.r.t. $W(t)$ [12]. In this framework the most relevant innovation w.r.t. the usual differential calculus is the fact that the differential $dW(t)$ of the Wiener process no longer can be deemed to be of the order dt , but we will have instead

$$[dW(t)]^2 = 2Ddt \quad \mathbf{E} [dW(t)] = 0 \quad \mathbf{E} [dW(t)dW(s)] = 2D\delta(s - t) ds dt \tag{85}$$

that are shorthand, symbolic notations for the following integral results:

$$\begin{aligned} \int_a^b X(t) [dW(t)]^2 &= 2D \int_a^b X(t) dt \\ \mathbf{E} \left[\int_a^b X(t) dW(t) \right] &= 0 \\ \mathbf{E} \left[\int_a^b X(t) dW(t) \int_a^b Y(s) dW(s) \right] &= 2D \int_a^b \mathbf{E} [X(t)Y(t)] dt \end{aligned}$$

Further conceivable differentials, as for instance

$$dW(t)dt \quad [dW(t)]^2dt \quad dW(t)(dt)^2 \quad \dots \quad (86)$$

will instead behave as higher order infinitesimals and will then be treated as zero in the sense that the corresponding Itô integrals vanish. These simple rules will be enough to formally deduce all the relevant results needed in our discussion.

In the following our one-dimensional processes $X(t), Y(t) \dots$ defined for $t \in [0, T]$ will be taken in H_2 , namely in the space of the processes such that

$$\mathbf{P} \left\{ \int_0^T X^2(t) dt < +\infty \right\} = 1$$

in order to make sure that – within a framework of suitable hypotheses pointed out in [12] – the Itô integral is well defined, and that in general our processes will have (again symbolically, as a surrogate for Itô integral expressions) *sdif*'s of the type

$$dX(t) = A(t)dt + B(t)dW(t) \quad (87)$$

where again the coefficients $\sqrt{|A(t)|}$ and $B(t)$ are taken in H_2 : of course the Wiener process $W(t)$ is retrieved when $A = 0$ and $B = 1$. Remark that when a process has the *sdif* (87) its infinitesimals follow rules similar to that of the Wiener process; in particular, as can be easily seen from (85) and (86), we have

$$[dX(t)]^2 = 2DB^2(t)dt \quad (88)$$

while further powers of $dX(t)$ and dt would be higher order infinitesimals, and hence will be neglected. In this context we also say that, with suitable initial conditions, a process $X(t)$ satisfies a *stochastic differential equations (SDE)*

$$dX(t) = a(X(t), t) dt + b(X(t), t) dW(t) \quad (89)$$

when for every $t \in [0, T]$ it has the previous *sdif* whose coefficients $A(t) = a(X(t), t)$ and $B(t) = b(X(t), t)$, \mathbf{P} -*a.s.* are contingent on the process $X(t)$ himself. In particular when the functions $a(x)$ and $b(x)$ are time-independent the *SDE*

$$dX(t) = a(X(t)) dt + b(X(t)) dW(t) \quad (90)$$

may also admit stationary solutions

Proposition B.1. (Itô formula) *If $X(t)$ has the *sdif* (87) and $h(x, t)$ is a fairly differentiable function, then the process $Y(t) = h(X(t), t)$ will have the *sdif**

$$\begin{aligned} dY(t) &= dh(X(t), t) \\ &= [h_t(X(t), t) + h_x(X(t), t) A(t) + D h_{xx}(X(t), t) B^2(t)] dt \\ &\quad + h_x(X(t), t) B(t) dW(t) \end{aligned} \quad (91)$$

When in particular $h(x)$ is time independent the Itô formula becomes

$$\begin{aligned} dY(t) &= dh(X(t)) \\ &= [h'(X(t)) A(t) + D h''(X(t)) B^2(t)] dt + h'(X(t)) B(t) dW(t) \end{aligned} \quad (92)$$

Proof: The formula (92) can be proved by using the Taylor formula for h together with (88), and neglecting the infinitesimals of order larger than 1:

$$\begin{aligned}
 dh(X(t)) &= h(X(t+dt)) - h(X(t)) = h(X(t) + dX(t)) - h(X(t)) \\
 &= \sum_{k \geq 0} \frac{h^{(k)}(X(t))}{k!} [dX(t)]^k - h(X(t)) \\
 &= h(X(t)) + h'(X(t))dX(t) + \frac{1}{2}h''(X(t)) [dX(t)]^2 \\
 &\quad + \sum_{k > 2} \frac{h^{(k)}(X(t))}{k!} [dX(t)]^k - h(X(t)) \\
 &= h'(X(t)) [A(t)dt + B(t)dW(t)] + h''(X(t)) DB^2(t)dt \\
 &= [h'(X(t))A(t) + D h''(X(t))B^2(t)] dt + h'(X(t))B(t) dW(t)
 \end{aligned}$$

The more general (91) can be proved – with minor additions – along the same lines. These results coincide with those of the usual calculus but for the second derivative terms that show up as soon as the diffusion coefficient D does not vanish ■

Proposition B.2. *Take two sdif's on the same Wiener process $W(t)$*

$$dX(t) = A_X(t)dt + B_X(t)dW(t) \qquad dY(t) = A_Y(t)dt + B_Y(t)dW(t)$$

then for their product we have the rule

$$d [X(t)Y(t)] = X(t) dY(t) + Y(t) dX(t) + 2B_X(t)B_Y(t)D dt \tag{93}$$

Proof: To prove (93) it would be enough to use (85), by neglecting other higher order infinitesimals:

$$\begin{aligned}
 d [X(t)Y(t)] &= X(t+dt)Y(t+dt) - X(t)Y(t) \\
 &= [X(t) + dX(t)][Y(t) + dY(t)] - X(t)Y(t) \\
 &= X(t)dY(t) + Y(t)dX(t) + dX(t)dY(t) \\
 &= X(t)dY(t) + Y(t)dX(t) + B_X(t)B_Y(t)[dW(t)]^2 \\
 &= X(t)dY(t) + Y(t)dX(t) + 2B_X(t)B_Y(t)Ddt
 \end{aligned}$$

which again shows an extra term with respect to the usual calculus. ■

B.2 SDE transformations

When a process $X(t)$ satisfies the *SDE* (181) we can often exploit the Itô formula (91) to transform this *SDE* in a more manageable form (see [12] p 33-39): take a fairly differentiable, monotonic (in x) function $y = h(x, t)$ and for every t denote $x = g(y, t)$ its spatial inverse, namely

$$h(g(y, t), t) = y \qquad g(h(x, t), t) = x$$

Then, if $X(t)$ satisfies the *SDE* (181) the Itô formula (91) implies that the transformed process $Y(t) = h(X(t), t)$ will satisfy a new *SDE*

$$dY(t) = \widehat{a}(Y(t), t) dt + \widehat{b}(Y(t), t) dW(t) \quad (94)$$

where now

$$\widehat{a}(y, t) = [h_t(x, t) + h_x(x, t)a(x, t) + D h_{xx}(x, t)b^2(x, t)]_{x=g(y, t)} \quad (95)$$

$$\widehat{b}(y, t) = [h_x(x, t)b(x, t)]_{x=g(y, t)} \quad (96)$$

or else, in an equivalent form,

$$\widehat{a}(h(x, t), t) = h_t(x, t) + h_x(x, t)a(x, t) + D h_{xx}(x, t)b^2(x, t) \quad (97)$$

$$\widehat{b}(h(x, t), t) = h_x(x, t)b(x, t) \quad (98)$$

When on the other hand $X(t)$ satisfies the *SDE* (90) with time-independent coefficients, the transformation $Y(t) = h(X(t))$ with monotonic $y = h(x)$ and $x = g(y) = h^{-1}(y)$ leads the *SDE*

$$dY(t) = \widehat{a}(Y(t)) dt + \widehat{b}(Y(t)) dW(t) \quad (99)$$

where now

$$\widehat{a}(y) = [a(x)h'(x) + D b^2(x)h''(x)]_{x=g(y)} \quad (100)$$

$$\widehat{b}(y) = [b(x)h'(x)]_{x=g(y)} \quad (101)$$

namely, in an equivalent form,

$$\widehat{a}(h(x)) = a(x)h'(x) + D b^2(x)h''(x) \quad (102)$$

$$\widehat{b}(h(x)) = b(x)h'(x) \quad (103)$$

We are now interested in transforming a *SDE* (181) into a new form (94) that turns out to be simpler. To this end we will analyze first what type of coefficients $\widehat{a}(y, t)$ and $\widehat{b}(y, t)$ lead to elementary solution, and then under which conditions the original coefficients $a(y, t)$ and $b(y, t)$ are susceptible to be transformed into that new form

B.2.1 Elementary solvable *SDE*'s

1. **Constant coefficients \widehat{a} and \widehat{b} :** In this case the *SDE* (94) becomes

$$dY(t) = \widehat{a} dt + \widehat{b} dW(t) \quad (104)$$

and its solution, with ***P***-a.s. initial condition $W(0) = 0$ and $Y(0) = Y_0$, apparently is

$$Y(t) = Y_0 + \widehat{a} t + \widehat{b} W(t) \quad (105)$$

Then the process $\bar{Y}(t) = Y(t) - Y_0 \sim \mathfrak{N}(\hat{a}t, 2D\hat{b}t)$ is a Wiener process with a \hat{b} re-scaled diffusion coefficient and a constant, deterministic drift \hat{a} . For a degenerate condition $Y_0 = y$, \mathbf{P} -a.s. at a time s the law of the process is then the transition *pdf*

$$\mathfrak{N}\left(y + \hat{a}(t - s), 2D\hat{b}(t - s)\right)$$

We will see in the following, however, that the compatibility conditions on $a(x, t)$ and $b(x, t)$ to be both transformable into new constant coefficients are in general excessively tight, so that the corresponding solutions discussed above are indeed of little practical interest

2. Smoluchowsky equation: We consider next the case of transformations leading to $\hat{b}(y, t) = 1$, namely to *SDE* (94) of the type

$$dY(t) = \hat{a}(Y(t), t) dt + dW(t) \tag{106}$$

that can always be considered as a Smoluchowsky equation (see [13], ch. 10), namely as the overdamped ($\gamma \rightarrow +\infty$) limit of an *OU* (Ornstein-Uhlenbeck) system of dynamic (Newton) *SDE*'s for a derivable process $Z(t)$

$$\begin{aligned} dZ(t) &= V(t) dt \\ dV(t) &= \gamma[\hat{a}(Z(t), t) - V(t)] dt + \gamma dW(t) \end{aligned}$$

where $\hat{a}(z, t)$ now represents an external field of forces, and the approximation is understood in the sense that

$$\lim_{\gamma \rightarrow +\infty} Z(t) = Y(t) \quad \mathbf{P}\text{-a.s.}$$

There is no simple, general way to solve even the equation (106), but when $\hat{a}(y)$ is time independent the *SDE* (106) takes the form

$$dY(t) = \hat{a}(Y(t)) dt + dW(t) \tag{107}$$

where $\hat{a}(z)$ is now a *stationary* external field of forces, so that we can reasonably hope to find at least some *stationary* solution. If we introduce indeed a potential $\phi(y)$ according to

$$\hat{a}(y) = -\frac{D}{kT} \phi'(y) \tag{108}$$

(here k is the Boltzmann constant and T the absolute temperature) then it easy to show that there is a stationary solution with an invariant Boltzmann distribution

$$\frac{e^{-\frac{\phi(y)}{kT}}}{\int_{\mathbb{R}} e^{-\frac{\phi(z)}{kT}} dz} \tag{109}$$

provided that $\phi(y)$ is such that $e^{-\frac{\phi(y)}{kT}}$ is an integrable function (for details see Appendix C.4)

3. **Process-independent coefficients $\widehat{a}(t)$ and $\widehat{b}(t)$:** When on the other hand $\widehat{a}(t)$ and $\widehat{b}(t)$ of the *SDE* (94) are both y -independent the equation is reduced to a *sdif* with deterministic coefficients

$$dY(t) = \widehat{a}(t) dt + \widehat{b}(t) dW(t) \quad (110)$$

and its solution is just the integral of the *sdif*, namely

$$Y(t) = Y_0 + \int_0^t \widehat{a}(u) du + \int_0^t \widehat{b}(u) dW(u) \quad (111)$$

with the initial condition $Y(0) = Y_0$, \mathbf{P} -*a.s.* This means that the process $\overline{Y}(t) = Y(t) - Y_0$ is Gaussian

$$\overline{Y}(t) \sim \mathfrak{N} \left(\int_0^t \widehat{a}(u) du, 2D \int_0^t \widehat{b}^2(u) du \right)$$

and by taking a degenerate condition $Y_0 = y$, \mathbf{P} -*a.s.* at a time s the law of the process is the transition *pdf* $f(x, t|y, s)$

$$\mathfrak{N} \left(y + \int_s^t \widehat{a}(u) du, 2D \int_s^t \widehat{b}^2(u) du \right) \quad (112)$$

4. **Process-linear coefficients:** We consider next the case of coefficients which are linear in y

$$\widehat{a}(y, t) = \widehat{a}_0(t) + \widehat{a}_1(t)y \quad \widehat{b}(y, t) = \widehat{b}_0(t) + \widehat{b}_1(t)y \quad (113)$$

namely of the *SDE*'s of the form

$$dY(t) = \left[\widehat{a}_0(t) + \widehat{a}_1(t)Y(t) \right] dt + \left[\widehat{b}_0(t) + \widehat{b}_1(t)Y(t) \right] dW(t) \quad (114)$$

and in particular its homogeneous counterpart with $\widehat{a}_0(t) = 0$ and $\widehat{b}_0(t) = 0$

$$dY(t) = \widehat{a}_1(t)Y(t)dt + \widehat{b}_1(t)Y(t)dW(t) \quad (115)$$

We look first for a solution of the **homogeneous equation** (115) and – as long as $Y(t) \neq 0$ – we define a new process $Z(t)$ through the transformation

$$\begin{aligned} z = h(y, t) = \ln y & \quad y = g(z, t) = e^z & \quad Z(t) = \ln Y(t) & \quad Y(t) = e^{Z(t)} \\ h_t(y, t) = 0 & \quad h_y(y, t) = \frac{1}{y} & \quad h_{yy}(y, t) = -\frac{1}{y^2} \end{aligned}$$

It is easy to see then from (95) and (96) that the *SDE* for $Z(t)$ is

$$dZ(t) = \left[\widehat{a}_1(t) - D \widehat{b}_1^2(t) \right] dt + \widehat{b}_1(t) dW(t) \quad (116)$$

with process independent coefficients as in (110), and hence from (111) the solution of (116) with $Z(0) = Z_0$ is

$$Z(t) = Z_0 + \int_0^t \left[\widehat{a}_1(s) - D \widehat{b}_1^2(s) \right] ds + \int_0^t \widehat{b}_1(s) dW(s)$$

The (never vanishing) solution of the homogeneous *SDE* (115) with $Y_0 = e^{Z_0}$ then is

$$Y(t) = Y_0 e^{\overline{Z}(t)} \tag{117}$$

where we have introduced the process

$$\overline{Z}(t) = Z(t) - Z_0 = \int_0^t \left[\widehat{a}_1(s) - D \widehat{b}_1^2(s) \right] ds + \int_0^t \widehat{b}_1(s) dW(s) \tag{118}$$

which is again a solution of (116), but with the initial condition $Z(0) = 0$. It turns out of course that $\overline{Z}(t)$ is a Gaussian process with

$$\overline{Z}(t) \sim \mathfrak{N} \left(\int_0^t \left[\widehat{a}_1(s) - D \widehat{b}_1^2(s) \right] ds, 2D \int_0^t \widehat{b}_1^2(s) ds \right)$$

Going back then to the **non homogeneous equation** (114) we first remark that from (92) and the expression (116) of the *sdif* $d\overline{Z}(t)$ we have

$$\begin{aligned} d \left(e^{-\overline{Z}(t)} \right) &= \left(- \left[\widehat{a}_1(t) - D \widehat{b}_1^2(t) \right] + D \widehat{b}_1^2(t) \right) e^{-\overline{Z}(t)} dt - \widehat{b}_1(t) e^{-\overline{Z}(t)} dW(t) \\ &= \left[-\widehat{a}_1(t) + 2D \widehat{b}_1^2(t) \right] e^{-\overline{Z}(t)} dt - \widehat{b}_1(t) e^{-\overline{Z}(t)} dW(t) \end{aligned} \tag{119}$$

and then that from (93), (114) and (119) it is

$$\begin{aligned} d \left(e^{-\overline{Z}(t)} Y(t) \right) &= e^{-\overline{Z}(t)} dY(t) + Y(t) d \left(e^{-\overline{Z}(t)} \right) \\ &\quad - \left[\widehat{b}_0(t) + \widehat{b}_1(t) Y(t) \right] \widehat{b}_1(t) e^{-\overline{Z}(t)} 2D dt \\ &= e^{-\overline{Z}(t)} \left(\left[\widehat{a}_0(t) + \widehat{a}_1(t) Y(t) \right] dt + \left[\widehat{b}_0(t) + \widehat{b}_1(t) Y(t) \right] dW(t) \right) \\ &\quad e^{-\overline{Z}(t)} Y(t) \left(\left[-\widehat{a}_1(t) + 2D \widehat{b}_1^2(t) \right] dt - \widehat{b}_1(t) dW(t) \right) \\ &\quad - \left[\widehat{b}_0(t) + \widehat{b}_1(t) Y(t) \right] \widehat{b}_1(t) e^{-\overline{Z}(t)} 2D dt \\ &= \left[\widehat{a}_0(t) - 2D \widehat{b}_0(t) \widehat{b}_1(t) \right] e^{-\overline{Z}(t)} dt + \widehat{b}_0(t) e^{-\overline{Z}(t)} dW(t) \end{aligned}$$

By taking now into account the initial condition $e^{-\overline{Z}(0)} Y(0) = Y(0) = Y_0$, the previous *sdif* can be integrated as

$$e^{-\overline{Z}(t)} Y(t) = Y_0 + \int_0^t \left[\widehat{a}_0(s) - 2D \widehat{b}_0(s) \widehat{b}_1(s) \right] e^{-\overline{Z}(s)} ds + \int_0^t \widehat{b}_0(s) e^{-\overline{Z}(s)} dW(s)$$

so that the general solution of the non homogeneous equation (114) finally is

$$Y(t) = Y_0 e^{\bar{Z}(t)} + \int_0^t \left[\hat{a}_0(s) - 2D \hat{b}_0(s) \hat{b}_1(s) \right] e^{\bar{Z}(t) - \bar{Z}(s)} ds \quad (120)$$

$$+ \int_0^t \hat{b}_0(s) e^{\bar{Z}(t) - \bar{Z}(s)} dW(s)$$

where the process $\bar{Z}(t)$ is defined in (118). We remark finally that in the case of **time-independent coefficients**, namely when

$$\hat{a}(y) = \hat{a}_0 + \hat{a}_1 y \quad \hat{b}(y) = \hat{b}_0 + \hat{b}_1 y \quad (121)$$

$$dY(t) = [\hat{a}_0 + \hat{a}_1 Y(t)] dt + [\hat{b}_0 + \hat{b}_1 Y(t)] dW(t) \quad (122)$$

the process (118) becomes

$$\bar{Z}(t) = (\hat{a}_1 - D \hat{b}_1^2) t + \hat{b}_1 W(t) \quad (123)$$

and the solution (120) is reduced to

$$Y(t) = Y_0 e^{\bar{Z}(t)} + \left(\hat{a}_0 - 2D \hat{b}_0 \hat{b}_1 \right) \int_0^t e^{\bar{Z}(t) - \bar{Z}(s)} ds + \hat{b}_0 \int_0^t e^{\bar{Z}(t) - \bar{Z}(s)} dW(s) \quad (124)$$

5. **Time-independent coefficients $\hat{a}(y)$ and $\hat{b}(y)$:** There are no general formulas available, but – as will be shown in the Section B.2.2 – it is always possible to manage the transformation $Y(t) = h(X(t))$ in such a way that $\hat{b}(y) = 1$ (different constant values could be easily subsumed in a redefinition of $W(t)$) so that the *SDE* (94) takes the form of a time independent Smoluchowsky equation (107). Even so we already remarked however that, beside a possible stationary solution, there is no simple, general way to solve the equation (107). If on the other hand we try to simplyfy our problem by taking $\hat{a}(y) = 0$ and an arbitrary $\hat{b}(y)$, then we are led to the equation

$$dY(t) = \hat{b}(Y(t)) dW(t) \quad (125)$$

but again, in the general case, we have no simple solution to show so that the problem must be dealt with on a case-by-case basis

B.2.2 Transformations to constant coefficients

We will first look for the transformations $y = h(x, t)$ leading to constant coefficients \hat{a} and \hat{b} : from (97) and (98) we have

$$\hat{a} = h_t(x, t) + h_x(x, t)a(x, t) + D h_{xx}(x, t)b^2(x, t) \quad (126)$$

$$\hat{b} = h_x(x, t)b(x, t) \quad (127)$$

From the equation (127) – as long as $b(x, t)$ does not change its sign as a function of x , and hence the resulting $h(x, t)$ turns out to be monotonic and invertible in x for every t – we immediately get the transformation

$$h_x(x, t) = \frac{\widehat{b}}{b(x, t)} \quad y = h(x, t) = \widehat{b} \int \frac{dx}{b(x, t)} \quad x = g(y, t) \quad (128)$$

and since this gives

$$h_{xx}(x, t) = -\widehat{b} \frac{b_x(x, t)}{b^2(x, t)} \quad h_t(x, t) = -\widehat{b} \int \frac{b_t(x, t)}{b^2(x, t)} dx$$

we also obtain from (126) the equation

$$\frac{\widehat{a}}{\widehat{b}} = - \int \frac{b_t(x, t)}{b^2(x, t)} dx + \frac{a(x, t)}{b(x, t)} - D b_x(x, t)$$

By x derivation we then find that $a(x, t)$ and $b(x, t)$ must satisfy the condition

$$a_x(x, t) - \frac{b_x(x, t)}{b(x, t)} a(x, t) = D b(x, t) b_{xx}(x, t) + \frac{b_t(x, t)}{b(x, t)}$$

that can be solved as a first order *ODE* (ordinary differential equation) for $a(x, t)$ giving the explicit relation ($\alpha(t)$ is an arbitrary integration function)

$$a(x, t) = \alpha(t) + b(x, t) \left[D b_x(x, t) + \int \frac{b_t(x, t)}{b^2(x, t)} dx \right] \quad (129)$$

that for time independent $a(x)$ and $b(x)$ becomes (α here is now an arbitrary integration constant)

$$a(x) = \alpha + D b(x) b'(x) \quad (130)$$

These are however very tight conditions that can be verified only in a few particular cases and hence we will rather go on discussing whether at least one of the two coefficients – either \widehat{a} , or \widehat{b} – can be reasonably made constant

We begin then by requiring only $\widehat{b}(y, t)$ to be constant and, without a loss of generality, we can take $\widehat{b}(y, t) = 1$: every other constant value can indeed be easily accounted for with a redefinition of the diffusion coefficient of $W(t)$. The transformation under discussion will then lead to the Smoluchowsky *SDE* (106) previously discussed in the Section B.2.1. From (127) with $\widehat{b} = 1$ we now get the previous transformation (128) in the form

$$y = h(x, t) = \int \frac{dx}{b(x, t)} \quad x = g(y, t) \quad (131)$$

and then, instead of requiring the condition (126), we must recall (95) to get the explicit expression for the new drift coefficient

$$\widehat{a}(y, t) = - \left[\int \frac{b_t(x, t)}{b^2(x, t)} dx - \frac{a(x, t)}{b(x, t)} + D b_x(x, t) \right]_{x=g(y, t)}$$

giving rise to the time dependent Smoluchowsky equation (106). When in particular $a(x)$ and $b(x)$ are time-independent the transformation becomes

$$h(x) = \int \frac{dx}{b(x)} \tag{132}$$

and the new drift is reduced to

$$\widehat{a}(y) = \left[\frac{a(x)}{b(x)} - D b'(x) \right]_{x=g(y)} \tag{133}$$

so that the transformed *SDE* takes the form (107) discussed in the point 5 of the previous Section B.2.1. As remarked therein, in this time-independent setting the drift function (133) represents a stationary dynamics for the transformed process $Y(t)$, and when it is deduced from a potential $\phi(y)$ it can lead to an *invariant Boltzmann distribution* (109) as stated in the Section B.2.1 (for details see Appendix C.4)

If we instead require only $\widehat{a}(y, t) = \widehat{a}$ to be a constant, then from (97) $h(x, t)$ must satisfy the *PDE* (partial differential equation)

$$h_t(x, t) + a(x, t)h_x(x, t) + D b^2(x, t)h_{xx}(x, t) = \widehat{a}$$

which, if $a(x)$ and $b(x)$ are time-independent, is reduced to the *ODE*

$$a(x)h'(x) + D b^2(x)h''(x) = \widehat{a}$$

that can be solved to give the explicit expression of the transformation $h(x)$ and of the coefficient $\widehat{b}(x)$. For simplicity, however, we will restrict ourselves to the case $\widehat{a} = 0$: for suitable integration constants c_1 and c_2 the general solution is

$$h(x) = c_1 + c_2 \int e^{-\int \frac{a(x)}{D b^2(x)} dx} dx$$

and from (96) the new diffusion coefficient will be

$$\widehat{b}(y) = c_2 \left[b(x) e^{-\int \frac{a(x)}{D b^2(x)} dx} \right]_{x=g(y)}$$

that will enter into the new *SDE* (125). As already remarked, however, we again have no general solutions available, so that these *SDE*'s could be tackled only case by case

B.2.3 Transformations to process-independent coefficients

Given the simple results (111) and (112) it would be interesting, as a next step, to find under what conditions we can transform an arbitrary *SDE* (181) into a new

equation with space-independent coefficients. To this end let us first remark that the two equations (97) and (98) become in this case

$$\widehat{a}(t) = h_t(x, t) + a(x, t)h_x(x, t) + D b^2(x, t)h_{xx}(x, t) \quad (134)$$

$$\widehat{b}(t) = h_x(x, t)b(x, t) \quad (135)$$

From (135) we have

$$h_x(x, t) = \frac{\widehat{b}(t)}{b(x, t)} \quad h_{xx}(x, t) = -\frac{\widehat{b}(t)}{b^2(x, t)}b_x(x, t) \quad (136)$$

$$h_{xt}(x, t) = \frac{\dot{\widehat{b}}(t)b(x, t) - \widehat{b}(t)b_t(x, t)}{b^2(x, t)} \quad (137)$$

while by deriving (134) with respect to x we find

$$h_{xt}(x, t) + \partial_x [a(x, t)h_x(x, t) + D b^2(x, t)h_{xx}(x, t)] = 0 \quad (138)$$

Then by substituting (136) and (137) into (138), after some manipulation we obtain

$$\frac{\dot{\widehat{b}}(t)}{\widehat{b}(t)} = b(x, t) \left[\frac{b_t(x, t)}{b^2(x, t)} - \partial_x \left(\frac{a(x, t)}{b(x, t)} \right) + D b_{xx}(x, t) \right] \quad (139)$$

By deriving (139) with respect to x again we finally get

$$\partial_x \left\{ b(x, t) \left[\frac{b_t(x, t)}{b^2(x, t)} - \partial_x \left(\frac{a(x, t)}{b(x, t)} \right) + D b_{xx}(x, t) \right] \right\} = 0 \quad (140)$$

a condition – involving only the initial coefficients $a(x, t)$ and $b(x, t)$, – that must be satisfied in order to ensure the existence of a transformation $h(x, t)$ bringing to an equation with space-independent coefficients. In fact, when (140) holds, the r.h.s. of (139) must depend only on t , and hence this equation enables us to find $\widehat{b}(t)$. Then by plugging this $\widehat{b}(t)$ into (135) we can find the transformation $h(x, t)$, and finally $\widehat{a}(t)$ comes out of (134) whose r.h.s. too will turn out to be dependent only on t

When in particular the coefficients $a(x)$ and $b(x)$ are time-independent the equation (139) becomes

$$\frac{\dot{\widehat{b}}(t)}{\widehat{b}(t)} = b(x) \left[D b''(x) - \frac{d}{dx} \left(\frac{a(x)}{b(x)} \right) \right] \quad (141)$$

and since now the l.h.s. depends only on t , and the r.h.s only on x , the two members of this equation must be both equal to a constant c . Hence instead of (140) the compatibility condition coming out of (141) is just

$$b(x) \left[D b''(x) - \frac{d}{dx} \left(\frac{a(x)}{b(x)} \right) \right] = c \quad (142)$$

while from

$$\frac{\widehat{b}'(t)}{\widehat{b}(t)} = c \quad (143)$$

we first get $\widehat{b}(t) = e^{ct}$, then from (135) we have

$$h(x, t) = e^{ct} \int \frac{dx}{b(x)} \quad (144)$$

and finally from (134)

$$\widehat{a}(t) = e^{ct} \left[c \int \frac{dx}{b(x)} + \frac{a(x)}{b(x)} - D b'(x) \right] \quad (145)$$

where the term in square brackets is in fact constant provided that the compatibility condition (142) is satisfied

B.2.4 Transformations to process-linear coefficients

Since we have shown that every *SDE* (114) with space-linear coefficients (113) can be explicitly solved, it will be important to find under what conditions we can transform an arbitrary *SDE* (181) in the new form (114). We will analyze in detail, however, only the case of **time-independent coefficients** $a(x)$, $b(x)$ and (121) when the new *SDE* is supposed to take the form (122). In this case the transformation $h(x)$ is time-independent too, and the conditions (97) and (98) become

$$\widehat{a}_0 + \widehat{a}_1 h(x) = h'(x)a(x) + D h''(x)b^2(x) \quad (146)$$

$$\widehat{b}_0 + \widehat{b}_1 h(x) = h'(x)b(x) \quad (147)$$

If we choose first to have $\widehat{b}_1 \neq 0$, from the equation (147) we at once have

$$h(x) = c e^{\widehat{b}_1 p(x)} - \frac{\widehat{b}_0}{\widehat{b}_1} \quad p(x) = \int \frac{1}{b(x)} dx \quad p'(x) = \frac{1}{b(x)} \quad (148)$$

and hence

$$h'(x) = c \widehat{b}_1 \frac{e^{\widehat{b}_1 p(x)}}{b(x)} \quad h''(x) = c \widehat{b}_1 \frac{e^{\widehat{b}_1 p(x)}}{b^2(x)} \left[\widehat{b}_1 - b'(x) \right]$$

By plugging all that into (146) after some algebra we get the equation

$$\frac{\widehat{a}_0 \widehat{b}_1 - \widehat{a}_1 \widehat{b}_0}{c \widehat{b}_1} = e^{\widehat{b}_1 p(x)} \left[\widehat{b}_1 q(x) + D \widehat{b}_1^2 - \widehat{a}_1 \right] \quad q(x) = \frac{a(x)}{b(x)} - D b'(x)$$

In order then to have a condition free from constants, by a first derivation we get

$$\begin{aligned} 0 &= \frac{d}{dx} \left(e^{\widehat{b}_1 p(x)} \left[\widehat{b}_1 q(x) + D \widehat{b}_1^2 - \widehat{a}_1 \right] \right) \\ &= \widehat{b}_1 e^{\widehat{b}_1 p(x)} \left(p'(x) \left[\widehat{b}_1 q(x) + D \widehat{b}_1^2 - \widehat{a}_1 \right] + q'(x) \right) \end{aligned}$$

namely from (148)

$$\widehat{a}_1 - D\widehat{b}_1^2 = \widehat{b}_1q(x) + b(x)q'(x)$$

With a second derivation we then have

$$0 = \frac{d}{dx} [\widehat{b}_1q(x) + b(x)q'(x)] = \widehat{b}_1q'(x) + \frac{d}{dx} [b(x)q'(x)]$$

that can be recast as

$$\widehat{b}_1 = -\frac{1}{q'(x)} \frac{d}{dx} [b(x)q'(x)] \tag{149}$$

and finally with a third derivation

$$\frac{d}{dx} \left(\frac{1}{q'(x)} \frac{d}{dx} [b(x)q'(x)] \right) = 0 \tag{150}$$

which is now the wanted condition involving only the coefficients $a(x)$ and $b(x)$ of the initial, non linear *SDE* (181). When (150) is satisfied, then we can take (149) as the value of the parameter \widehat{b}_1 , and $h(x) = ce^{\widehat{b}_1p(x)}$ for some suitable value of the constant c as the transformation able to reduce our equation to its linear form

If instead we require $\widehat{b}_1 = 0$, from the equation (147) and within the same notations we simply get

$$h(x) = \widehat{b}_0p(x) + c \quad h'(x) = \frac{\widehat{b}_0}{b(x)} \quad h''(x) = -\widehat{b}_0 \frac{b'(x)}{b^2(x)}$$

and hence after some manipulation from (146) we have

$$q(x) = \widehat{a}_1p(x) + c\widehat{a}_1 + \frac{\widehat{a}_0}{\widehat{b}_0}$$

then by derivation we obtain $q'(x) = \widehat{a}_1p'(x)$ namely $b(x)q'(x) = \widehat{a}_1$, and by further derivation we finally get the condition

$$\frac{d}{dx} [b(x)q'(x)] = 0 \tag{151}$$

It is apparent that this condition (151) also implies the condition (150) which however remains the most general requirement whose compliance is needed in order to be able to transform the *SDE* (181) into one with space-linear, time-independent coefficients

C Solving Fokker-Planck equations

Given the *SDE* (181) the *pdf* of its solutions can be obtained by solving the (forward) Fokker-Planck equation

$$\partial_t f(x, t) = -\partial_x [a(x, t)f(x, t)] + D \partial_x^2 [b^2(x, t)f(x, t)] \tag{152}$$

that for time-independent coefficients (namely for an *SDE* of the form (90)) becomes

$$\partial_t f(x, t) = -\partial_x [a(x)f(x, t)] + D \partial_x^2 [b^2(x)f(x, t)] \quad (153)$$

Standard solution methods are well known, as that of the eigenfunction expansion discussed later. We will however first give a look to a few results giving the transition *pdf* without solving (152), but using a few results about the expectations that however are not easily calculated explicitly

C.1 Semi-explicit transition *pdf*'s

We will first recall a formal procedure (see [12], §13, p. 91) used to find the transition *pdf* of the *FPE* (152), namely of a process solution of a the *SDE* (181). Define – in a notation only partially coherent with our previous one – the functions

$$y = h(x, t) = \int \frac{dx}{b(x, t)} \quad x = g(y, t) \quad h(g(y, t), t) = y$$

then the function

$$\begin{aligned} \hat{a}(y, t) &= h_t(g(y, t), t) + h_x(g(y, t), t) a(g(y, t), t) + D h_{xx}(g(y, t), t) b^2(g(y, t), t) \\ &= \left[-\int \frac{b_t(x, t)}{b^2(x, t)} dx + \frac{a(x, t)}{b(x, t)} - D b_x(x, t) \right]_{x=g(y, t)} \end{aligned}$$

and finally

$$\alpha(y, t) = \frac{1}{2D} \int \hat{a}(y, t) dy \quad \beta(y, t) = -\frac{\hat{a}^2(y, t)}{4D} - \frac{\hat{a}_y(y, t)}{2} - \frac{1}{2D} \int \hat{a}_t(y, t) dy$$

Consider now for $0 \leq s \leq t$ the two-times process

$$Z(s, t) = \int_0^1 \beta(\overline{W}_{st}(u) + \overline{h}_{st}(u), s + (t-s)u) du$$

where for $0 \leq u \leq 1$ we defined

$$\begin{aligned} \overline{h}_{st}(u) &= u h(x, t) + (1-u)h(y, s) \\ \overline{W}_{st}(u) &= W(s + (t-s)u) - (uW(t) + (1-u)W(s)) \end{aligned}$$

Remark that $\overline{W}_{st}(u)$ is now a *rectilinear Brownian bridge* with $\overline{W}_{st}(0) = \overline{W}_{st}(1) = 0$ (see Appendix C.2). Then (we just recall the results without proofs) the transition *pdf* solution of (152) is

$$f(x, t|y, s) = \frac{\mathbf{E} [e^{(t-s)Z(s,t)}]}{b(x, t)\sqrt{4\pi D(t-s)}} e^{-\frac{[h(x,t)-h(y,s)]^2}{4D(t-s)} + \alpha(h(x,t),t) - \alpha(h(y,s),s)} \quad (154)$$

When moreover the coefficients are time-independent the *SDE* takes the form (90) and the result (154) can be simplified: we have indeed now

$$y = h(x) = \int \frac{dx}{b(x)} \quad x = g(y) \quad h(g(y)) = y \quad \hat{a}(y) = \left[\frac{a(x)}{b(x)} - D b'(x) \right]_{x=g(y)}$$

and hence

$$\alpha(y) = \frac{1}{2D} \int \hat{a}(y) dy \quad \beta(y) = -\frac{\hat{a}^2(y)}{4D} - \frac{\hat{a}'(y)}{2}$$

In particular we find

$$\alpha(h(x)) = \frac{1}{2D} \int \left[\frac{a(x)}{b(x)} - D b'(x) \right] \frac{dx}{b(x)} = \frac{1}{2D} \int \frac{a(x)}{b^2(x)} dx - \frac{1}{2} \ln b(x)$$

As a consequence, by redefining now

$$Z(s, t) = \int_0^1 \beta(\bar{W}_{st}(u) + \bar{h}(u)) du \quad \bar{h}(u) = uh(x) + (1-u)h(y)$$

the transition *pdf* solution of (153) will become

$$f(x, t|y, s) = \frac{\mathbf{E} [e^{(t-s)Z(s,t)}]}{b(x)\sqrt{4\pi D(t-s)}} \sqrt{\frac{b(y)}{b(x)}} \exp \left\{ \frac{1}{2D} \int_y^x \frac{a(z)}{b^2(z)} dz - \frac{1}{4D(t-s)} \left(\int_y^x \frac{dz}{b(z)} \right)^2 \right\} \quad (155)$$

Remark however that the results (154) and (155) constitute only a semi-explicit form of the *pdf* because they are apparently contingent on an expectation not easy to calculate as it is pointed out in the Section 3.2

C.2 Random bridges

C.2.1 Non-random interpolation

Brownian bridge *SDE*'s are stochastic versions of *ODE*'s for trajectories interpolating two, or more, fixed points (see for example [14, 15] pp. 358-360). In general the non-random interpolating trajectories, coinciding with the expectation of the corresponding random bridges, are supposed to be *linear functions* of the time *t*, but we will argue here that there is no really compelling reason for this choice.

Let us start with a trajectory *x(t)* connecting two possible values *a* and *b* at the endpoints of a compact time interval $[0, T]$, namely

$$x(t) = a g\left(\frac{t}{T}\right) + b h\left(\frac{t}{T}\right) \quad 0 \leq t \leq T \quad (156)$$

where we will suppose that $g(s)$ and $h(s)$ are derivable at least once in $[0, T]$, and

$$\begin{cases} g(0) = 1 \\ g(1) = 0 \end{cases} \quad \begin{cases} h(0) = 0 \\ h(1) = 1 \end{cases} \quad (157)$$

so that we trivially get that

$$x(0) = a \quad x(T) = b \quad (158)$$

As a matter of fact, every possible function $x(t)$ complying with the extremal conditions (158) can be cast in the form (156): for a given $x(t)$ just choose an arbitrary $g(s)$ satisfying (157), and then it will be enough to take

$$h(s) = \frac{x(Ts) - a g(s)}{b}$$

in order to obtain (156). Remark by the way that nothing forbids an explicit dependence of $g(s)$ and $h(s)$ from a and b , in so far as the conditions (157) are preserved. Therefore the expression (156) can be considered as general enough for our purposes

We will look then for a first order *ODE* such that the trajectory (156) will be its (unique) solution for the initial condition $x(0) = a$: it is straightforward to understand that the form of this equation, albeit independent from the initial condition a , will however explicitly depend on the final condition b aimed at by our trajectory. A first order *ODE* allows indeed a free choice just for one initial condition, while in general no independent final condition can be arbitrarily added if we want to have a chance to find solutions. As a consequence an *ODE* admitting both the extremal conditions (158) must depend on one of them: in other words there is no unique equation fitting both the conditions (158) for arbitrary values of a and b . In order to eliminate the initial condition $x(0) = a$ from our *ODE* let us remark that from (156) we get

$$\dot{x}(t) = \frac{a}{T} \dot{g}\left(\frac{t}{T}\right) + \frac{b}{T} \dot{h}\left(\frac{t}{T}\right) \quad (159)$$

so that from (156) and (159) we have

$$\frac{\dot{x}(t) - \frac{b}{T} \dot{h}\left(\frac{t}{T}\right)}{\frac{1}{T} \dot{g}\left(\frac{t}{T}\right)} = a = \frac{x(t) - b h\left(\frac{t}{T}\right)}{g\left(\frac{t}{T}\right)}$$

and rearranging the terms we find the *ODE*

$$\dot{x}(t) = \frac{b \left[g\left(\frac{t}{T}\right) \dot{h}\left(\frac{t}{T}\right) - \dot{g}\left(\frac{t}{T}\right) h\left(\frac{t}{T}\right) \right] + x(t) \dot{g}\left(\frac{t}{T}\right)}{T g\left(\frac{t}{T}\right)} \quad (160)$$

whose solutions (156) will connect every possible initial condition a to the same final value b inscribed into it.

The simplest example, used for the most widespread stochastic generalization to Brownian bridges (see [14, 15] for details), adopts the following linear functions

$$g(s) = 1 - s \quad h(s) = s \quad (161)$$

so that the corresponding connecting trajectories

$$x(t) = a \left(1 - \frac{t}{T}\right) + b \frac{t}{T}$$

are t -linear and satisfy the *ODE*

$$\dot{x}(t) = \frac{b - x(t)}{T - t}$$

But this by no means constitutes a unique possibility. We could for instance take

$$g(s) = (1 - s)^2 \quad h(s) = s^2$$

and in this case we would have the parabolic trajectories

$$x(t) = a \left(1 - \frac{t}{T}\right)^2 + b \left(\frac{t}{T}\right)^2$$

solutions of the *ODE*

$$\dot{x}(t) = \frac{2}{T} \frac{bt - Tx(t)}{T - t}$$

As final examples among many we can either put

$$g(s) = \cos \frac{\pi s}{2} \quad h(s) = \sin \frac{\pi s}{2}$$

and get as trajectories

$$x(t) = a \cos \frac{\pi t}{2T} + b \sin \frac{\pi t}{2T}$$

and as *ODE*

$$\dot{x}(t) = \frac{\pi}{2T} \frac{b - x(t) \sin \frac{\pi t}{2T}}{\cos \frac{\pi t}{2T}}$$

or instead

$$g(s) = \cos^2 \frac{\pi s}{2} \quad h(s) = \sin^2 \frac{\pi s}{2}$$

and get as trajectories

$$x(t) = a \cos^2 \frac{\pi t}{2T} + b \sin^2 \frac{\pi t}{2T}$$

and as *ODE*

$$\dot{x}(t) = \frac{b - x(t)}{T} \pi \tan \frac{\pi t}{2T}$$

C.2.2 Brownian bridges

A straightforward stochastic generalization of the *ODE* (160) discussed in the previous section is obtained just by adding a Brownian noise $W(t)$ with constant diffusion coefficient $2D$

$$dX(t) = \frac{b \left[g\left(\frac{t}{T}\right) \dot{h}\left(\frac{t}{T}\right) - \dot{g}\left(\frac{t}{T}\right) h\left(\frac{t}{T}\right) \right] + X(t) \dot{g}\left(\frac{t}{T}\right)}{T g\left(\frac{t}{T}\right)} dt + dW(t) \quad (162)$$

and apparently results in a *SDE* with linear coefficients and initial condition $X(0) = a$, \mathbf{P} -*a.s.* Therefore, according to the notations adopted in the Appendix B.2.1, the equation (162) is a *SDE* of the form (114) with

$$\begin{aligned} \widehat{a}_0(t) &= \frac{b}{T} \frac{g\left(\frac{t}{T}\right) \dot{h}\left(\frac{t}{T}\right) - \dot{g}\left(\frac{t}{T}\right) h\left(\frac{t}{T}\right)}{g\left(\frac{t}{T}\right)} = b g\left(\frac{t}{T}\right) \frac{d}{dt} \left[\frac{h\left(\frac{t}{T}\right)}{g\left(\frac{t}{T}\right)} \right] \\ \widehat{a}_1(t) &= \frac{1}{T} \frac{\dot{g}\left(\frac{t}{T}\right)}{g\left(\frac{t}{T}\right)} = \frac{d}{dt} \left[\ln g\left(\frac{t}{T}\right) \right] \\ \widehat{b}_0(0) &= 1 \\ \widehat{b}_1(t) &= 0 \end{aligned}$$

and hence it is easy to find that

$$\overline{Z}(t) = \int \widehat{a}_1(t) dt = \ln g\left(\frac{t}{T}\right) \quad e^{\overline{Z}(t)} = g\left(\frac{t}{T}\right)$$

so that its solution after some algebra becomes

$$X(t) = a g\left(\frac{t}{T}\right) + b h\left(\frac{t}{T}\right) + g\left(\frac{t}{T}\right) \int_0^t \frac{1}{g\left(\frac{s}{T}\right)} dW(s) \quad (163)$$

Remark that the first two terms of this solution exactly coincide with the non-random interpolating trajectories (156). In particular when we take the linear functions (161) we get the solution

$$X(t) = a \left(1 - \frac{t}{T}\right) + b \frac{t}{T} + (T - t) \int_0^t \frac{1}{T - s} dW(s) \quad (164)$$

which is exactly the *rectilinear Brownian bridge* discussed in [14]

Following the same line of reasoning of [14] we could now prove that the solution (163) is Gaussian with \mathbf{P} -*a.s.* continuous paths, with expectation

$$m(t) = \mathbf{E} [X(t)] = a g\left(\frac{t}{T}\right) + b h\left(\frac{t}{T}\right)$$

and with covariance

$$C(s, t) = \mathbf{cov} [X(s), X(t)] = D g\left(\frac{s}{T}\right) g\left(\frac{t}{T}\right) \int_0^{s \wedge t} \frac{1}{g^2\left(\frac{u}{T}\right)} du$$

so that its laws can now be deemed completely known. In particular for the variance we have

$$\mathbf{V} [X(t)] = C(t, t) = D g^2\left(\frac{t}{T}\right) \int_0^t \frac{1}{g^2\left(\frac{s}{T}\right)} ds$$

In the case of the rectilinear Brownian bridge (164) these formulas give

$$\begin{aligned} m(t) &= a \left(1 - \frac{t}{T}\right) + b \frac{t}{T} \\ C(s, t) &= D(T-s)(T-t) \int_0^{s \wedge t} \frac{du}{(T-u)^2} = D \left((s \wedge t) - \frac{st}{T} \right) \end{aligned}$$

and its distributions coincide with that of a Wiener process conditioned at both the endpoints with $X(0) = a$ and $X(T) = b$: in fact, if we take

$$\phi(x, t|y) = \frac{e^{-\frac{(x-y)^2}{2Dt}}}{\sqrt{2\pi Dt}}$$

the finite dimensional distributions of the rectilinear Brownian bridge coincide with the following conditional *pdf*'s of a Wiener process for $0 = t_0 < t_1 < \dots < t_n < T$

$$\begin{aligned} f(x_1, t_1; \dots; x_n, t_n | a, 0; b, T) &= \frac{f(x_1, t_1; \dots; x_n, t_n; b, T | a, 0)}{f(b, T | a, 0)} \\ &= \frac{f(b, T | x_n, t_n) \dots f(x_2, t_2 | x_1, t_1) f(x_1, t_1 | a, 0)}{f(b, T | a, 0)} \\ &= \frac{\phi(b, T - t_n | x_n) \dots \phi(x_2, t_2 - t_1 | x_1) \phi(x_1, t_1 | a)}{\phi(b, T | a)} \end{aligned}$$

In fact we will call *rectilinear Brownian bridge* every stochastic process with such finite dimensional distributions. In particular it could be shown that

$$a \left(1 - \frac{t}{T}\right) + b \frac{t}{T} + \left(W(t) - \frac{t}{T}W(T)\right)$$

also is a rectilinear Brownian bridge

C.3 Eigenfunction expansion

For a *FPE* (152) with both $a(x)$ and $b(x)$ time-independent coefficients we will here replace $a(x)$ with $\vec{v}(x)$ (the symbol that we will adopt in the Section D for the forward velocity within the framework of the stochastic mechanics), while for

short we will adopt the notation $B(x) = D b^2(x)$. The pdf $f(x, t)$ of our continuous Markov process $X(t)$ is then a solution of the FPE

$$\partial_t f = \partial_x^2 (Bf) - \partial_x (\vec{v} f) = \partial_x [\partial_x (Bf) - \vec{v} f] \quad (165)$$

defined for $x \in [a, b]$ (beware also the temporary new meaning of the symbols a and b) and $t \geq s$. We will further suppose that $\vec{v}(x)$ has no singularities in (a, b) , and that $\vec{v}(x)$ and $B(x)$ are both continuous and differentiable functions. The conditions imposed on the probabilistic solutions are of course

$$f(x, t) \geq 0 \quad \int_a^b f(x, t) dx = 1 \quad a < x < b \quad s \leq t \quad (166)$$

while from (165) the second condition (166) also takes the form

$$[\partial_x (Bf) - \vec{v} f]_a^b = 0, \quad s \leq t$$

Suitable initial conditions will be added to produce the required evolution: for example a transition pdf $f(x, t|y, s)$ will be selected by the initial condition

$$\lim_{t \rightarrow s^+} f(x, t) = f(x, s^+) = \delta(x - y) \quad (167)$$

It is also possible to show by direct calculation that

$$\tilde{f}(x) = Z^{-1} e^{-\int \frac{B'(x) - \vec{v}(x)}{B(x)} dx} \quad Z = \int_a^b e^{-\int \frac{B'(x) - \vec{v}(x)}{B(x)} dx} dx \quad (168)$$

is an invariant solution of (165) satisfying the conditions (166) (for its coherence with the Boltzmann distribution (109) see Appendix C.4). Remark that (165) is not in the standard self-adjoint form, but if we define the new function $g(x, t)$ by means of

$$f(x, t) = \sqrt{\tilde{f}(x)} g(x, t)$$

it would be easy to show [5, 6] that $g(x, t)$ obeys now an equation of the form

$$\partial_t g = \mathcal{L}[g]$$

where the operator \mathcal{L} defined on a test function φ as

$$\mathcal{L}[\varphi] = \frac{d}{dx} \left[p(x) \frac{d\varphi(x)}{dx} \right] - q(x)\varphi(x)$$

$$p(x) = B(x) > 0 \quad q(x) = \frac{[B'(x) - \vec{v}(x)]^2}{4B(x)} - \frac{[B'(x) - \vec{v}(x)]'}{2}$$

is now self-adjoint. Then, by separating the variables by means of $g(x, t) = \gamma(t)G(x)$ we have $\gamma(t) = e^{-\lambda t}$ while G must be solution of a typical Sturm-Liouville problem associated to the equation

$$\mathcal{L}[G(x)] + \lambda G(x) = 0 \quad (169)$$

with the boundary conditions

$$[B'(a) - \vec{v}(a)]G(a) + 2B(a)G'(a) = 0 \tag{170}$$

$$[B'(b) - \vec{v}(b)]G(b) + 2B(b)G'(b) = 0 \tag{171}$$

It is easy to see that $\lambda = 0$ is always an eigenvalue for the problem (169) with (170) and (171), and that the corresponding eigenfunction is $G_0(x) = \sqrt{\tilde{f}(x)}$. The other simple eigenvalues λ_n will then constitute an infinite, increasing sequence and the corresponding eigenfunction $G_n(x)$ will have n simple zeros in (a, b) . This also means that $\lambda_0 = 0$, corresponding to the eigenfunction $G_0(x)$ which never vanishes in (a, b) , is the lowest eigenvalue so that all other eigenvalues are strictly positive. The eigenfunctions will constitute a complete orthonormal set of functions in $L^2([a, b])$ so that the general solution of (165) with (166) will have the form

$$f(x, t) = \sum_{n=0}^{\infty} c_n e^{-\lambda_n t} \sqrt{\tilde{f}(x)} G_n(x) = \sum_{n=0}^{\infty} c_n e^{-\lambda_n t} G_0(x) G_n(x) \tag{172}$$

with $c_0 = 1$ for normalization (remember that $\lambda_0 = 0$). The coefficients c_n for a particular solution selected by an initial condition

$$f(x, s^+) = f_0(x)$$

are then calculated from the orthonormality relations as

$$c_n = \int_a^b f_0(x) \frac{G_n(x)}{G_0(x)} dx$$

and in particular for the transition *pdf* we have from (167) that

$$c_n = \frac{G_n(x_0)}{G_0(x_0)}$$

Since $\lambda_0 = 0$ and $\lambda_n > 0$ for $n \geq 1$, the general solution (172) has a precise time evolution: all the exponential factors vanish with $t \rightarrow +\infty$ with the only exception of the term $n = 0$ which is constant, so that exponentially fast we will always have

$$\lim_{t \rightarrow +\infty} f(x, t) = c_0 \sqrt{\tilde{f}(x)} G_0(x) = \tilde{f}(x)$$

namely the general solution will relax toward the invariant solution $\tilde{f}(x)$

C.4 Invariant Boltzmann distributions

Let us consider the *SDE* with time-independent coefficients

$$dX(t) = a(X(t)) dt + b(X(t)) dW(t)$$

and the corresponding *FPE* (see also Section C)

$$\begin{aligned}\partial_t f_X(x, t) &= -\partial_x [a(x)f_X(x, t)] + D \partial_x^2 [b^2(x)f_X(x, t)] \\ &= -\partial_x [a(x)f_X(x, t) - D \partial_x (b^2(x)f_X(x, t))]\end{aligned}\quad (173)$$

We know then from the Section B.2.2 that the transformation $Y(t) = h(X(t))$ with a monotonic $h(x)$ and

$$y = h(x) = \int \frac{dx}{b(x)} \quad h'(x) = \frac{1}{b(x)} \quad x = g(y) = h^{-1}(x) \quad (174)$$

will bring us to a new *SDE*

$$dY(t) = \hat{a}(Y(t)) dt + dW(t)$$

and to a new *FPE*

$$\partial_t f_Y(y, t) = -\partial_y [\hat{a}(y)f_Y(y, t)] + D \partial_y^2 f_Y(y, t) = -\partial_y [\hat{a}(y)f_Y(y, t) - D \partial_y f_Y(y, t)] \quad (175)$$

where now

$$\hat{a}(y) = \left[\frac{a(x)}{b(x)} - D b'(x) \right]_{x=g(y)}$$

We claimed in the Section B.2.1 that if we introduce a potential $\phi(y)$ according to

$$\hat{a}(y) = -\frac{D}{kT} \phi'(y)$$

then – provided that $\phi(y)$ is such that $e^{-\frac{\phi(y)}{kT}}$ is an integrable function – it is possible to show that

$$\tilde{f}_Y(y) = Z^{-1} e^{-\frac{\phi(y)}{kT}} = Z^{-1} e^{\frac{1}{D} \int \hat{a}(y) dy} \quad Z = \int_{\mathbb{R}} e^{-\frac{\phi(z)}{kT}} dz \quad (176)$$

is an invariant Boltzmann distribution. It is apparent indeed that, since it is

$$\partial_y \tilde{f}_Y(y) = \frac{\hat{a}(y)}{D} \tilde{f}_Y(y)$$

then $\tilde{f}_Y(y)$ is a stationary solution of (175). On the other hand, when $Y(t) = h(X(t))$ according to the transformation (174), the respective *pdf*'s are also connected by the transformations

$$f_Y(h(x), t) = |b(x)| f_X(x, t) \quad f_Y(y, t) = [|b(x)| f_X(x, t)]_{x=g(y)}$$

and hence in particular we have

$$\tilde{f}_X(x) = \frac{\tilde{f}_Y(h(x))}{|b(x)|} \quad (177)$$

that we claim to be a stationary solution for the original *FPE* (173)

In order to check our last statement, we will first express $\tilde{f}_X(x)$ in terms of the coefficients $a(x)$ and $b(x)$ of (173): to this end let us remark that

$$\int \hat{a}(y) dy = \int \left[\frac{a(x)}{b(x)} - Db'(x) \right]_{x=g(y)} dy = \left[\int \left(\frac{a(x)}{b(x)} - Db'(x) \right) \frac{dx}{b(x)} \right]_{x=g(y)}$$

because, taking for short

$$A(x) = \frac{a(x)}{b(x)} - Db'(x)$$

we have³ for the change of variable $y = h(x)$

$$\begin{aligned} \int \hat{a}(y) dy &= \int A(g(y)) dy = \left[\int A(x)h'(x) dx \right]_{x=g(y)} \\ &= \left[\int \left(\frac{a(x)}{b(x)} - Db'(x) \right) \frac{dx}{b(x)} \right]_{x=g(y)} \end{aligned}$$

or also the equivalent formulation

$$\left[\int \hat{a}(y) dy \right]_{y=h(x)} = \int \left(\frac{a(x)}{b(x)} - Db'(x) \right) \frac{dx}{b(x)}$$

³Given a function $f(x)$, its primitives are the functions $F(x)$ such that

$$F(x) = \int f(x) dx \quad F'(x) = f(x)$$

Let us suppose now to have an invertible transformation of variables $y = h(x)$ with

$$y = h(x) \quad x = h^{-1}(y) = g(y) \quad g(h(x)) = x \quad g'(y) = \frac{1}{h'(g(y))}$$

It is then easy to show that the function

$$F_1(x) = \left[\int f(g(y))g'(y) dy \right]_{y=h(x)} = G(h(x)) \quad G(y) = \int f(g(y))g'(y) dy$$

is again a primitive of $f(x)$ because

$$F_1'(x) = G'(h(x))h'(x) = h'(x) [f(g(y))g'(y)]_{y=h(x)} = h'(x) \left[\frac{f(g(y))}{h'(g(y))} \right]_{y=h(x)} = f(x)$$

As a consequence the rules for the change of variables in the indefinite integrals are

$$\int f(x) dx = \left[\int f(g(y))g'(y) dy \right]_{y=h(x)} \quad \left[\int f(x) dx \right]_{x=g(y)} = \int f(g(y))g'(y) dy$$

To have the formula in the text just take $f(x) = A(x)h'(x)$

so that from (176) and (177) we find

$$\tilde{f}_X(x) = \frac{e^{\frac{1}{D} \int (\frac{a(x)}{b(x)} - D b'(x)) \frac{1}{b(x)} dx}}{Z |b(x)|}$$

On the other hand, since it is easy to see that

$$e^{\int \frac{b'(x)}{b(x)} dx} = e^{\int \frac{d}{dx} |\ln b(x)| dx} = e^{\ln |b(x)|} = |b(x)|$$

we finally have

$$\tilde{f}_X(x) = Z^{-1} e^{\frac{1}{D} \int (\frac{a(x)}{b(x)} - 2D b'(x)) \frac{1}{b(x)} dx} \quad (178)$$

Now it is easy to check that this $\tilde{f}_X(x)$ is the invariant solution of the *FPE* (173): we find indeed that

$$\begin{aligned} D\partial_x \left[b^2(x) \tilde{f}_X(x) \right] &= \tilde{f}_X(x) \left[2Db(x)b'(x) + b^2(x) \left(\frac{a(x)}{b(x)} - 2D b'(x) \right) \frac{1}{b(x)} \right] \\ &= a(x) \tilde{f}_X(x) \end{aligned}$$

It is finally apparent that the *pdf* (178) also coincides with the invariant *pdf* (168) because within the notation of the Section C.3 we have

$$-\frac{B'(x) - \vec{v}(x)}{B(x)} = \left(\frac{a(x)}{b(x)} - 2D b'(x) \right) \frac{1}{b(x)}$$

By summarizing we can say that, if the *pdf* (176) is the invariant solution (if it exists) of the stationary *FPE* (175) for the process $Y(t) = h(X(t))$, then the *pdf* (178) is the corresponding invariant solution of the *FPE* (173) for the process $X(t)$

D An *anamnesis* of stochastic mechanics

Initially proposed as a possible interpretation of quantum mechanics [13, 22] with the challenging aim of shedding new light on its enduring mysteries, over the years the stochastic mechanics evolved into a more general theory dealing with conservative diffusion processes [23]. Its tools are therefore valuable nowadays even beyond the strict quantum precinct, and in particular they have been employed in the broad field of the stochastic control [24]. We will refrain however from giving here a comprehensive review of these topics referring the interested readers to the quoted literature, and we will rather confine ourselves in the present appendix to recall just the few results deemed to be instrumental for the follow-up of the present enquiry

From a quantum wave function $\psi(x, t)$ solution of a (one-dimensional) Schrödinger equation

$$i\hbar\partial_t\psi(x, t) = -\frac{\hbar^2}{2m}\partial_x^2\psi(x, t) + V(x, t)\psi(x, t) \quad (179)$$

we can deduce the form of the forward velocity $\vec{v}(x, t)$ – here the upper arrow just means *forward*, and does not denote a *vector*, while $\overleftarrow{v}(x, t)$ will be understood as a *backward* velocity – appearing in both the *FPE*

$$\begin{aligned} \partial_t f(x, t) &= -\partial_x[\vec{v}(x, t)f(x, t)] + D \partial_x^2 f(x, t) \\ &= \partial_x \left[D \partial_x f(x, t) - \vec{v}(x, t)f(x, t) \right] \end{aligned} \quad (180)$$

for the *pdf* $f(x, t) = |\psi(x, t)|^2$, and the associated Itô *SDE*

$$dX(t) = \vec{v}(X(t), t)dt + dW(t) \quad (181)$$

for the corresponding Markov process $X(t)$ in the framework of the stochastic mechanics: here $W(t)$ is a Wiener process with a constant diffusion coefficient $2D = \frac{\hbar}{m}$, namely such that $\mathbf{E}[W(t)^2] = 2Dt$. If $\psi(x, t)$ is an arbitrary solution of (179), it is well known indeed that with the usual *Ansatz*

$$\psi(x, t) = R(x, t) e^{iS(x, t)/\hbar} \quad (182)$$

where R and S are real functions, $R^2 = |\psi|^2$ comes out to be a particular solution of the *FPE* (180) with forward velocity field of the form

$$\vec{v}(x, t) = \frac{\partial_x S(x, t)}{m} + \frac{\hbar}{2m} \partial_x [\ln R^2(x, t)] \quad (183)$$

as it is deduced by separating the real and imaginary parts of (179). Remark that the explicit dependence of \vec{v} on the form of R clearly indicates that to have a solution of (180) which makes quantum sense we must pick-up just one, suitable solution. In the stochastic mechanical framework, indeed, the system is ruled not only by the *FPE* (180), but also by a second, dynamical equation (the imaginary part)

$$\partial_t S(x, t) + \frac{[\partial_x S(x, t)]^2}{2m} + V(x, t) - \frac{\hbar^2}{2m} \frac{\partial_x^2 R(x, t)}{R(x, t)} = 0 \quad (184)$$

known as Hamilton-Jacobi-Madelung equation

Let us consider now the Schrödinger equation (179) in the case of a time-independent potential $V(x)$, with a Hamiltonian

$$\widehat{H}\psi(x) = -\frac{\hbar^2}{2m} \partial_x^2 \psi(x) + V(x)\psi(x)$$

with purely discrete spectrum and stationary, normalizable states, and let us use the following notations for these states, and their eigenvalues and eigenfunctions:

$$\begin{aligned} \psi_n(x, t) &= \phi_n(x) e^{-iE_n t/\hbar} \\ \widehat{H}\phi_n(x) &= -\frac{\hbar^2}{2m} \phi_n''(x) + V(x)\phi_n(x) = E_n \phi_n(x) \end{aligned}$$

For later convenience we will also introduce the constant

$$D = \frac{\hbar}{2m} \tag{185}$$

so that the previous eigenvalue equation can be recast in the following form

$$D\phi_n''(x) = \frac{V(x) - E_n}{\hbar} \phi_n(x)$$

For a stationary solution $\psi_n(x, t)$ the *Ansatz* (182) will give

$$S(x, t) = -E_n t, \quad R(x, t) = \phi_n(x)$$

so that for our stationary states the velocity fields are

$$\vec{v}_n(x) = 2D \frac{\phi_n'(x)}{\phi_n(x)} \tag{186}$$

D.1 The *FPE* for stationary states

From (186) we see that the forward velocities $\vec{v}_n(x)$ for stationary states are time-independent, and that they have singularities in the zeros (nodes) of the eigenfunction. Since the n -th eigenfunction of a quantum system with bound states has exactly n simple nodes x_1, \dots, x_n , the coefficients of the *FPE*

$$\partial_t f(x, t) = -\partial_x [\vec{v}_n(x) f(x, t)] + D \partial_x^2 f(x, t) = \partial_x \left[D \partial_x f(x, t) - \vec{v}_n(x) f(x, t) \right] \tag{187}$$

diverge in these n points and we will be obliged to solve it in separate intervals by imposing suitable boundary conditions connecting the different sections (see Appendix C for further details). As a matter of fact, these singularities effectively separate the real axis in $n + 1$ sub-intervals with impenetrable (to the probability current) walls. Hence the process will not have an *unique* invariant measure and will never cross the boundaries fixed by the singularities of $\vec{v}_n(x)$: if we start at t_0 in one of the intervals in which the axis is divided we will always remain therein. As a consequence, with an arbitrary initial distribution, we must require that the integrals

$$\int_{x_k}^{x_{k+1}} f(x, t) dx$$

be kept at a constant value for $t \geq t_0$: this values will not, in general, be equal to 1 (only their sum will amount to 1) and, since the separate intervals can not communicate, they will be fixed by the choice of the initial conditions. The boundary conditions are hence imposed by the conservation of the probability in $[x_k, x_{k+1}]$ and that entails the vanishing of the probability current in (187) at the end points of the intervals:

$$\left[D \partial_x f(x, t) - \vec{v}_n(x) f(x, t) \right]_{x_k, x_{k+1}} = 0, \quad t \geq t_0$$

Therefore, since every particular solution is selected by the initial conditions, we are first interested in finding the transition *pdf* $f(x, t|y, s)$ which is singled out by the condition

$$\lim_{t \rightarrow s^+} f(x, t) = f(x, s^+) = \delta(x - y) \quad (188)$$

The evolution of any other initial condition $f_0(x)$ at $t = s$ is subsequently ruled by the Chapman-Kolmogorov equation

$$f(x, t) = \int_{-\infty}^{+\infty} f(x, t|y, s) f_0(y) dy \quad (189)$$

To solve (187) in every interval $[x_k, x_{k+1}]$ (both finite or infinite), when we already know the invariant, time-independent solution $\phi_n^2(x)$, we usually put

$$f(x, t) = \phi_n(x)g(x, t)$$

in order to reduce (187) to the form

$$\partial_t g = \mathcal{L}_n g \quad (190)$$

where \mathcal{L}_n is now the self-adjoint operator defined on $[x_k, x_{k+1}]$ as

$$\mathcal{L}_n g(x) = \frac{d}{dx} \left[p(x) \frac{dg(x)}{dx} \right] - q_n(x)g(x)$$

$$p(x) = D > 0 \quad q_n(x) = \frac{\vec{v}_n^2(x)}{4D} + \frac{\vec{v}'_n(x)}{2}$$

To solve (190) it is then advisable to separate the variables into $g(x, t) = \gamma(t)G(x)$, so that we immediately have $\gamma(t) = e^{-\lambda t}$, while $G(x)$ must be a solution of the Sturm-Liouville eigenvalue problem associated to the equation

$$\mathcal{L}_n G(x) + \lambda G(x) = 0 \quad (191)$$

with the boundary conditions

$$\left[2DG'(x) - \vec{v}_n(x)G(x) \right]_{x_k, x_{k+1}} = 0 \quad (192)$$

The general behavior of the solutions obtained as expansions in the system of the eigenfunctions of (191) has already been discussed elsewhere [5, 6]

D.2 The Itô *SDE* for stationary states

If on the other hand we would like to see the problem from the standpoint of the Itô *SDE*'s for some Markov process $X(t)$ we will be confronted with the Smoluchowsky equations of the type (107)

$$dX(t) = \vec{v}_n(X(t))dt + dW(t) \quad (193)$$

which are the path-wise counterparts of the *FPE*'s (187). It is well known from our discussion in the Appendix C.4 that in this case, if

$$e^{\frac{1}{D} \int \vec{v}_n(x) dx}$$

is an integrable function, then

$$\frac{e^{\frac{1}{D} \int \vec{v}_n(x) dx}}{\int_{\mathbf{R}} e^{\frac{1}{D} \int \vec{v}_n(x) dx} dx} \tag{194}$$

is the *pdf* of the stationary solution of (193). On the other hand we also know that there is no simple way to find the general solutions of (193) without scrutinizing it in particular cases

D.3 QHO stationary states

Let us then consider in detail the particular example of a *QHO* with the potential

$$V(x) = \frac{m}{2} \omega^2 x^2$$

It is well-known that its eigenvalues are

$$E_n = \hbar\omega \left(n + \frac{1}{2} \right) ; \quad n = 0, 1, 2 \dots$$

while, with the notation

$$\sigma^2 = \frac{\hbar}{2m\omega} = \frac{D}{\omega} \tag{195}$$

the eigenfuncions are

$$\phi_n(x) = \frac{1}{\sqrt{\sigma\sqrt{2\pi}2^n n!}} e^{-x^2/4\sigma^2} H_n \left(\frac{x}{\sigma\sqrt{2}} \right)$$

where H_n are the Hermite polynomials (see [7] 8.95). The corresponding forward velocity fields are then easily calculated from (186)

$$\vec{v}_n(x) = \omega\sigma\sqrt{2} \frac{H'_n \left(\frac{x}{\sigma\sqrt{2}} \right)}{H_n \left(\frac{x}{\sigma\sqrt{2}} \right)} - \omega x = \omega\sigma\sqrt{2} \left[\frac{2nH_{n-1}(z)}{H_n(z)} - z \right]_{z=\frac{x}{\sigma\sqrt{2}}} \tag{196}$$

and the first examples are

$$\begin{aligned} \vec{v}_0(x) &= -\omega x \\ \vec{v}_1(x) &= -\omega x + \omega\sigma \frac{2\sigma}{x} \\ \vec{v}_2(x) &= -\omega x + \omega\sigma \frac{4\sigma x}{x^2 - \sigma^2} \\ \vec{v}_3(x) &= -\omega x + \omega\sigma \frac{6\sigma(x^2 - \sigma^2)}{x(x^2 - 3\sigma^2)} \\ \vec{v}_4(x) &= -\omega x + \omega\sigma \frac{8\sigma x(x^2 - 3\sigma^2)}{x^4 - 6\sigma^2 x^2 + 3\sigma^4} \end{aligned}$$

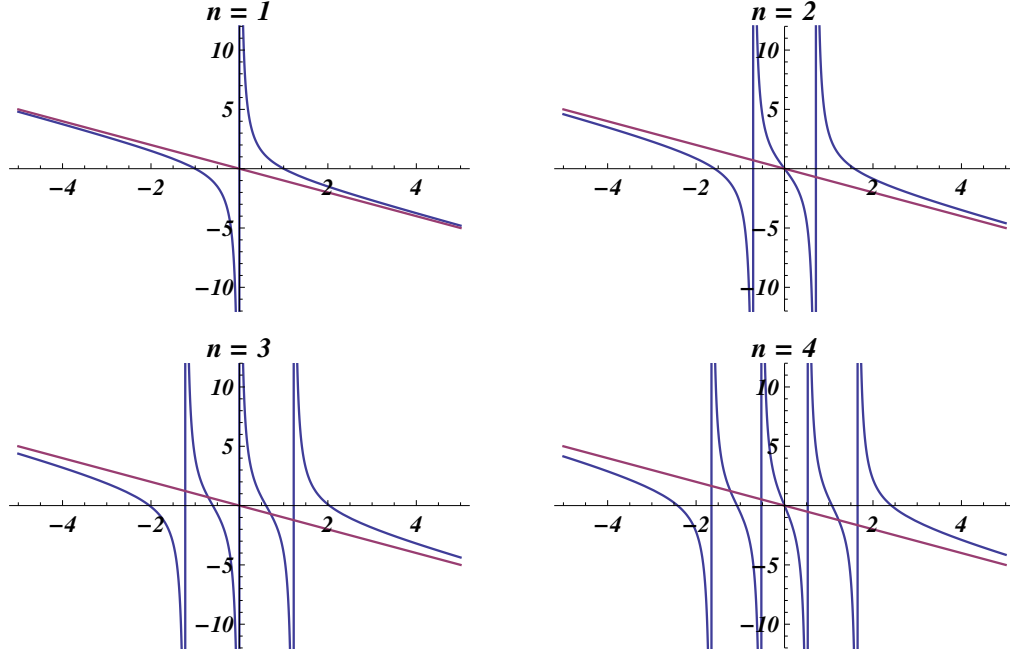


Figure 4: The dimensionless forward velocities $\frac{\vec{v}_n}{\omega\sigma\sqrt{2}}$ for $n = 1, 2, 3, 4$ plotted in function of the dimensionless variable $\frac{x}{\sigma\sqrt{2}}$. The case $n = 0$ leads to the straight line

with singularities in the zeros x_k of the Hermite polynomials. These velocities are *piecewise* (between two subsequent singularities) *monotonic decreasing* functions as displayed in the Figure 4

D.4 The processes for the QHO stationary states

We first recall a few general remarks about the solution methods of the eigenvalue problem (191) which for our forward velocities $\vec{v}_n(x)$, with $\epsilon = \hbar\lambda$, can be written as

$$-\frac{\hbar^2}{2m} G''(x) + \left(\frac{m}{2} \omega^2 x^2 - \hbar\omega \frac{2n+1}{2} \right) G(x) = \epsilon G(x)$$

in every interval $[x_k, x_{k+1}]$, $k = 0, 1, \dots, n$ between two subsequent singularities of $\vec{v}_n(x)$, with the boundary conditions

$$[\phi_n(x)G'(x) - \phi'_n(x)G(x)]_{x_k, x_{k+1}} = 0$$

Since $\phi_n(x)$ (but not $\phi'_n(x)$) vanishes in the x_k 's, the actual boundary conditions to impose are

$$G(x_k) = G(x_{k+1}) = 0$$

where it is understood that in $x_0 = -\infty$ and $x_{n+1} = +\infty$ this respectively means

$$\lim_{x \rightarrow -\infty} G(x) = 0 \quad \lim_{x \rightarrow +\infty} G(x) = 0$$

In a dimensionless form, by using $z = \frac{x}{\sigma}$, $\mu = \frac{\epsilon}{\hbar\omega} = \frac{\lambda}{\omega}$ and $\chi(z) = G(\sigma z)$, our eigenvalue problem then becomes

$$\chi''(z) - \left(\frac{z^2}{4} - \frac{2n+1}{2} - \mu \right) \chi(z) = 0 \quad \chi(z_k) = \chi(z_{k+1}) = 0 \quad (197)$$

where z_k, z_{k+1} are the new dimensionless endpoints. If now μ_m and $\chi_m(z)$ are the eigenvalues and eigenfunctions, the general solution of the *FPE* (187) will be

$$f(x, t) = \sum_{m=0}^{\infty} c_m e^{-\mu_m \omega t} \phi_n(x) \chi_m \left(\frac{x}{\sigma} \right)$$

where the coefficients c_m will be fixed by the initial conditions and by the non-negativity and normalization requirements for $f(x, t)$ along all its evolution. We finally remember that two linearly independent solutions of the ordinary differential equation (197) are

$$\begin{aligned} \chi^{(1)}(z) &= e^{-z^2/4} M \left(-\frac{\mu+n}{2}, \frac{1}{2}; \frac{z^2}{2} \right) \\ \chi^{(2)}(z) &= z e^{-z^2/4} M \left(-\frac{\mu+n-1}{2}, \frac{3}{2}; \frac{z^2}{2} \right) \end{aligned}$$

where $M(a, b; z)$ are the confluent hypergeometric functions [7]

D.4.1 The ground state $n = 0$

When $n = 0$ the *FPE* (187) takes the form

$$\partial_t f(x, t) = \omega \sigma^2 \partial_x^2 f(x, t) + \partial_x [\omega x f(x, t)] \quad (198)$$

while the corresponding Itô *SDE* (193) is

$$dX(t) = -\omega X(t) dt + dW(t) \quad (199)$$

where $W(t)$ is a Wiener process with diffusion coefficient $D = \omega \sigma^2$. In both cases we at once recognize an OU process with transition *pdf*

$$f(x, t|y, s) = \frac{1}{\beta(t-s)\sqrt{2\pi}} e^{-[x-\alpha(t-s)]^2/2\beta^2(t-s)} \quad t \geq s \quad (200)$$

where we used the notation

$$\alpha(t) = ye^{-\omega t} \quad \beta^2(t) = \sigma^2 (1 - e^{-2\omega t}) \quad (201)$$

This solution of (198), which obeys the initial condition $f(x, s) = \delta(x - y)$, is also the *pdf* of the solution of the *SDE* (199) with initial condition $X(s) = y$, **P**-*a.s.*, namely

$$X(t) = ye^{-\omega(t-s)} + \int_s^t e^{-\omega(t-t')} dW(t')$$

The stationary process is instead selected by the initial condition $X(0) \sim \mathfrak{N}(0, \sigma^2)$ namely by the invariant initial *pdf*

$$f(x, 0) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}$$

which is also the asymptotic *pdf* for every initial condition when the evolution is ruled by (200), so that the invariant distribution plays also the role of the limit distribution. It is remarkable that this invariant *pdf* also coincides with the quantum stationary *pdf* $\phi_0^2(x) = |\psi_0(x, t)|^2$: in other words, the process associated by the stochastic mechanics to the ground state of a *QHO* is a stationary *OU* process

D.4.2 The the first excited state $n = 1$

In the case $n = 1$ the forward velocity $\vec{v}_1(x)$ has a singularity in $x = 0$, the *FPE* is

$$\partial_t f(x, t) = \omega\sigma^2 \partial_x^2 f(x, t) + \partial_x \left[\left(\omega x - \frac{2\omega\sigma^2}{x} \right) f(x, t) \right] \quad (202)$$

while the corresponding Itô *SDE* (193) is

$$dX(t) = \left(-\omega X(t) + \frac{2\omega\sigma^2}{X(t)} \right) dt + dW(t) \quad (203)$$

The *FPE* (202) is then reduced to the eigenvalue problem (197) with $x_0 = -\infty$, $x_1 = 0$ and $x_2 = +\infty$, namely (with $z = x/\sigma$)

$$\chi''(z) - \left(\frac{z^2}{4} - \frac{3}{2} - \mu \right) \chi(z) = 0 \quad \chi(-\infty) = \chi(0) = \chi(+\infty) = 0 \quad (204)$$

This problem should be separately solved for $z \leq 0$ and for $z \geq 0$. The eigenvalues turn out to be $\mu_m = 2m$ with $m = 0, 1, \dots$ and the complete set of eigenfunctions (for both $z \geq 0$, and $z \leq 0$) is

$$\chi_m(z) = ze^{-z^2/4} M \left(-m, \frac{3}{2}; \frac{z^2}{2} \right) = \frac{(-1)^m m!}{\sqrt{2}(2m+1)!} e^{-z^2/4} H_{2m+1} \left(\frac{z}{\sqrt{2}} \right)$$

In particular it is easy to see that the relation with the quantum eigenfunction ϕ_1 is

$$\phi_1(x) = \frac{\chi_0(x/\sigma)}{\sqrt{\sigma\sqrt{2}}} \quad \chi_0(z) = ze^{-z^2/4}$$

and that the solution of (202) for an initial condition imposed at the time s is

$$f(x, t) = \sum_{m=0}^{\infty} c_m e^{-2m\omega(t-s)} \phi_1(x) \chi_m \left(\frac{x}{\sigma} \right) \quad t \geq s$$

where the c_m 's are fixed by the initial condition. For the transition *pdf* from (188) we have

$$c_m = \frac{2}{\sqrt{\sigma\sqrt{2\pi}}} \frac{(2m+1)!!}{(2m)!!} \frac{\chi_m(y/\sigma)}{\chi_0(y/\sigma)}$$

and by summing up the series [5, 6] we will have with the notations (201)

$$f(x, t|y, s) = \Theta(xy) \frac{x}{\alpha(t-s)} \frac{e^{-\frac{[x-\alpha(t-s)]^2}{2\beta^2(t-s)}} - e^{-\frac{[x+\alpha(t-s)]^2}{2\beta^2(t-s)}}}{\beta(t-s)\sqrt{2\pi}} \quad (205)$$

where $\Theta(z)$ is the Heaviside function. In particular we have

$$\lim_{t \rightarrow +\infty} f(x, t|y, s) = 2\Theta(xy) \frac{x^2 e^{-x^2/2\sigma^2}}{\sigma^3 \sqrt{2\pi}} = 2\Theta(xy) \phi_1^2(x)$$

and for an arbitrary initial condition $f(x, s^+) = f_0(x)$ we have

$$\begin{aligned} \lim_{t \rightarrow +\infty} f(x, t) &= \lim_{t \rightarrow +\infty} \int_{-\infty}^{+\infty} f(x, t|y, s) f_0(y) dy \\ &= 2\phi_1^2(x) \int_{-\infty}^{+\infty} \Theta(xy) f_0(y) dy = \Gamma(q; x) \phi_1^2(x) \end{aligned}$$

where we have defined the function

$$\Gamma(q; x) = q\Theta(x) + (2-q)\Theta(-x); \quad q = 2 \int_0^{+\infty} f_0(y) dy$$

Remark that $q = 1$ when the initial probability is equally shared on the two (positive and negative) half-lines, and in this case we have $\Gamma(1; x) = 1$ so that the asymptotical *pdf* exactly coincides with the quantum stationary pdf $\phi_1^2(x)$. If on the other hand $q \neq 1$ the asymptotical *pdf* has the same shape of $\phi_1^2(x)$ but with different weights on the two half-lines. The transition *pdf* (205) is however less elementary of what it looks like at first sight. It is possible for instance to calculate expectations, but the results are not very simple: for instance we have

$$\begin{aligned} \mathbf{E}[X(t) | X(0) = y] &= \int_0^\infty \frac{x^2}{\alpha(t)} \frac{e^{-\frac{[x-\alpha(t)]^2}{2\beta^2(t)}} - e^{-\frac{[x+\alpha(t)]^2}{2\beta^2(t)}}}{\beta(t)\sqrt{2\pi}} \\ &= \sqrt{\frac{2}{\pi}} \beta(t) e^{-\frac{\alpha^2(t)}{2\beta^2(t)}} + \frac{\alpha^2(t) + \beta^2(t)}{\alpha(t)} \left[2\Phi\left(\frac{\alpha(t)}{\beta(t)}\right) - 1 \right] \end{aligned}$$

where as usual

$$\Phi(z) = \int_{-\infty}^z \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx$$

while $\alpha(t)$ and $\beta(t)$ are defined in (201). If on the other hand $X(0)$ is not degenerate in y but has a *pdf* $f_0(y)$, the expectation $\mathbf{E}[X(t)]$ would be calculated as an y -integral of $\mathbf{E}[X(t) | X(0) = y] f_0(y) dy$, by recalling of course that the variable y is hidden into $\alpha(t)$

The transition *pdf* (205) anyhow completely solves the problem of the *FPE* (202), and hence also completely defines the law of the Markov process solution of the (non linear) *SDE* (203): it is then puzzling to remark that apparently we do not know any simple procedure to solve (203) as a *SDE*. For instance we could think of changing (203) in some other, more manageable form by means of a transformation $Y(t) = h(X(t))$ with the Itô formula (see Appendix B). Given the form of our equation, the simplest idea could seem to be to take $Y(t) = X^2(t)$. Now (203) is an Itô *SDE* with the following coefficients

$$a(x) = \vec{v}_1(x) = -\omega x + \frac{2\omega\sigma^2}{x} \qquad b(x) = 1 \qquad (206)$$

but our transformation $y = h(x) = x^2 \geq 0$ (with $h'(x) = 2x$, $h''(x) = 2$) is only *piecewise monotonic* separately on the two real half-lines, so that we should separate the process in two regions according to the sign of $x = g(y) = \pm\sqrt{y}$: a procedure a bit confusing if done by hand. This of course corresponds to the fact that the $a(x)$ has a singularity in $x = 0$ which effectively separates the two half-lines. The Itô calculus implies that $Y(t)$ will satisfy a new *SDE* with the coefficients (100) and (101), namely (with $D = \omega\sigma^2$)

$$\hat{a}(y) = -2\omega y + 6\omega\sigma^2 \qquad \hat{b}(y) = \pm 2\sqrt{y}$$

so that we will have one of the two equations for $Y(t) \geq 0$

$$dY(t) = (6\omega\sigma^2 - 2\omega Y(t)) dt \pm 2\sqrt{Y(t)}dW(t)$$

which in fact are not much easier to handle than the original one, and also seem not to be on a very firm ground because of the piecewise monotonicity of $h(x)$

On the other hand there not seems to be any hope of transforming (203) in a *SDE* with *linear coefficients* by means of some other clever transformation because the compatibility conditions [12] are not satisfied: in fact, from (206) and with

$$q(x) = \frac{a(x)}{b(x)} - Db'(x) = a(x) = -\omega x + \frac{2\omega\sigma^2}{x}$$

we have

$$\frac{1}{q'(x)} \frac{d}{dx} [b(x)q'(x)] = \frac{-4\omega\sigma^2}{\omega x(x^2 + 2\sigma^2)}$$

which apparently is not x -independent, as instead it is required by the compatibility conditions (see [12] p 39). In any case, since we completely know the law of $X(t)$ through the transition *pdf* (205), some effort could be produced in the hope of

gaining an insight into the process $X(t)$ solution of (203) from the fact that (205) turns out to be some kind of combination of symmetrically separated OU solutions. The proposed transformation $y = x^2$, however, looks rather preposterous because $h(x) = x^2$ is not a monotonic function: we will discuss a possible monotonic modification in the Section D.6 of the present appendix

D.4.3 The second excited state $n = 2$

If $n = 2$ the velocity $\vec{v}_2(x)$ has singularities in $x = \pm\sigma$, the FPE is

$$\partial_t f(x, t) = \omega\sigma^2 \partial_x^2 f(x, t) + \partial_x \left[\left(\omega x - \frac{4\omega\sigma^2 x}{x^2 - \sigma^2} \right) f(x, t) \right] \quad (207)$$

while the corresponding Itô SDE (193) is

$$dX(t) = \left(-\omega X(t) + \frac{4\omega\sigma^2 X(t)}{X^2(t) - \sigma^2} \right) dt + dW(t) \quad (208)$$

The FPE (207) is then reduced to the eigenvalue problem (197) with $x_0 = -\infty$, $x_1 = -\sigma$, $x_2 = \sigma$ and $x_3 = +\infty$, namely (with $z = x/\sigma$)

$$\chi''(z) - \left(\frac{z^2}{4} - \frac{5}{2} - \mu \right) \chi(z) = 0 \quad \chi(-\infty) = \chi(\pm 1) = \chi(+\infty) = 0 \quad (209)$$

that should be separately solved in the three intervals $(-\infty, -1]$, $[-1, 1]$ and $[1, +\infty)$. The two linearly independent solutions are now

$$\chi^{(1)}(z) = e^{-z^2/4} M \left(-\frac{\mu+2}{2}, \frac{1}{2}; \frac{z^2}{2} \right) \quad \chi^{(2)}(z) = ze^{-z^2/4} M \left(-\frac{\mu+1}{2}, \frac{3}{2}; \frac{z^2}{2} \right)$$

and it is easy to check that $\mu = 0$ is an eigenvalue for all the three intervals with eigenfunction

$$\chi_0(z) = e^{-z^2/4} M \left(-1, \frac{1}{2}; \frac{z^2}{2} \right) = e^{-z^2/4} H_2 \left(\frac{z}{\sqrt{2}} \right) = 2e^{-z^2/4}(z^2 - 1)$$

so that the relation with the quantum eigenfunction now is

$$\phi_2(x) = \frac{\chi_0(x/\sigma)}{\sqrt{8\sigma\sqrt{2\pi}}}$$

As for the other eigenvalues and eigenfunction they can be obtained only numerically: for example it can be shown that, beyond $\mu_0 = 0$, the first eigenvalues in the interval $[-1, 1]$ can be calculated as the first values such that

$$M \left(-\frac{\mu+1}{2}, \frac{3}{2}; \frac{1}{2} \right) = 0$$

and are $\mu_1 \sim 7.44$, $\mu_2 \sim 37.06$, $\mu_3 \sim 86.41$. Also for the unbounded interval $[1, +\infty)$ (the analysis is similar for $(-\infty, -1]$) the eigenvalues are derivable only numerically.

D.5 The macroscopic limit

From the *SDE* (193) one could hope to derive some macroscopic, deterministic equation describing the behavior of some global characteristic of the process. However it is easy to see that by taking the expectation of (193) we just have

$$\frac{d\mathbf{E}[X(t)]}{dt} = \mathbf{E}\left[\vec{v}_n(X(t))\right]$$

but that we can not deduce any equation for $\mathbf{E}[X(t)]$ because the velocities $\vec{v}_n(x)$ are not linear functions (with the only exception of $\vec{v}_0(x)$), and hence the term $\mathbf{E}\left[\vec{v}_n(X(t))\right]$ can not be put in the form $\vec{v}_n(\mathbf{E}[X(t)])$. We could surmise that (at least when the form of $\vec{v}_n(x)$ is explicitly given, as for the *QHO*) some function $h_n(x)$ can be found such that

$$\mathbf{E}\left[\vec{v}_n(X(t))\right] = h_n(\mathbf{E}[X(t)])$$

but this seems not to be an easy task, even for the simplest case of $\vec{v}_1(x)$. The outlook seems not to be much brighter if we take the *medians* of (193) instead of the expectations: as a matter of fact in this case – because of the non linearity of the functional $\mathbf{M}[\dots]$ – we can not even suppose that $\mathbf{M}[dX(t)]$ is simply reduced to $d\mathbf{M}[X(t)]$, and hence the hope to obtain a differential equation for the median is dashed from the beginning

The unique viable way to have deterministic equations from (193) seems then to be to *switch off* the Wiener noise by taking a vanishing diffusion coefficient $D = 0$, while still keeping a non-zero Planck constant \hbar in order to have non-trivial values for σ in the $\vec{v}_n(x)$: this means of course that the relation (185) no longer holds, so that now in some sense we are out of the framework of the stochastic mechanics. In any case it could be instructive to see what kind of deterministic trajectories are solutions of the dynamical systems

$$\dot{x}(t) = \vec{v}_n(x(t)) \tag{210}$$

associated to the forward velocities of the stationary states of a *QHO*. Since our forward velocities (196) are time-independent functions, the *ODE*'s (210) can be solved by separation of the variables for $t \geq 0$ with initial condition $x(0) = y$

$$\int_{y/\sigma\sqrt{2}}^{x/\sigma\sqrt{2}} \frac{H_n(z)}{2nH_{n-1}(z) - zH_n(z)} dz = \omega t \tag{211}$$

but since there is no general formula giving the solutions for every n we will be obliged to show them one by one: for $n = 0$ the equation is

$$\dot{x}(t) = -\omega x(t)$$

and from (211) we have

$$[\ln z]_{y/\sigma\sqrt{2}}^{x/\sigma\sqrt{2}} = -\omega t$$

so that every initial condition will eventually go to $x(+\infty) = 0$ according to

$$x(t) = ye^{-\omega t} \tag{212}$$

For $n = 1$ the equation is

$$\dot{x}(t) = -\omega x(t) + \frac{2\omega\sigma^2}{x(t)}$$

the solution is

$$[\ln(z^2 - 1)]_{y/\sigma\sqrt{2}}^{x/\sigma\sqrt{2}} = -2\omega t$$

and hence

$$x^2(t) = 2\sigma^2 + (y^2 - 2\sigma^2)e^{-2\omega t} \tag{213}$$

The trajectory $x(t)$ will then exponentially go from y to $\pm\sigma\sqrt{2}$ where the sign of the square root will be decided according to the sign of the initial value y : in this second case we have indeed two attracting points. For $n = 2$ on the other hand the equation is

$$\dot{x}(t) = -\omega x(t) + \frac{4\omega\sigma^2 x}{x^2(t) - \sigma^2}$$

and we have

$$[\ln z(2z^2 - 5)^2]_{y/\sigma\sqrt{2}}^{x/\sigma\sqrt{2}} = -5\omega t$$

namely the trajectories are implicitly defined by

$$x(t) [x^2(t) - 5\sigma^2]^2 = y(y^2 - 5\sigma^2)^2 e^{-5\omega t} \tag{214}$$

and, while not easy to be calculated explicitly, they will have now three asymptotic attracting points in $x = 0$ and $x = \pm\sigma\sqrt{5}$. The solutions are much less elementary for $n \geq 3$: for instance with $n = 3$ the trajectories are implicitly defined by

$$\left[\ln(\sqrt{57} + 9 - 4z^2)^{19+\sqrt{57}}(\sqrt{57} - 9 + 4z^2)^{19-\sqrt{57}} \right]_{y/\sigma\sqrt{2}}^{x/\sigma\sqrt{2}} = -76\omega t$$

and we are able to find explicitly only the four asymptotic attracting points

$$x = \pm\sigma\sqrt{\frac{9 + \sqrt{57}}{2}} = \pm 2.8766\sigma \quad x = \pm\sigma\sqrt{\frac{9 - \sqrt{57}}{2}} = \pm 0.8515\sigma$$

Subsequent solutions would grow increasingly complicated and will not be displayed here

D.6 Looking deeper into the $n = 1$ case

By looking into the transition *pdf* (205) for the $n = 1$ eigenstate we see that it is a combination of two *OU* transition *pdf*'s (200) with opposite expectations $\pm\alpha$. This combination, however, does not qualify as a proper *mixture* because of the opposite sign of the two terms. Remark in any case that, these signs notwithstanding, the *pdf* (205) is assembled in such a way that it turns out to be always non-negative over \mathbf{R} , as it must be for a *pdf*. It would be interesting then to understand the nature of this combination because this could possibly shed some light on the kind of combination of *OU* processes constituting a solution of (203)

Proposition D.1. *If $p(x)$ is a pdf with finite expectation $\alpha = \int_{\mathbf{R}} xp(x) dx$, and if*

$$p(x) \geq p(-x) \quad \forall x \geq 0 \quad (215)$$

then $\alpha \geq 0$, and when $\alpha > 0$

$$f(x) = \Theta(x) \frac{x}{\alpha} [p(x) - p(-x)] = \begin{cases} \frac{x}{\alpha} [p(x) - p(-x)] & x \geq 0 \\ 0 & x \leq 0 \end{cases} \quad (216)$$

is a *pdf*, while

$$\bar{f}(x) = \frac{x}{2\alpha} [p(x) - p(-x)] \quad (217)$$

is a symmetric *pdf* in the sense that $\bar{f}(-x) = \bar{f}(x)$. On the other hand, if $\alpha = 0$ we have $p(x) = p(-x)$ (namely $p(x)$ must be symmetric) and f, \bar{f} must be defined – if possible – as a limit for $\alpha \rightarrow 0$

Proof: First of all we have

$$\begin{aligned} \alpha &= \int_{-\infty}^{\infty} xp(x) dx = \int_{-\infty}^0 xp(x) dx + \int_0^{\infty} xp(x) dx \\ &= - \int_0^{\infty} xp(-x) dx + \int_0^{\infty} xp(x) dx = \int_0^{\infty} x[p(x) - p(-x)] dx \geq 0 \end{aligned}$$

because all the terms in the integral are non-negative. Then it is immediate to check that the function

$$q(x) = x[p(x) - p(-x)] \quad x \in \mathbf{R}$$

is symmetric (namely $q(x) = q(-x)$) and non-negative for every $x \in \mathbf{R}$ with $q(0) = 0$. Now it is easy to see that

$$\int_0^{\infty} q(x) dx = \int_0^{\infty} xp(x) dx - \int_0^{\infty} xp(-x) dx = \int_{-\infty}^{\infty} xp(x) dx = \alpha$$

and hence also that

$$\int_{-\infty}^{\infty} q(x) dx = 2\alpha$$

It is apparent then that (216) is a *pdf* concentrated on the positive half-line, while (217) is a symmetric *pdf* defined on \mathbf{R} . Finally, since we have seen that

$$\alpha = \int_0^\infty x[p(x) - p(-x)] dx = \int_0^\infty q(x) dx$$

and $q(x) \geq 0$, then α can not be zero unless $q(x) = 0$, namely $p(x) = p(-x)$. In this case f and \bar{f} can not be defined as (216) and (217) and we must resort to a limit for $\alpha \rightarrow 0$ hoping that the Theorem of l'Hôpital brings it to a finite result ■

It is apparent then that the *pdf* (205) is a particular case of (216) where $p(x)$ is the Gaussian law (200) of an *OU* process. It is not clear, instead, what kind of combination of *rv*'s – if any – admits either $f(x)$ or $\bar{f}(x)$ as their *pdf*'s: let us suppose that X is a *rv* with *pdf* $p(x)$. Then $p(-x)$ will play the role of the *pdf* for $-X$, and the condition (215) could be formulated as

$$\mathbf{P}\{X \in B\} \geq \mathbf{P}\{X \in -B\}$$

where $B \in \mathcal{B}(\mathbf{R}_+)$ is a Borelian on the positive half-line \mathbf{R}_+ , while we define $-B = \{x \in \mathbf{R} \mid -x \in B\}$. Our problem then can be formulated as follows: is either $f(x)$, or $\bar{f}(x)$ the *pdf* of some combination of X and $-X$? Could such a combination obey some simpler form of our *SDE* (203)? We do not know an answer at this point, but we can add just a final remark about the expectations of f and \bar{f} : because of the symmetry we immediately have

$$\int_{-\infty}^\infty x\bar{f}(x) dx = 0$$

while on the other hand

$$\begin{aligned} \int_{-\infty}^\infty xf(x) dx &= \int_0^\infty \frac{x^2}{\alpha}[p(x) - p(-x)] dx = \int_0^\infty \frac{x^2}{\alpha}p(x) dx - \int_0^\infty \frac{x^2}{\alpha}p(-x) dx \\ &= \int_0^\infty \frac{x^2}{\alpha}p(x) dx - \int_{-\infty}^0 \frac{x^2}{\alpha}p(x) dx = \int_{-\infty}^\infty \frac{x|x|}{\alpha}p(x) dx \end{aligned}$$

In other words the expectation of a *rv* Y with *pdf* $f(x)$ seems to coincide with the expectation of the *rv* $\frac{1}{\alpha}X|X|$, if X has the *pdf* $p(x)$ (namely the *pdf* which defines $f(x)$ according to (216) in the Proposition E.2). Remark however that, if X has the *pdf* $p(x)$, $f(x)$ defined in (216) would not be the *pdf* of $Y = X|X|$ which instead, after a short calculation, would be

$$f_Y(y) = \frac{1}{2\sqrt{|y|}}p\left(\frac{|y|}{y}\sqrt{|y|}\right)$$

This discussion, however, in some sense suggests a role for the transformation $Y(t) = h(X(t))$

$$y = h(x) = x|x| = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x \leq 0 \end{cases}$$

which, at variance with x^2 , is now a monotonic transformation with

$$h'(x) = 2|x| \quad h''(x) = 2\frac{|x|}{x} \quad x = g(y) = \begin{cases} \sqrt{y} & y \geq 0 \\ -\sqrt{-y} & y \leq 0 \end{cases}$$

Now, if $X(t)$ is a solution of the Itô *SDE* (203) with the coefficients (206), the Itô calculus implies that $Y(t) = h(X(t))$ will satisfy a new *SDE* with the coefficients (100) and (101), namely (with $D = \omega\sigma^2$)

$$\hat{a}(y) = -2\omega y + 6\omega\sigma^2\frac{|y|}{y} \quad \hat{b}(y) = 2\sqrt{|y|}$$

so that we will have

$$dY(t) = \left(6\omega\sigma^2\frac{|Y(t)|}{Y(t)} - 2\omega Y(t) \right) dt + 2\sqrt{|Y(t)|}dW(t)$$

E Quantiles and medians: a reminder

E.1 Definitions

The law of a *rv* (*random variable*) X whatsoever is characterized by a *cdf* (*cumulative distribution function*) $F(x) = \mathbf{P}\{X \leq x\}$ which is a monotonic, non decreasing function of x confined between 0 and 1, and right-continuous wherever it jumps. It can also show flat spots where no probability is present. The *qf* (*quantile function*) is then usually defined as

$$Q(p) = \inf\{x \in \mathbf{R} : p \leq F(x)\} \quad 0 \leq p \leq 1 \quad (218)$$

This results in a well defined, *one-valued* function with $Q(0) = -\infty$, while $Q(1) = +\infty$ when $F(x)$ only asymptotically reaches the value 1. In the case of continuous laws (no jumps), however, the definition (218) can be reduced to

$$Q(p) = \inf\{x \in \mathbf{R} : p = F(x)\} \quad (219)$$

and when $F(x)$ is also strictly increasing (no flat spots) we finally have

$$Q(p) = F^{-1}(p) \quad (220)$$

It is apparent now that $Q(p)$ jumps wherever $F(x)$ has flat spots, while it has flat spots wherever $F(x)$ jumps. A closer inspection of the definition (218) shows moreover that in its discontinuities $Q(p)$ is left-continuous. In fact every non-decreasing, left-continuous function is a possible *qf*. An extensive presentation of both the properties and the statistical applications of (220) can be found in [16]. In the framework of this notation all the *quantiles* – and in particular the *median* – have a non ambiguous definition

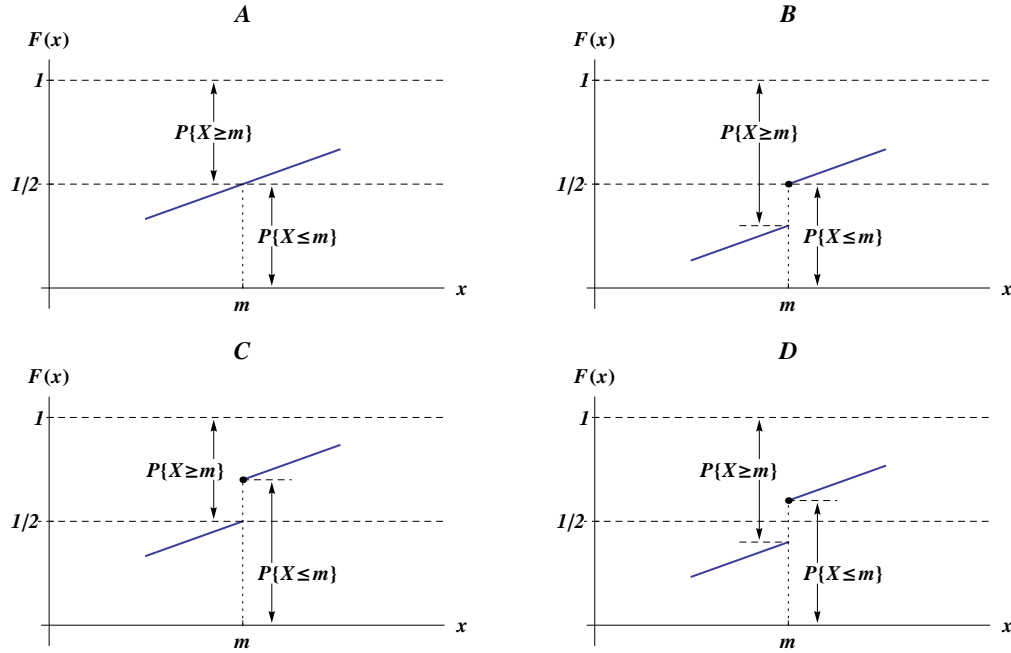


Figure 5: Medians according to the Definition E.2 for *cdf*'s $F(x)$ with possible jumps, but without flat spots: the value of m is unique and coincides with the m_X of the Definition E.1

Definition E.1. The *median* $M[X]$ of a *rv* X is the quantile of order $1/2$, namely

$$m_X = M[X] = Q(1/2) \tag{221}$$

There is however another, more general definition which allows for multi-valued medians in the sense that they can also coincide with a full, closed interval of numbers (the *median segment*), this definition being of interest mostly when we deal with the limits of sums of independent random variables [17, 18]

Definition E.2. The *median* $M[X]$ of a *rv* X is any number m such that

$$P\{X \leq m\} \geq 1/2 \quad \text{and} \quad P\{X \geq m\} \geq 1/2 \tag{222}$$

It is possible to see in fact that when the median defined as in (222) corresponds to an interval of numbers, then – due to the presence of the $\inf\{\dots\}$ – the median m_X defined as in (221) coincides with the left endpoint of the said interval. On the other hand, when the Definition E.2 gives rise to a unique value, the two definitions apparently coincide. This shows that in any case m_X of Definition E.1 always take one (the smallest) of the possible values m of the Definition E.2. Examples of applications of the Definition E.2 are displayed in the Figures 5 and 6, and in particular it is easy to see that, according to the two definitions, for a Bernoulli *rv*

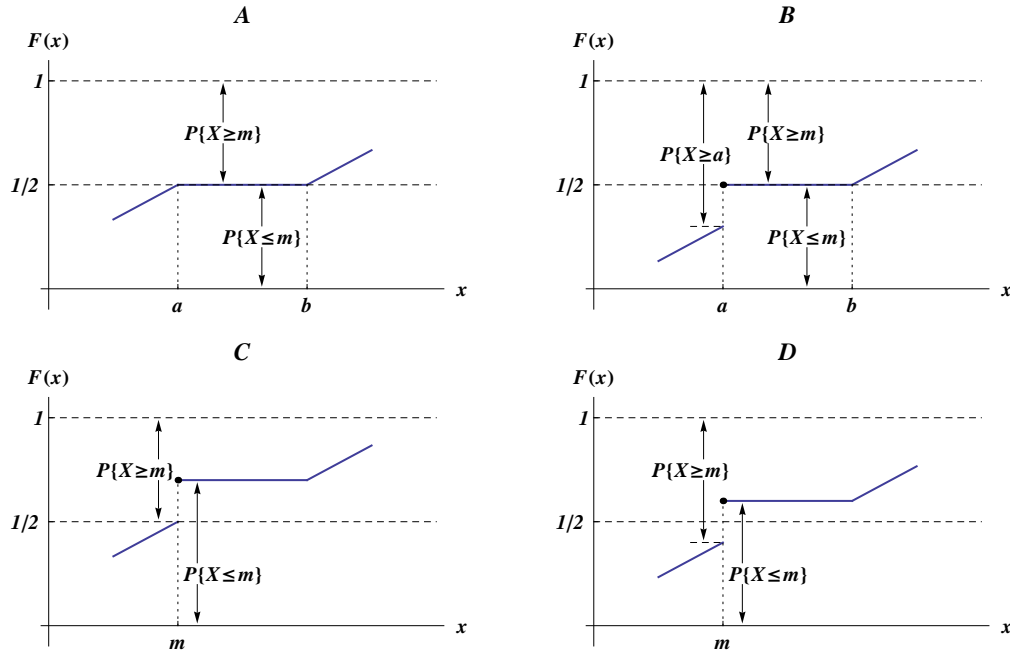


Figure 6: Medians according to the Definition E.2 for *cdf*'s $F(x)$ with flat spots and possible jumps: the values of m fill a non degenerate interval $[a, b]$ whenever the flat spot falls at level $1/2$ as in A and B. In these cases the median m_X of the Definition E.1 coincides with the left endpoint a of the said interval

X taking values 1,0 with probabilities respectively p and $1 - p$ (for $0 \leq p \leq 1$) we have

$$\begin{array}{cc}
 \text{Definition E.1} & \text{Definition E.2} \\
 \mathbf{M}[X] = \begin{cases} 0 & 0 \leq p \leq \frac{1}{2} \\ 1 & \frac{1}{2} < p \leq 1 \end{cases} & \mathbf{M}[X] = \begin{cases} 0 & 0 \leq p < \frac{1}{2} \\ [0,1] & p = \frac{1}{2} \\ 1 & \frac{1}{2} < p \leq 1 \end{cases}
 \end{array}$$

We finally remember that sometimes in statistics, when the median segment does not degenerate in a single point, the median is not its left endpoint but rather some other intermediate point, for example its middle point: we will not however elaborate more on these additional possibilities here

E.2 Properties

Proposition E.3. *Every rv X admits a median according to the Definition E.2, and if $T(x)$ is a monotonic function then it is*

$$\mathbf{M}[T(X)] = T(\mathbf{M}[X])$$

In particular, with $\lambda, \eta \in \mathbf{R}$, we always have

$$\mathbf{M}[-X] = -\mathbf{M}[X] \qquad \mathbf{M}[\lambda + \eta X] = \lambda + \eta \mathbf{M}[X]$$

Proof: If m is a median for X , then from Definition E.2 both the following inequalities must hold

$$\mathbf{P}\{X \leq m\} \geq 1/2 \qquad \text{and} \qquad \mathbf{P}\{X \geq m\} \geq 1/2$$

But with $T(x)$ monotonic these are also equivalent to the pair of inequalities

$$\mathbf{P}\{T(X) \leq T(m)\} \geq 1/2 \qquad \text{and} \qquad \mathbf{P}\{T(X) \geq T(m)\} \geq 1/2$$

so that, always from Definition E.2, $T(m)$ is a median for $T(X)$ ■

On the other hand it is neither easy to compute the medians of a sum in terms of the medians of the summands, nor to relate the medians of an integrable random variable to its mean value. Nonetheless many relevant results can be deduced and we will select here a few among them

Proposition E.4. *Given a rv X with $\mathbf{E}[|X|^p] < +\infty$ for some $p \geq 1$ (namely endowed at least with the expectation, and possibly also with higher order moments), for every possible value of $\mathbf{M}[X]$ according to the Definition E.2, and for every $a \in \mathbf{R}$ we always have*

$$|\mathbf{M}[X] - a| \leq (2 \mathbf{E}[|X - a|^p])^{\frac{1}{p}} \tag{223}$$

In particular for $p = 2$ and $a = \mathbf{E}[X]$ it is straightforward to see that

$$|\mathbf{M}[X] - \mathbf{E}[X]| \leq \sqrt{2\mathbf{V}[X]}$$

Proof: From the Chebyshev inequality for every $p \geq 1$, $a \in \mathbf{R}$ and $\epsilon > 0$ we first of all have

$$\mathbf{P}\{|X - a| \geq \epsilon\} = \mathbf{P}\{|X - a|^p \geq \epsilon^p\} \leq \frac{\mathbf{E}[|X - a|^p]}{\epsilon^p}$$

so that by taking $\epsilon^p = 2\mathbf{E}[|X - a|^p]$ we get

$$\mathbf{P}\left\{|X - a| \geq (2\mathbf{E}[|X - a|^p])^{\frac{1}{p}}\right\} \leq \frac{1}{2}$$

and since

$$\begin{aligned} & \mathbf{P}\left\{|X - a| \geq (2\mathbf{E}[|X - a|^p])^{\frac{1}{p}}\right\} \\ &= \mathbf{P}\left\{X \geq a + (2\mathbf{E}[|X - a|^p])^{\frac{1}{p}}\right\} + \mathbf{P}\left\{X \leq a - (2\mathbf{E}[|X - a|^p])^{\frac{1}{p}}\right\} \end{aligned}$$

both the following two inequalities hold simultaneously

$$\mathbf{P}\left\{X \geq a + (2\mathbf{E}[|X - a|^p])^{\frac{1}{p}}\right\} \leq \frac{1}{2} \qquad \mathbf{P}\left\{X \leq a - (2\mathbf{E}[|X - a|^p])^{\frac{1}{p}}\right\} \leq \frac{1}{2}$$

Now it is apparent that $a - (2\mathbf{E}[|X - a|^p])^{\frac{1}{p}} \leq a + (2\mathbf{E}[|X - a|^p])^{\frac{1}{p}}$, so that from the Definition E.2 we deduce that $\mathbf{M}[X]$ must fall somewhere in between these two numbers, namely

$$a - (2\mathbf{E}[|X - a|^p])^{\frac{1}{p}} \leq \mathbf{M}[X] \leq a + (2\mathbf{E}[|X - a|^p])^{\frac{1}{p}}$$

and hence (223) is completely proved ■

Remark however that there are many, perfectly legitimate, laws which possess a median, but do not have a well defined expectation (such as, for example, the Cauchy law), and hence are not in the framework of the Proposition E.4 hypotheses: as a consequence the scope of this result is less wide than it looks like at first sight.

It is well known that for *rv*'s with $\mathbf{E}[|X|^2] < +\infty$ the expectation $\mathbf{E}[X]$ can be characterized as the value of the variable $a \in \mathbf{R}$ that minimizes the mean *square* error $\mathbf{E}[|X - a|^2]$. A similar result holds for the medians of *rv*'s X with $\mathbf{E}[|X|] < +\infty$, but in terms of the mean *absolute* error

Proposition E.5. *Given a rv X with $\mathbf{E}[|X|] < +\infty$, a number m is a possible value of the median $\mathbf{M}[X]$ according to the Definition E.2 if and only if*

$$\mathbf{E}[|X - m|] = \min_{a \in \mathbf{R}} \mathbf{E}[|X - a|]$$

Proof: See [18] p. 43 ■

A *rv* X and its law are said to be **μ -symmetric** if $\mu - X$ has the same distribution as $X - \mu$ for some parameter μ . This means that for 0-symmetric (or *symmetric* tout-court) *rv*'s we have

$$\mathbf{P}\{X \leq x\} = \mathbf{P}\{-X \leq x\} \quad \forall x \in \mathbf{R}$$

namely (by changing for convenience the explicit sign of x)

$$\mathbf{P}\{X \leq -x\} = \mathbf{P}\{-X \leq -x\} = \mathbf{P}\{X \geq x\} = 1 - \mathbf{P}\{X < x\} \quad \forall x \in \mathbf{R}$$

or, in terms of the *cdf* $F(x)$,

$$F(-x) + F(x^-) = 1 \quad \forall x \in \mathbf{R} \tag{224}$$

In particular for $x = 0$ the equation (224) implies that

$$F(0) + F(0^-) = 1$$

and hence either $F(0) = 1/2$ (when $F(x)$ is continuous in $x = 0$), or $F(0^-)$ and $F(0)$ are symmetrically located around the central value $1/2$ (when $F(x)$ jumps in $x = 0$). Obviously, when the expectations exist, we have $\mathbf{E}[X] = 0$ for every symmetric *rv*, and also the most natural choice for the median of a symmetric random variable seems to be 0, but in this case some qualification is in order

Proposition E.6. *Given a symmetric rv X , either $\mathbf{M}[X] = 0$ if the median is unique, or the non-degenerate median segment is $[-a, a]$ for some suitable $a > 0$*

Proof: When $F(x)$ is continuous in $x = 0$ (without being constant in a neighborhood of it) then we have seen that $F(0) = 1/2$ and hence the median has the unique value $\mathbf{M}[X] = 0$. The same result holds when $F(x)$ jumps in $x = 0$ because in this case we know that $F(0-) < 1/2$ and $F(0) > 1/2$. On the other hand when $F(x)$ is continuous and constantly takes the value $1/2$ in a neighborhood of $x = 0$, then we have a non-degenerate median segment which must be symmetric around $x = 0$. If indeed $m > 0$ belongs to this segment, also $-m$ must be a median value because

$$F(-m) = 1 - F(m^-) = 1 - F(m) = 1 - 1/2 = 1/2$$

Then the median segment apparently is $[-a, a]$ for some suitable $a > 0$ ■

When a *cdf* $F(x)$ is continuous without flat spots on all its support the median is always uniquely defined, the two Definitions E.1 and E.2 give rise to the same value, and we can simply take $Q(1/2)$ as the median. In this case the *qf* $Q(p)$ has many other properties that can be found for example in [16]. In particular it is possible to describe – by means of suitable parameters – entire *types* of *qf*'s starting from some *basic form* $S(p)$: if for instance $S(p)$ is the *qf* of some rv X , then it is easy to see that all the functions

$$Q(p) = \lambda + \eta S(p)$$

with $\lambda \in \mathbf{R}$ and $\eta > 0$, would be good *qf*'s for the *rv*'s the type $\lambda + \eta X$ spanned by X . If in fact $F(x)$ and $G(x)$ respectively are the *cdf*'s of $Q(p)$ and $S(p)$, it is easy to see that from the previous equation we also have

$$F(x) = G\left(\frac{x - \lambda}{\eta}\right)$$

namely $F(x)$ belongs to the type spanned by $G(z)$: if indeed X is distributed as $G(x)$, then $\lambda + \eta X$ is distributed as $F(x)$. This property however can be put in a more general form as follows

Proposition E.7. *If $Q_X(p)$ is the *qf* of the continuous rv X , and $Z = T(X)$ we have*

$$Q_Z(p) = \begin{cases} T(Q_X(p)) & \text{if } T(x) \text{ is continuous, monotonic increasing} \\ T(Q_X(1-p)) & \text{if } T(x) \text{ is continuous, monotonic decreasing} \end{cases}$$

Proof: When $T(x)$ is increasing we have

$$p = F_Z(z) = \mathbf{P}\{Z \leq z\} = \mathbf{P}\{T(X) \leq z\} = \mathbf{P}\{X \leq T^{-1}(z)\} = F_X(T^{-1}(z))$$

and hence $T^{-1}(z) = Q_X(p)$, namely $z = T(Q_X(p))$, so that finally $Q_Z(p) = z = T(Q_X(p))$. If on the other hand $T(x)$ is decreasing, being our *rv*'s continuous we find that

$$p = F_Z(z) = \mathbf{P}\{Z \leq z\} = \mathbf{P}\{T(X) \leq z\} = \mathbf{P}\{X \geq T^{-1}(z)\} = 1 - F_X(T^{-1}(z))$$

and hence, as before, we get $Q_Z(p) = T(Q_X(1 - p))$ ■

Remark finally that, when $T(x)$ is simply monotonic, from both the previous results we get

$$Q_Z(1/2) = T(Q_X(1/2))$$

so that

$$\mathbf{M}[T(X)] = \mathbf{M}[Z] = m_Z = T(m_X) = T(\mathbf{M}[X])$$

namely we again obtain the result about the medians stated in the Proposition E.3

E.3 Expectations and medians for symmetric laws

To avoid possible ambiguities in this section we will confine ourselves to *continuous* (namely *absolutely continuous*) laws equipped with a non vanishing *pdf*. Generally speaking expectations and medians do not coincide, except in particular cases: for instance, the *pdf* $f(x)$ of a μ -symmetric *rv* X is an even function around the parameter μ

$$f(\mu + x) = f(\mu - x)$$

then (if the expectation exists) it is easy to see that $\mathbf{E}[X] = \mathbf{M}[X] = \mu$. We know on the other hand from the Proposition E.3 that the medians show a particular property not shared with the expectations: when $y = T(x)$ is a monotonic function defined on the set of values of the *rv* X and we define $Y = T(X)$, we get

$$\mathbf{M}[T(X)] = T(\mathbf{M}[X]) \tag{225}$$

It is apparent then that if X is μ -symmetric we also have

$$\mathbf{M}[T(X)] = T(\mathbf{E}[X]) \tag{226}$$

and in particular this is true for the **Gaussian** *rv*'s $X \sim \mathfrak{N}(\mu, \sigma^2)$ that are famously μ -symmetric and will be briefly discussed herein

If, to begin with, X is a *dimensionless* Gaussian *rv*, then $Y = e^X \sim \ln\mathfrak{N}(\mu, \sigma^2)$ is a dimensionless log-normal, and hence we have

$$\mathbf{E}[X] = \mathbf{M}[X] = \mu \qquad \mathbf{V}[X] = \sigma^2 \tag{227}$$

$$\mathbf{E}[Y] = e^{\mu + \frac{\sigma^2}{2}} \qquad \mathbf{M}[Y] = e^\mu \qquad \mathbf{V}[Y] = e^{2\mu + \sigma^2} (e^{\sigma^2} - 1) \tag{228}$$

In particular, in agreement with (226), we have the following relations

$$\mathbf{E}[\ln Y] = \mathbf{E}[X] = \ln \mathbf{M}[Y] \qquad \mathbf{M}[e^X] = \mathbf{M}[Y] = e^{\mathbf{E}[X]} \tag{229}$$

that are instrumental in the discussion of the present paper, and then will deserve a few additional remarks

First of all the equations (229) apparently hold for dimensionless Gaussian *rv*'s X and for their log-normal counterparts $Y = e^X$, but a dimensionally complete

formulation is always possible: let us suppose that our Gaussian $X \sim \mathfrak{N}(\mu, \sigma^2)$ is no longer dimensionless, but it is for instance a length. In this case, by taking advantage of the standard deviation σ that is a length too, we remark first that $X/\sigma \sim \mathfrak{N}(\mu/\sigma, 1)$ is now dimensionless, and then that $Z = e^{X/\sigma} \sim \ln\mathfrak{N}(\mu/\sigma, 1)$ is a dimensionless log-normal with

$$\mathbf{E}[Z] = e^{\frac{\mu}{\sigma} + \frac{1}{2}} \quad \mathbf{M}[Z] = e^{\frac{\mu}{\sigma}} \quad \mathbf{V}[Z] = e^{\frac{2\mu}{\sigma} + 1} (e - 1)$$

Going then to the dimensional variable $Y = \sigma Z = \sigma e^{X/\sigma}$, we at once have

$$\mathbf{E}[Y] = \sigma e^{\frac{\mu}{\sigma} + \frac{1}{2}} \quad \mathbf{V}[Y] = \sigma^2 e^{\frac{2\mu}{\sigma} + 1} (e - 1)$$

while from the Proposition E.3 for the median we have

$$\mathbf{M}[Y] = \sigma \mathbf{M}[Z] = \sigma e^{\frac{\mu}{\sigma}}$$

In particular the relations (229) are accordingly changed into

$$\mathbf{E}\left[\sigma \ln \frac{Y}{\sigma}\right] = \mathbf{E}[X] = \sigma \ln \frac{\mathbf{M}[Y]}{\sigma} \quad \mathbf{M}\left[\sigma e^{\frac{X}{\sigma}}\right] = \mathbf{M}[Y] = \sigma e^{\frac{\mathbf{E}[X]}{\sigma}} \quad (230)$$

Remark that these results should be suitably adjusted when dealing with a Gaussian process $X(t)$ because its variance could possibly be time-dependent: in this case it would be appropriate to find some other constant parameter to play the role of σ

We must remark moreover that, if face the problem of writing down some deterministic evolution either as expectation, or as median of a process, it is possible to consider several alternatives by taking for instance into account also the variances. For example we found that the expectation of $Y = \sigma e^{X/\sigma}$

$$\mathbf{E}[Y] = \sigma e^{\frac{\mu}{\sigma} + \frac{1}{2}} \quad (231)$$

is a function of the expectation μ and of the variance σ^2 of the original normal *rv* X : as a consequence, since for the processes discussed in the present paper, both $\mathbf{E}[X(t)]$ and $\mathbf{V}[X(t)]$ are explicitly known, for our required evolutions we can always resort to the expectations rather than to the medians

Of course the relations (229), (230) and (231) perfectly agree with (225) drawn from the Proposition E.3 but for the fact that here, being X a Gaussian *rv*, we also take advantage of the relation $\mathbf{E}[X] = \mathbf{M}[X]$ which only holds for μ -symmetric distributions as discussed at the beginning of this section: for example with X arbitrarily distributed and $Y = \sigma e^{X/\sigma}$, we always have

$$\mathbf{M}[Y] = \mathbf{M}\left[\sigma e^{X/\sigma}\right] = \sigma e^{\frac{\mathbf{M}[X]}{\sigma}} \quad (232)$$

while (230) only holds for a Gaussian X

References

- [1] *E. De Lauro, S. De Martino, S. De Siena and V. Giorno*, Physica A 401 (**2014**) 207
- [2] *C.H. Skiadas*, Meth Comput Appl Prob 12 (**2010**) 261
- [3] *S. Pasquali*, Rend Sem Mat Univ Padova 106 (**2001**) 165
- [4] *M. Yor*, EXPONENTIAL FUNCTIONALS OF BROWNIAN MOTION AND RELATED PROCESS (Springer, Berlin **2001**)
- [5] *N Cufaro Petroni, S De Martino and S De Siena*, Phys Lett A 245 (**1998**) 1
- [6] *N Cufaro Petroni, S De Martino, S De Siena and F Illuminati*, J Phys A 32 (**1999**) 7489
- [7] *I.S. Gradshteyn and I.M. Ryzhik*, TABLE OF INTEGRALS, SERIES AND PRODUCTS (Academic Press, Burlington **2007**)
- [8] *D. Dufresne*, Adv Appl Prob 33 (**2001**) 223
- [9] *M. Schröder*, Adv Appl Prob 35 (**2003**) 159
- [10] *P. Carr and M. Schröder*, Th Prob Appl 48 (**2004**) 400
- [11] *C.C. Heyde*, J Royal Stat Soc, Series B (Methodological) 25 (**1963**) 392
P. Holgate, Comm in Stat - Th Meth 18 (**1989**) 4539
R. Barakat, J Opt Soc of America 66 (**1976**) 211
E. Barouch and G.M. Kaufman and M.L. Glasser, St App Math 75 (**1986**) 37
R.B. Leipnik, J Austr Math Soc Series B 32 (**1991**) 327
- [12] *I.I. Gihman and A.V. Skorohod*, STOCHASTIC DIFFERENTIAL EQUATIONS (Springer, Berlin **1972**)
- [13] *E. Nelson*, DYNAMICAL THEORIES OF BROWNIAN MOTION (Princeton UP, Princeton **1967**), also available now in a new *pdf* version on web.math.princeton.edu/~nelson/books.html
- [14] *I. Karatzas and S.E. Shreve*, BROWNIAN MOTION AND STOCHASTIC CALCULUS (Springer, New York, **1991**)
- [15] *D. Revuz and M. Yor*, CONTINUOUS MARTINGALES AND BROWNIAN MOTION (Springer, Berlin, **1999**)
- [16] *W.G. Gilchrist*, STATISTICAL MODELLING WITH QUANTILE FUNCTIONS (Chapman&Hall, Boca Raton **2000**)
- [17] *M. Loève*, PROBABILITY THEORY I–II (Springer, Berlin, **1977-8**).

- [18] *D.W. Stroock*, PROBABILITY THEORY: AN ANALYTIC VIEW (Cambridge U.P., New York **2011**)
- [19] *W. Mückenheim*, Phys Rep 133 (**1986**) 337
- [20] *R.P. Feynman*, Int J Th Phys 21 (**1982**) 467
- [21] *R.P. Feynman*, in QUANTUM IMPLICATIONS: ESSAYS IN HONOUR OF DAVID BOHM, B.J. Hiley and F.D. Peat eds (Routledge, New York **1987**) p. 235
- [22] *F. Guerra*, Phys Rep 77 (**1981**) 263
- [23] *F. Guerra and L. Morato*, Phys Rev D 27 (**1983**) 1774
- [24] *D. de Falco, S. De Martino, and S. De Siena*, Phys Rev Lett 49 (**1982**) 181
S. De Martino, S. De Siena, G. Vitiello and F. illuminati, Mod Phys Lett B 08 (**1994**) 977
N. Cufaro Petroni, S. De Martino, and S. De Siena, Phys Rev E 63 (**2000**) 016501
N. Cufaro Petroni, S. De Martino, and S. De Siena, Phys Rev ST Accel Beams 6 (**2003**) 034206