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Logistic and θ-logistic models in population dynamics: general analysis and exact results

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Abstract

Stochastic logistic and θ-logistic models have many applications in biological and physical contexts, and investigating their structure is of great relevance. In the present paper we provide the closed form of the path-like solutions for the logistic and θ-logistic stochastic differential equations, along with the exact expressions of both their probability density functions and their moments. We simulate in addition a few typical sample trajectories, and we provide a few examples of numerical computation of the said closed formulas at different noise intensities: this shows in particular that an increasing randomness—while making the process more unpredictable—asymptotically tends to suppress in average the logistic growth. These main results are preceded by a discussion of the noiseless, deterministic versions of these models: a prologue which turns out to be instrumental—on the basis of a few simplified but functional hypotheses—to frame the logistic and θ-logistic equations in a unified context, within which also the Gompertz model emerges from an anomalous scaling.

Keywords: population dynamics, logistic equations, stochastic growth models

(Some figures may appear in colour only in the online journal)

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1. Introduction

Investigation of population dynamics can be traced back to the Fibonacci series in thirteenth century, and have been then developed until the present day [1–3] with the introduction of various models designed to describe a very large number of systems with both theoretical and practical relevance [4, 5]. Phenomenological equations have been proposed to account for the macroscopic behaviors resulting from a suitable averaging.

On a macroscopic level, two approaches became very popular along the years and can now be considered as prototypical: the Verhulst (logistic) model [6] and the Gompertz model [7], both introduced in the first half of the nineteenth century, and then resumed and developed in the first half of the twentieth century. The θ-logistic equation (Richards model) [8, 9] was subsequently added as a flexible generalization of the logistic evolution. The corresponding laws can indeed be obtained resorting to a proportionality between the differential increment of the size of a system and its current size, and then suitably correcting it by adding a nonlinear factor that prevents an un-physical (Malthusian) explosion allowed only in the first stage of the evolution: this will eventually drive the system toward a finite asymptotic dimension, namely to a stable equilibrium point. The said correction is in fact related to the finite amount of resources available for a given system, and to its growing density, two features both leading to a reduction of the resources allotted individually. As a matter of fact, any growing organism is an *active matter* system [10, 11], i.e. an open dynamical system getting resources in an exchange with the surrounding environment (e.g. metabolic exchanges in the case of biological systems), and only unbounded resources and no spatial limitations could allow for indefinite growth. In studying these systems, various approaches have been developed within the large research field usually denoted as *population dynamics* (a term actually including many different topics). These approaches include standard statistical mechanics methods [12–14], entropic techniques [15–17], and stochastic models [18–22]. On the other hand, although in general the description of a system depends on the specific scale that has been chosen, and on different scales different descriptions are established, for all biological systems multiscale problems are a permanent feature. This requires a multiscale approach as that described in [23] in which the proper selection of a microscopic dynamics leads to an accurate derivation of the mesoscopic and macroscopic models with the identification of a unified framework that allows to deduce these structures at each scale using the same principles and similar parameters.

The present paper is mainly focused on the stochastic approach, and in particular on the stochastic logistic and θ-logistic models, whose exploitation leads also to interesting applications in specific physical frameworks (for recent examples see [24–28]). Investigating the mathematical structures of these models can thus be very useful, besides being interesting in itself. It is to be remarked that the Gompertz stochastic model is already rather well established: its distributions are indeed log–normal and it has been shown that its macroscopic evolution is properly described by the median of the process [29]. The same cannot be said, instead, for the logistic and θ-logistic models, whose solution procedures are rather more tangled. The key point is that the logistic and θ-logistic solutions are expressed in terms of exponential functionals of Brownian motion (EFBM), convoluted processes of relevant interest in the financial context [30–32]. Exploiting however their explicit distribution available in the literature [31], we are able to provide a closed form for the distributions at one time of the logistic and θ-logistic stochastic processes, and the exact expressions of their associated moments. We provide also the time plots of the sample trajectories, and a few numerical evaluations of the exact formulas for the most relevant moments (expectation and variance) to explore their behavior and their changes at different levels of randomness.

These results are preceded by an analysis of the logistic, θ-logistic and Gompertz equations in their noiseless, deterministic layout, with the aim of getting first a perspicuous and unified interpretation of their structure, and then a more definite identification of the underlying hypotheses leading to the macroscopic evolutions. After a look to the form of the equations with a focus on the important role of time scales, we start again from the very beginning, i.e. from the task of describing how the average growth of a system, made up by many individuals, leads to the macroscopic laws. We show that this result can be deduced from rather simplified—but working—assumptions, with macroscopic laws connecting percentage increments, and then realizing a self-controlled evolution. Within this framework we recognize a θ-hierarchy in dissipating resources, and we also suggest a unifying procedure accounting for the emergence of the—seemingly eccentric—Gompertz term, by providing a more defined physical meaning to a known mathematical approach, and by including in so doing the Gompertz growth in the θ-logistic frame as a limiting case.

The paper is organized as follows: in the section 2 we present the preliminary analysis of the deterministic logistic and θ-logistic equations in a unified context, with the inclusion in the same framework of the Gompertz model as a limiting case. The next section 3 contains our main results with respect to the stochastic implementations of the logistic and θ-logistic models. Here, after summarizing the state of the art including the explicit stationary distributions and the path-wise solutions of the stochastic differential equations (SDEs), we show, by exploiting a few trajectories simulations and some numerical computation, the strong impact of the noise intensity on both the process predictability and its asymptotic expectation. After that we also provide the exact expressions (in integral form) of the distributions and moments of the stochastic logistic and θ-logistic processes, along with some numerical plot of the most important issues (mean and variance) in the logistic instance, and a concise examination of them. Discussion and conclusions finally follow in the section 4.

2. Deterministic growth models

2.1. An overview of known results

In this section we will briefly summarize the main features of the logistic and Gompertz equations, and we will find out their general structure in what we regard as their most revealing setting, a formulation that will provides a hint for later developments. At the same time we will also put in evidence the important role played by the time scales. In our models the main variable will be the macroscopic size of the system $n(t)$, namely the (dimensionless) number of elementary components (e.g. the cells in a biological systems) at the instant t. The θ-logistic equation then usually takes the form

$$\frac{\mathrm{d}n(t)}{\mathrm{d}t} = \omega_e n(t) - \omega_f n^{\theta+1}(t) \tag{1}$$

(the *simple logistic* is recovered for $\theta = 1$), while the Gompertz equation reads

$$\frac{\mathrm{d}n(t)}{\mathrm{d}t} = \omega_e n(t) - \omega_f n(t) \ln\ n(t) \tag{2}$$

where the constants $\omega_e = {}^1/_{\tau_e}$ and $\omega_f = {}^1/_{\tau_f}$ are the reciprocal of the characteristic times τ_e and τ_f. The θ-logistic equation can also be recast in the form

$$\frac{dn(t)}{dt} = \omega_e n(t) \left[1 - \left(\frac{n(t)}{K}\right)^{\theta}\right], \quad K = \left(\frac{\tau_f}{\tau_e}\right)^{\frac{1}{\theta}} = \left(\frac{\omega_e}{\omega_f}\right)^{\frac{1}{\theta}} \tag{3}$$

while in the Gompertz case we have

$$\frac{dn(t)}{dt} = \omega_e n(t) \left(1 - \frac{\ln\ n(t)}{\ln\ K}\right), \quad K = e^{\frac{\tau_f}{\tau_e}} = e^{\frac{\omega_e}{\omega_f}} \tag{4}$$

that for later convenience can also be written as

$$\frac{d\ln\ n(t)}{dt} = -\omega_f\ \ln\ \frac{n(t)}{K} \tag{5}$$

The quantity K in the previous equations is the asymptotic value of $n(t)$ when $t \to \infty$, i.e. the value of n that sets its derivative to zero, and that is also known as *carrying capacity*. It is known that the solutions of our equations for $n(0) = n_0$ respectively are (see for example [1, 3, 22])

$$n(t) = \frac{Kn_0}{n_0 + (K - n_0)e^{-\omega_e t}} \qquad \text{(simple logistic)} \tag{6}$$

$$n(t) = \frac{Kn_0}{\sqrt[\theta]{n_0^{\theta} + (K^{\theta} - n_0^{\theta})e^{-\theta\omega_e t}}} \qquad (\theta\text{-logistic}) \tag{7}$$

$$n(t) = K\ \exp\{\alpha_0\ e^{-\omega_f t}\} \quad \alpha_0 = \ln(n_0/K) \qquad \text{(Gompertz)} \tag{8}$$

Looking back now at the equations (3) and (4), we see that they are all of the general form

$$\frac{dn(t)}{dt} = \omega_e n(t)\, [1 - h(n(t))] \tag{9}$$

where $0 < h(n(t)) < 1$, and therefore also $0 < 1 - h(n(t)) < 1$, because we always have $n(t) < K$ if—as it is realistic in our investigation—we take $n_0 < K$. The second member in the equations is a product of two terms: the first term, that by himself would produce an exponential explosion $n_0\ e^{\omega_e t}$, is corrected by the second one (a negative feedback, usually known as *individual growth rate*): it is this counteraction that drives the system toward its finite asymptotic size. Remark that, accordingly, one can assume almost vanishing values of $h(n(t))$ at the early stage of the evolution, the region of time where Malthusian growth dominates, while the value 1 is asymptotically approached for $t \to \infty$, when the number attains its maximum value and stops growing.

As for the two characteristic times, it is apparent that τ_e is the time scale of the purely exponential growth, while, as emerges from (1) and (2), τ_f characterizes the *strength* or *speed* of the correcting term. Obviously it will be $\tau_f > \tau_e$, and usually also $\tau_f \gg \tau_e$. The carrying capacity emerges from the competition between the correction and exponential trends, and it is in fact connected with their ratio: the slower the action of the feedback w.r.t. the explosion, the larger the carrying capacity. In the Gompertz case the carrying capacity is the exponential of the said ratio. Since moreover the whole growth is controlled by the individual growth rate, the braking mechanism must be linked to the decrease of resources available for an elementary component of the system.

Before concluding the section, it is useful for later convenience to introduce a rescaled variable $x(\tau) = x(\omega_e t) = n(t)/K$ and a rescaled time $\tau = \omega_e t$ so that the form of the logistic and θ-logistic equations respectively become

$$\dot{x}(\tau) = x(\tau)\,(1 - x(\tau)) \qquad \dot{x}(\tau) = x(\tau)\left(1 - x^{\theta}(\tau)\right) \tag{10}$$

while the corresponding solutions with $x_0 = n_0/K$ are

$$x(\tau) = \frac{x_0}{x_0 + (1 - x_0)\mathrm{e}^{-\tau}} \qquad x(\tau) = \left(\frac{x_0^{\theta}}{x_0^{\theta} + (1 - x_0^{\theta})\mathrm{e}^{-\theta\,\tau}}\right)^{1/\theta} \tag{11}$$

2.2. Merging the equations

2.2.1. General principles of a unified model. The form of the previous equations, and more specifically the nonlinear term $h(t)$, is usually chosen by resorting to phenomenological criteria depending on the specific system to be described. Otherwise it can emerge—again phenomenologically—by coupling differential equations, as happens for example to the logistic case in the epidemiological context. In this section, before proceeding to deal with the stochastic models, we therefore deem instrumental to linger a bit longer on these identification criteria, looking for a description derived from suitable—albeit still phenomenological—general assumptions. To this end we will reboot our procedure starting again from the beginning, i.e. from the generally recognized main goal of a population dynamics inquiry: taken an evolving natural system consisting, at a given time, of a large number of components, address the problem of forecasting the growth of this number at later times. Our aim is not to obtain a thorough unified view of a system on different scales, as in the multiscale approach mentioned in the introduction, but rather to get a general framework that provides well defined criteria accounting for the phenomenological evolutions here considered. The idea is to get an intermediate description—as is usually done in the physical contexts—by resorting to a simplified model that, although based on somewhat rough premises, can catch the essential features of the evolution mechanisms, providing in this way a conceptually clear link with the deep characteristics of the natural phenomena we are studying. What we want to achieve is better clarified by starting from a few basic questions to be answered: being the components active particles, how their evolution could be explicitly constrained by the exchanges of *energy* (this term conventionally summarizes any kind of resources), and how the description could then be translated only in terms of the numbers of components? How could these constraints induce a self-conditioning mechanism? And eventually, and most important, can the logistic or θ-logistic evolutions and, possibly, also the Gompertz evolution, be framed in a common, unified context? In order to answer these questions we will introduce below our simplified model where, besides obviously assuming a bounded amount of resources, the key points are first to introduce an explicit energy dependence, and then to connect the relative values (percentages) of the relevant quantities, automatically providing in so doing a self controlling mechanism that leads to an asymptotic final size. Furthermore, the different possible scalings with the number of components of the resources exploitation, linked to what is interpreted as the occurrence of different levels of *cooperation* (*coherence*) among the components themselves, lead to a description only in terms of numbers of active particles, and account for different θ-logistic evolutions. Finally this scheme turns out to be sufficiently general to include, as a singular limit case, the Gompertz evolution in the same framework.

Moving on to construct our model, if we denote with $n(t)$ the average number of active particles of our system at the generic instant t, the main point is to compute its increment $\Delta n(t) \doteq n(t + \Delta t) - n(t)$ at a subsequent time $t + \Delta t$. Here $\Delta n(t)$ will be supposed to result from the accumulation of many microscopic increments produced by the possible occurrence of random events (the birth or death of one individual, one mitosis, and so on) between t and $t + \Delta t$: at this stage of the inquiry, however, we will keep this underlying microscopic *probabilistic* mechanism only in the background. Without yet assuming a fully stochastic model, indeed, we will only surmise the existence of this random underworld as a background justification of our coarse grained deterministic equations. Remembering now the criteria above exposed, we are thus led to the following simplified hypotheses:

(a) At each instant, the system can rely on a finite and fixed (mean) amount of resources, of different origins, conventionally denoted by E_T, whose specific nature is not relevant in our scheme because, eventually, all the quantities will be translated in terms of number of components.
(b) Within the system the active particles exploit these resources both to *survive* and to *grow*, but survival *takes precedence* in the sense that, at each stage, the resources available for growth are what is left of E_T once the resources for survival have been taken out. Furthermore, at each step every active particle needs on average a quantity ϵ_s of resources to survive.
(c) Growth stops when the total amount of resources E_T is only sufficient to the survival of all the active particles: in that case the population achieves its maximum, finite dimension K a.k.a. *carrying capacity*.
(d) There is a constant, *average rate of increment per unit time* $\omega_e = \tau_e^{-1}$ of the number of active particles, so that the average rate of increase in $\mathrm{d}t$ will be $\omega_e \mathrm{d}t$. In the literature ω_e is often called *probability per unit time* and has been already introduced in very different contexts as, for example, in the Drude simplified model of conduction [33]

Before further developing our model from the previous assumptions, we consider first an ideal case to provide some suggestions for the more realistic ones. We will suppose then that there are no limitations to the available resources ($E_\mathrm{T} = \infty$) and to the available space. In this case, whatever the need for survival resources, at any instant the availability of growth resources would be boundless, and thus the population increment would be obtained by simply applying the average rate of increase to the whole number $n(t)$

$$\mathrm{d}n(t) = \omega_e \, n(t) \, \mathrm{d}t \tag{12}$$

with a resulting Malthusian explosion $n(t) = n_0 \, \mathrm{e}^{\omega_e t}$. Here of course n_0 denotes the system size at time zero. The previous relation can however be also written as

$$\frac{\mathrm{d}n(t)}{n(t)} = \omega_e \, \mathrm{d}t \times 1$$

On the lhs we find the (infinitesimal) percentage increment of the number, while from the rhs we see that this increment results from the product of the average rate of increment in $\mathrm{d}t$ and 1. Being in our case the available resources not bounded, the factor 1 can be simply interpreted as the fraction of resources available for growth at any instant. On the basis of this consideration we are led then to propose the following principle:

> *A growth equation is obtained by imposing that the percentage increment of a population in a small time interval* $\mathrm{d}t$ *is equal to the product between the average rate of increment in the same time interval, and the percentage of resources (w.r.t. the total ones) that is left available after the survival resources have been used*

We will see soon that this latter percentage depends only on the population size.

Going now to more realistic instances, we start from the simplest case by supposing that *at each instant the resources are evenly distributed among all the* $n(t)$ *individuals*. Being ϵ_s the mean amount of resources exploited by an active particle to survive, in our approximation we first of all have

$$E_\mathrm{T} = \epsilon_s K$$

Then, according to our hypotheses, if $n(t) < K$ is the number of active particles at the instant t, the resources exploited for survival at that instant are $E_\mathrm{s}(t) = \epsilon_s n(t) < E_\mathrm{T}$, and those available for growth are $E_\mathrm{g}(t) = E_\mathrm{T} - E_\mathrm{s}(t) = \epsilon_s(K - n(t))$ so that

$$\frac{\mathrm{d}n(t)}{n(t)} = \omega_e\,\mathrm{d}t\frac{E_\mathrm{g}(t)}{E_\mathrm{T}} = \omega_e\,\mathrm{d}t\frac{K-n(t)}{K} = \omega_e\,\mathrm{d}t\left(1 - \frac{n(t)}{K}\right) \tag{13}$$

and finally in terms of the reduced number and time

$$\frac{\mathrm{d}x(\tau)}{x(\tau)} = \mathrm{d}\tau(1 - x(\tau)) \tag{14}$$

that can be easily rearranged into the simple logistic equation (10) ($\theta = 1$). The result (1) can then be quickly retrieved by reintroducing the variable $n(t)$ and the characteristic time τ_e, and defining the time $\tau_f = \tau_e K$.

On the other hand—according to whether the system has a coherent character, with consequent collective and synergistic behaviors, or, on the contrary, it displays inefficiencies and *non-collaborating* components—resource scalings different from the linear one are allowed. A generalized scaling $E_\mathrm{T} = (\epsilon_s K)^\theta$ and $E_\mathrm{s}(t) = \epsilon_s n^\theta(t)$ can thus be introduced, giving rise to the θ-logistic equation

$$\frac{\mathrm{d}x(\tau)}{x(\tau)} = \mathrm{d}\tau(1 - x^\theta(\tau)) \tag{15}$$

In this formulation, however, the Gompertz model still seems to stand apart: would it be possible to recover even this equation within the framework of the previous scheme? In the next section we will provide a path to a positive answer.

2.2.2. Retrieving the Gompertz equation.

To explain in the above context the eccentric logarithmic term of the Gompertz model, we must at once recognize that we can no longer start from some kind of proportionality between the percentage increase of $n(t)$ and the time interval Δt. We will instead suppose more in general for the reduced quantities

$$\frac{\Delta x(\tau)}{x(\tau)} = w(x(\tau), \Delta\tau) \tag{16}$$

where $w(x(\tau), \Delta\tau)$ is a function still to be determined. To this purpose we preliminarily remark that, to be consistent, the procedure we will establish must anyway lead to a final result that fulfills some obvious constraints:

- $w(x(\tau), \Delta\tau)$ must become *small* for large times, and must *approach* 1 for small times
- $w(x(\tau), \Delta\tau)$ must go to zero with $\Delta\tau$ as a continuity requirement

We also expect moreover that, at the end of our procedure, at the rhs of the equation we will find again the product of an infinitesimal probability times a percentage term constraining the growth.

We go on now by assuming that $w(x(\tau), \Delta\tau)$ generalizes the θ-logistic term with the *anomalous scaling* $\theta(\Delta\tau) = \omega_f \Delta\tau + o(\Delta\tau)$, where $\tau_f = \omega_f^{-1}$ is the characteristic time-scale. We therefore take the function

$$w(x(\tau), \Delta\tau) = 1 - x(\tau)^{\omega_f \Delta\tau + o(\Delta\tau)} \tag{17}$$

which apparently fulfills the required constraints: since indeed K is the maximum asymptotic value of $n(t)$, for $t \to \infty$ we find $x(\tau) \to 1$ and the increment of the number (i.e. the correcting term) tends to become small, while in a very early stage of evolution $x(\tau) \ll 1$ and $w \approx 1$. The requirement $w(x(\tau), \Delta\tau) \approx 0$ when $\Delta\tau \approx 0$, is clearly fulfilled as well. We can then take advantage of a power expansion to write

$$w(x(\tau), \Delta\tau) = 1 - \mathrm{e}^{\left(\omega_f \Delta\tau + o(\Delta\tau)\right)\ \ln\ x(\tau)} = 1 - \left(1 + \omega_f \Delta\tau\ \ \ln\ \ x(\tau)\right) + o(\Delta\tau) \tag{18}$$

finding first

$$w(x(\tau), \mathrm{d}\tau)) = -\omega_f\ \ \ln\ \ x(\tau)\mathrm{d}\tau \tag{19}$$

and then finally the Gompertz equation (5) for the reduced variables

$$\frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau} = -\omega_f\, x(\tau) \ln\ \ x(\tau), \tag{20}$$

If we remember that $x(\tau) = n(t)/K$, and $\tau_e \doteq (\ln K)^{-1}\tau_f$, we can also retrace the factorized form of (4) as a product of the probability per unit time and a reduced percentage of available resources. This concludes the retrieval of the Gompertz model within the framework of our general scheme.

Remark that the Gompertz growth is obtained when $\theta \to 0$ in a suitable sense, justifying in this way its *maximally coherent* character. Moreover, some physical sense can be ascribed to the well known mathematical result $1 - x^\theta = -\theta \ln x + o(\theta)$ when $\theta \to 0$ often recalled in the literature when the Gompertz model is investigated: the meaning indeed is that scaling in the Gompertz growth depends on the microscopic scales (times) of the system. In turn this fact can clarify once again the origin of the extremely coherent character of Gompertz evolution, because the *cooperation level* extends on the microscopic domain.

By summarizing, the conceptual frame introduced above allows to clarify the growth rules, and in particular to explicitly highlight their link with the exchange of energy between the active particles and the surrounding environment, naturally inducing in this way a self controlling mechanism. Furthermore, it suggests that different ways of resources exploitation, due to the different levels of connection and collaboration among the active particles, give rise to different scaling with the number of components, thus offering the possibility of inserting all the θ-logistic evolutions in a single framework in which the different levels of coherence are indexed by the parameter θ: this also allows to include in the same hierarchy the Gompertz model, opening the way to a physical interpretation of a well known mathematical limit in terms of *maximal coherence*. Moreover, the simplified model, here introduced, also offers more interesting perspectives, as will be illustrated at the end of the paper within the conclusive remarks.

3. Stochastic growth models

We will now discuss a few questions arising from the introduction of fluctuations and leading to stochastic growth models. Here, the reduced number $x(\tau)$ will be promoted to a full-fledged stochastic process $X(\tau)$ in the reduced, dimensionless time $\tau = \omega_e t$, but since from now on there will be no risk of ambiguity we will revert in the following to the simpler notation $X(t)$ where it will be always understood that t is the dimensionless time.

In our scheme it will be rather natural to take fluctuations on the fraction

$$Q_\mathrm{g} = \frac{E_\mathrm{g}}{E_\mathrm{T}} = \frac{E_\mathrm{T} - E_\mathrm{s}}{E_\mathrm{T}}$$

of the resources available for the growth. Considering indeed the general θ-logistic case and following an usual procedure [22], we will simply add to Q_g a *white noise* $\dot{W}(t)$ (namely a process such that $\boldsymbol{E}\left[\dot{W}(t)\right] = 0$, $\boldsymbol{E}\left[\dot{W}(t)\dot{W}(s)\right] = 2D\,\delta(t-s)$, where D is a constant diffusion coefficient and $\boldsymbol{E}\left[\,\cdot\,\right]$ denotes the expectation) and therefore (15) will become

$$\frac{\mathrm{d}X(t)}{X(t)} = \left(Q_\mathrm{g} + \dot{W}(t)\right)\ \mathrm{d}t = \left[X(t)(1 - X^\theta(t)) + \dot{W}(t)\right]\ \mathrm{d}t \tag{21}$$

giving rise finally to the SDE

$$\mathrm{d}X(t) = X(t)\left(1 - X^\theta(t)\right)\mathrm{d}t + X(t)\mathrm{d}W(t) \tag{22}$$

where we exploited the well known fact that the white noise $\dot{W}(t)$ is the (distributional) derivative of a Wiener process $W(t) \sim \mathfrak{N}(0, 2Dt)$ in the sense that $\dot{W}(t)\mathrm{d}t$ is in fact the increment $\mathrm{d}W(t)$ where $\boldsymbol{E}\left[\mathrm{d}W(t)\right] = 0$ and $\boldsymbol{E}\left[\mathrm{d}W(t)\mathrm{d}W(s)\right] = 2D\,\delta(t-s)\mathrm{d}t\mathrm{d}s$. Remark that with this procedure, whatever the growth law considered, the stochastic term is always given by $X\mathrm{d}W$: this term is widely adopted in the literature about the logistic and θ-logistic cases, although multiplicative noises, or even more complex additive stochastic terms, have been introduced both in discrete and continuous time versions [20–22, 34–41]. In the Gompertz instance, adding this noise term directly leads to the a geometric Wiener process and, as pointed out in the introduction, in this case all the aspects of the model, and its connection with the macroscopic equation, are completely defined. For the stochastic logistic and θ-logistic models instead only a few aspects have been completely elaborated, while others, and very important too, still are not. In the following, we first summarize the results already obtained in the literature, and then we discuss our main new results.

3.1. A few preliminary results about the logistic models

Many aspects of the logistic and θ-logistic stochastic models have been already systematically discussed (see for instance [42]): we will recall here just a few relevant results useful in the following sections. First, the *stationary distributions* have been computed and their stability has been studied too [36]; also quasi-stationary distributions have been investigated in the discrete case [2, 37, 38]. The stationary distribution for the stochastic θ-logistic equation is the generalized gamma law $\mathfrak{G}_\theta\left(\frac{1-D}{D}, \frac{1}{(\theta D)^{1/\theta}}\right)$ with probability density functions (pdf)

$$f_s(x) = \frac{\theta\, x^{\frac{1-D}{D}-1}\ \mathrm{e}^{-\frac{x^\theta}{\theta D}}}{(\theta D)^{\frac{1-D}{\theta D}}\,\Gamma\left(\frac{1-D}{\theta D}\right)} \tag{23}$$

provided that $D < 1$. This last condition ensures normalization, and defines the region of stability of the system. The simple logistic case is obtained by choosing $\theta = 1$ (for computational

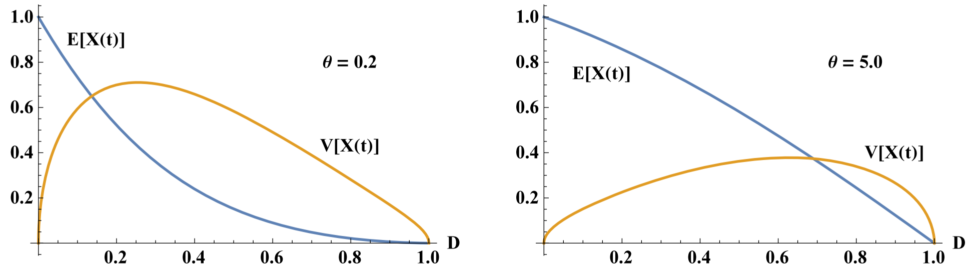

Figure 1. Expectations and variances of stationary θ-logistic processes as a function of D and for several values of θ.

details, see also [42]). It is also easy to see then that the moments in the stationary distribution (23) are

$$\boldsymbol{E}\left[X^k(t)\right] = (\theta D)^{\frac{k}{\theta}} \frac{\Gamma\left(\frac{1+(k-1)D}{\theta D}\right)}{\Gamma\left(\frac{1-D}{\theta D}\right)} \tag{24}$$

and in particular for the simple logistic ($\theta = 1$) we have $\boldsymbol{E}[X(t)] = 1 - D$ and $\boldsymbol{V}[X(t)] = D(1 - D)$. These simple results (and their generalizations for the θ-logistic cases shown in the figure 1) suggest that the asymptotic (ergodic) stationary level of a random logistic is in average suppressed by high noise intensity (D near to 1). In other words, the noise acts as an effective disruption on the logistic growth: a relevant point that will be resumed later.

Even the *path-wise solutions* of the processes are explicitly known [21, 42]. If indeed we define the following Wiener process with constant drift

$$Z(t) = (1 - D)t + W(t) \sim \mathfrak{N}((1 - D)t,\, 2Dt) \tag{25}$$

it is possible to show that the solution of the θ-logistic SDE (22) with initial condition $X(0) = X_0, \boldsymbol{P} -$ a.s. is

$$X(t) = \left(\frac{X_0^\theta\, \mathrm{e}^{\,\theta Z(t)}}{1 + \theta X_0^\theta \int_0^t \mathrm{e}^{\,\theta Z(u)} \mathrm{d}u}\right)^{1/\theta} \tag{26}$$

that is correctly brought back to the noiseless, deterministic solution (11) by switching off the noise ($D = 0$ and $W(t) = 0, \boldsymbol{P} -$ a.s., namely $Z(t) = t$) and by taking a degenerate initial condition $X_0 = x_0, \boldsymbol{P} -$ a.s. The solution of the simple logistic SDE (22) with $\theta = 1$ finally is

$$X(t) = \frac{X_0\, \mathrm{e}^{Z(t)}}{1 + X_0 \int_0^t \mathrm{e}^{\,Z(u)} \mathrm{d}u} \tag{27}$$

3.2. Sample paths, distributions and moments

Despite the expressions (26) and (27) being fully explicit, to compute the (non-stationary) expectation $\boldsymbol{E}[X(t)]$ and the higher moments $\boldsymbol{E}\left[X^k(t)\right]$ is not at all a simple task, and since not even a perturbative approach in terms of small noisy disturbances seems to be available [43], the fully non-perturbative tools will be in fact required. Looking at the expressions (26) and

(27) we see on the other hand that the integrals in the denominators (the terms hardest to crack) are indeed processes usually called EFBM of the type

$$\int_0^t \mathrm{e}^{\,aW(u)+b\,u}\mathrm{d}u \tag{28}$$

that have been extensively studied in the financial context [30–32]. Remark that since the Wiener process is Gaussian we have $W(t) \sim \mathfrak{N}(0, 2Dt)$, and therefore it is also $\theta Z(t) \sim \mathfrak{N}\left(\theta(1-D)t, 2\theta^2 Dt\right)$. As a consequence the integrand of our EFBM is log–normal $\mathrm{e}^{\,\theta Z(t)} \sim \mathfrak{ln}\ \mathfrak{N}\left(\theta(1-D)t, 2\theta^2 Dt\right)$ and the following expectations are easily calculated

$$\boldsymbol{E}\left[\mathrm{e}^{\,\theta Z(t)}\right] = \mathrm{e}^{\theta\,[1+(\theta-1)D]\,t} \qquad \boldsymbol{E}\left[\int_0^t \mathrm{e}^{\,\theta Z(u)}\mathrm{d}u\right] = \frac{\mathrm{e}^{\theta\,[1+(\theta-1)D]\,t}-1}{\theta\,[1+(\theta-1)D]} \tag{29}$$

Many other results about these EFBM are collected in the literature [30–32], but their exact distributions are rather convoluted, and on the other hand the determination of the moments of (26) and (27) requires precisely the utilization of these tangled joint distributions of $Z(t)$ with its corresponding EFBM. In the following we will therefore provide a few exact formulas for the pdf and the moments of our process $X(t)$, along with some numerical estimate of the values of these moments.

3.2.1. Trajectories simulations. We will stop first, however, to present a few numerical simulations of the sample trajectories of the process $X(t)$ confining ourselves for clarity to the simple logistic case (27) with $\theta = 1$. We will progressively turn the noise on by increasing the diffusion coefficient D, and we will compare the random paths of the process with both its deterministic behavior (the smooth, monotonic black curve) and its asymptotic, stationary expectation (the horizontal, red line). It is apparent then from the first pair of plots in the figures 2 and 3 that for a reasonably low level of noise (here D is either 0.005 or 0.05) the random paths fluctuate close to the deterministic curve, and then asymptotically stabilize around their ergodic expectation. Moreover the stationary variance grows with D. When on the other hand the value of the diffusion coefficient increases toward 0.5 or 0.7 as in the figures 4 and 5 the behavior of the trajectories begins to be much more irregular with spikes and flat spots surrounding a decreasing asymptotic expectation. If finally D approaches the value 1 (we remember that in order to find a possible stationary solution we must suppose $D < 1$) the random samples in the figure 6 become quite unpredictable with paths that mostly never take off, while a few other trajectories briefly explode to larger values: asymptotically however the paths crash near to zero. Finally in the figure 7 the ergodic relaxation toward the stationary fluctuation (the variability of the paths looks indeed to be stabilized) is apparent when we consider a somewhat longer time span. As a matter of fact our pictures display just a few examples, but the general conduct of the trajectories seems in fact to be already well sketched out and is in perfect agreement with the remarks about the stationary solutions put forward in the section 3.1.

3.2.2. A reformulation in terms of the standard Brownian motion. In order to be able to take advantage more easily of the results existing in the literature we will first convert our previous formulas into a slightly different, customary notation [31]: to this purpose we introduce the standard Brownian motion $B_t \sim \mathfrak{N}(0,\, t)$ and its corresponding EFBM

$$B_t^{(\nu)} = B_t + \nu t \sim \mathfrak{N}(\nu t,\, t) \qquad 2B_t^{(\nu)} = 2B_t + 2\nu t \sim \mathfrak{N}(2\nu t,\, 4t) \tag{30}$$

$$A_t^{(\nu)} = \int_0^t \mathrm{e}^{2B_s^{(\nu)}}\mathrm{d}s = \int_0^t \mathrm{e}^{2(B_s+\nu s)}\mathrm{d}s \qquad A_t = A_t^{(0)} \tag{31}$$

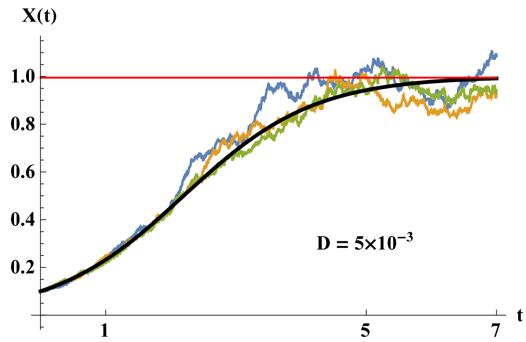

Figure 2. Sample paths of a simple logistic $X(t)$ with $D = 0.005$. The horizontal red line represents the asymptotic, stationary expectation.

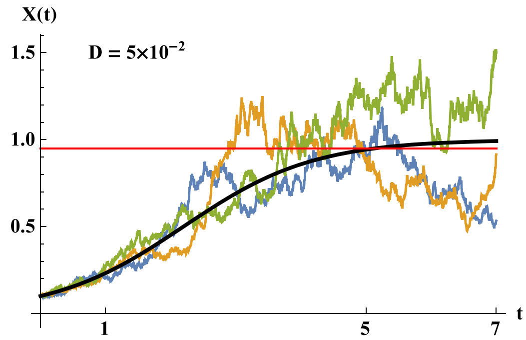

Figure 3. Sample paths of a simple logistic $X(t)$ with $D = 0.05$. The horizontal red line represents the asymptotic, stationary expectation.

and then using the self-similarity properties of a Wiener process

$$\sqrt{\lambda}\, W(t) = W(\lambda t) \quad B_s = \frac{W(s)}{\sqrt{2D}} = W\left(\frac{s}{2D}\right) \qquad \sqrt{2D}\, B_t = B_{2Dt} = W(t)$$

we can reduce our previous formulas to this new notation. First with the change of integration variable

$$s = \frac{D\theta^2}{2}\, u$$

we have

$$\int_0^t \mathrm{e}^{\,\theta Z(u)} \mathrm{d}u = \frac{2}{D\theta^2} \int_0^{D\theta^2 t/2} \mathrm{e}^{\theta Z\left(\frac{2s}{D\theta^2}\right)} \mathrm{d}s = \frac{2}{D\theta^2} \int_0^{\tau} \mathrm{e}^{\theta Z\left(\frac{2s}{D\theta^2}\right)} \mathrm{d}s \quad \tau = \frac{D\theta^2}{2}\, t$$

On the other hand we have

$$\theta Z\left(\frac{2s}{D\theta^2}\right) = \theta W\left(\frac{2s}{D\theta^2}\right) + \frac{1-D}{D\theta} 2s = 2W\left(\frac{s}{2D}\right) + \frac{1-D}{D\theta} 2s$$
$$= 2B_s + \frac{1-D}{D\theta} 2s = 2(B_s + \nu s) = 2B_s^{(\nu)} \qquad \nu = \frac{1-D}{D\theta}$$

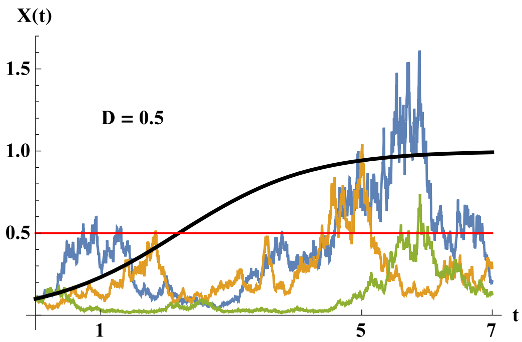

Figure 4. Sample paths of a simple logistic $X(t)$ with $D = 0.5$. The horizontal red line represents the asymptotic, stationary expectation.

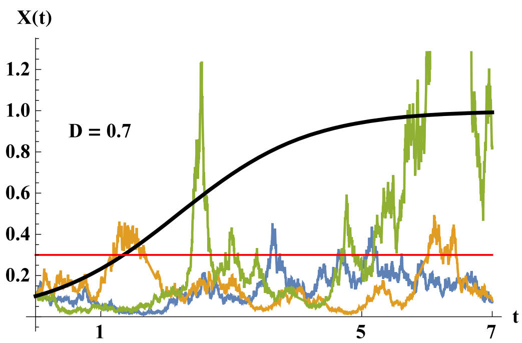

Figure 5. Sample paths of a simple logistic $X(t)$ with $D = 0.7$. The horizontal red line represents the asymptotic, stationary expectation.

and hence

$$\int_0^t \mathrm{e}^{\theta Z(u)}\mathrm{d}u = \frac{2}{D\theta^2}\int_0^\tau \mathrm{e}^{2B_s^{(\nu)}}\mathrm{d}s = \frac{2A_\tau^{(\nu)}}{D\theta^2} \qquad \tau = \frac{D\theta^2}{2}\,t \;\; \nu = \frac{1-D}{D\theta} \tag{32}$$

This puts the denominator of (26) in terms of (31). Now we must reduce also the numerator to a function of the exponential of $B_\tau^{(\nu)}$ with the same τ and ν of $A_\tau^{(\nu)}$. Since we have

$$\theta Z(t) = \theta W(t) + (1-D)\theta t = 2W\left(\frac{\theta^2 t}{4}\right) + (1-D)\theta t = 2B_{\frac{D\theta^2}{2}t} + (1-D)\theta t$$
$$= 2\left(B_\tau + \frac{(1-D)\theta}{2}t\right) = 2\left(B_\tau + \nu\tau\right) = 2B_\tau^{(\nu)}$$

the formula (26) for the process paths in terms of $A_\tau^{(\nu)}$ and $B_\tau^{(\nu)}$ finally becomes

$$X(t) = \left(\frac{x_0^\theta\,\mathrm{e}^{2B_\tau^{(\nu)}}}{1+\frac{2x_0^\theta}{D\theta}A_\tau^{(\nu)}}\right)^{\frac{1}{\theta}} = \left(\frac{D\theta\,x_0^\theta\,\mathrm{e}^{2B_\tau^{(\nu)}}}{D\theta + 2x_0^\theta A_\tau^{(\nu)}}\right)^{\frac{1}{\theta}} \qquad \tau = \frac{D\theta^2}{2}t \;\; \nu = \frac{1-D}{D\theta} \tag{33}$$

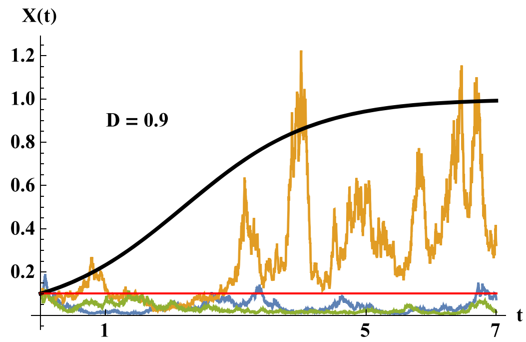

Figure 6. Sample paths of a simple logistic $X(t)$ with $D = 0.9$. The horizontal red line represents the asymptotic, stationary expectation.

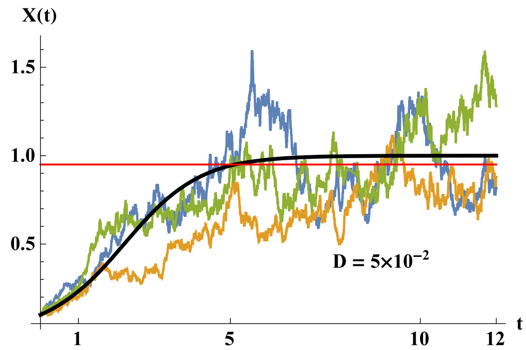

Figure 7. Ergodic relaxation in time toward stationary fluctuations around the asymptotic expectation (red line).

where $D > 0$, $\tau > 0$ and $\nu > -1$. This will give us in the following the possibility of directly exploiting a few preexisting results.

3.2.3. Probability density functions. We know (see for instance [31]) that the joint pdf of $A_\tau^{(\nu)}$, $B_\tau^{(\nu)}$ in their respective values a and b is

$$
\begin{aligned}
g(a,b) &= \frac{e^{\nu b - \frac{\nu^2\tau}{2} - \frac{1+e^{2b}}{2a}}}{a}\,\vartheta\left(\frac{e^b}{a},\tau\right) \\
&= \frac{e^{-\frac{\nu^2\tau}{2}+\frac{\pi^2}{2\tau}}\; e^{(\nu+1)\,b - \frac{1+e^{2b}}{2a}}}{a^2\sqrt{2\pi^3\tau}} \int_0^\infty e^{-\frac{e^b}{a}\cosh\, s}\, \sinh\, s\; e^{-\frac{s^2}{2\tau}}\, \sin\,\frac{\pi s}{\tau}\, ds \qquad (34)
\end{aligned}
$$

$$
\vartheta(r,v) = \frac{r\, e^{\frac{\pi^2}{2v}}}{\sqrt{2\pi^3 v}} \int_0^\infty e^{-\frac{s^2}{2v} - r\cosh\, s}\, \sinh\, s\, \sin\,\frac{\pi s}{v}\, ds \qquad (35)
$$

and therefore, in addition to being able to simulate trajectories, we are also in a position to calculate both the pdf of $X(t)$ and its moments. We see indeed from (33) that $X(t)$ is a function of $A_\tau^{(\nu)}$ and $B_\tau^{(\nu)}$, and being apparently $A_\tau^{(\nu)} \geqslant 0$ it is also easy to realize that

$$Y(t) = \mathrm{e}^{B_\tau^{(\nu)}} \geqslant \left(\frac{X(t)}{x_0}\right)^{\frac{\theta}{2}}$$

We can then first find the joint pdf $h(x, y)$ of $X(t)$ and $Y(t)$ with the following monotone variable transformation

$$\begin{cases} x = \left(\dfrac{D\,\theta\, x_0^\theta\ \mathrm{e}^{\,2b}}{D\,\theta + 2x_0^\theta a}\right)^{1/\theta} \geqslant 0 \\ y = \mathrm{e}^{\,b} \geqslant \left(\dfrac{x}{x_0}\right)^{\theta/2} \geqslant 0 \end{cases} \qquad \begin{cases} a = \dfrac{D\,\theta}{2}\left(\dfrac{y^2}{x^\theta} - \dfrac{1}{x_0^\theta}\right) \geqslant 0 \\ b = \ln\ y \end{cases} \tag{36}$$

and afterward calculate the univariate pdf of $X(t)$ by simple marginalization. The Jacobian of the transformation being

$$\begin{aligned} J &= \begin{vmatrix} \partial x/\partial a & \partial x/\partial b \\ \partial y/\partial a & \partial y/\partial b \end{vmatrix} = \begin{vmatrix} \partial x/\partial a & \partial x/\partial b \\ 0 & \mathrm{e}^{\,b} \end{vmatrix} \\ &= \mathrm{e}^{\,b}\frac{\partial x}{\partial a} = -\frac{2\ \mathrm{e}^{-b}}{D\theta^2}\left(\frac{D\,\theta\, x_0^\theta\ \mathrm{e}^{\,2b}}{D\,\theta + 2x_0^\theta a}\right)^{\frac{1+\theta}{\theta}} = -\frac{2x^{1+\theta}}{D\theta^2\, y} \end{aligned}$$

the new joint pdf is

$$h(x, y) = \frac{g\,(a(x, y),\ b(x, y))}{|J(x, y)|} \tag{37}$$

so that from (34) with $y \geqslant (x/x_0)^{\theta/2}$ we have

$$\begin{aligned} h(x, y) &= \frac{\mathrm{e}^{-\frac{\nu^2\tau}{2}+\frac{\pi^2}{2\tau}}}{\sqrt{2\pi^3\tau}}\ \frac{2x_0^{2\theta}\ x^{\theta-1}y^{\nu+2}}{D\,(x_0^\theta y^2 - x^\theta)^2}\ \mathrm{e}^{-\frac{x_0^\theta x^\theta(1+y^2)}{D\theta(x_0^\theta y^2 - x^\theta)}} \\ &\quad\times \int_0^\infty \mathrm{d}s\ \mathrm{e}^{-\frac{2\,x_0^\theta x^\theta y}{D\theta(x_0^\theta y^2 - x^\theta)}\cosh\ s}\ \mathrm{e}^{-\frac{s^2}{2\tau}}\ \sinh\ s\ \sin\ \frac{\pi s}{\tau} \end{aligned} \tag{38}$$

and finally, with the further change of variable $u = x_0^\theta y^2 - x^\theta$, the pdf of $X(t)$ is

$$\begin{aligned} f\,(x, t) &= \int_0^\infty h(x, y)\,\mathrm{d}y\ =\ \frac{\mathrm{e}^{-\frac{\nu^2\tau}{2}+\frac{\pi^2}{2\tau}}}{\sqrt{2\pi^3\tau}}\int_{\left(\frac{x}{x_0}\right)^{\theta/2}}^\infty \mathrm{d}y\frac{2x_0^{2\theta}\ x^{\theta-1}y^{\nu+2}}{D\,(x_0^\theta y^2 - x^\theta)^2} \\ &\quad\times \int_0^\infty \mathrm{d}s\ \mathrm{e}^{-\frac{x_0^\theta x^\theta(1+2y\ \cosh\ z+y^2)}{D\theta(x_0^\theta y^2 - x^\theta)}}\ \mathrm{e}^{-\frac{s^2}{2\tau}}\ \sinh\ s\ \sin\ \frac{\pi s}{\tau} \\ &= \frac{x_0^{\frac{(1-\nu)\theta}{2}}\ \mathrm{e}^{-\frac{\nu^2\tau}{2}+\frac{\pi^2}{2\tau}}}{D\sqrt{2\pi^3\tau}}\ x^{\theta-1}\int_0^\infty \mathrm{d}u\frac{(u + x^\theta)^{\frac{\nu+1}{2}}}{u^2} \\ &\quad\times \int_0^\infty \mathrm{d}s\ \mathrm{e}^{-\frac{x^\theta}{D\,\theta u}\left(x_0^\theta + 2x_0^{\theta/2}\sqrt{u+x^\theta}\cosh\ s + u + x^\theta\right)}\mathrm{e}^{-\frac{s^2}{2\tau}}\ \sinh\ s\ \sin\ \frac{\pi s}{\tau} \end{aligned} \tag{39}$$

In particular, in the case of a simple logistic ($\theta = 1$) we have

$$f(x,t) = \frac{x_0^{\frac{1-\nu}{2}}\ \mathrm{e}^{-\frac{\nu^2\tau}{2}+\frac{\pi^2}{2\tau}}}{D\sqrt{2\pi^3\tau}} \int_0^\infty \mathrm{d}u \frac{(u+x)^{\frac{\nu+1}{2}}}{u^2} \times \int_0^\infty \mathrm{d}s\ \mathrm{e}^{-\frac{x}{D\,u}\left(x_0+2\sqrt{x_0(u+x)}\cosh\ s+u+x\right)} \mathrm{e}^{-\frac{s^2}{2\tau}}\ \sinh\ s\ \sin\ \frac{\pi s}{\tau} \tag{40}$$

3.2.4. Moments of X(t). The moments of $X(t)$ can now be calculated either directly form (33) and (34) as

$$\begin{aligned}\boldsymbol{E}\left[X^k(t)\right] &= \int_0^\infty \mathrm{d}a \int_{-\infty}^\infty \mathrm{d}b \left(\frac{D\theta\, x_0^\theta\ \mathrm{e}^{\,2b}}{D\theta + 2x_0^\theta a}\right)^{\frac{k}{\theta}} g(a,b) \\ &= \int_0^\infty \mathrm{d}a \int_{-\infty}^\infty \mathrm{d}b \left(\frac{D\theta\, x_0^\theta\ \mathrm{e}^{\,2b}}{D\theta + 2x_0^\theta a}\right)^{\frac{k}{\theta}} \frac{\mathrm{e}^{-\frac{\nu^2\tau}{2}+\frac{\pi^2}{2\tau}}\ \mathrm{e}^{\,(\nu+1)\,b\,-\frac{1+e^{2b}}{2a}}}{a^2\sqrt{2\pi^3\tau}} \\ &\quad\times \int_0^\infty \mathrm{e}^{-\frac{e^b}{a}\cosh\ s}\ \mathrm{e}^{-\frac{s^2}{2\tau}}\ \sinh\ s\ \sin\ \frac{\pi s}{\tau}\ \mathrm{d}s\end{aligned} \tag{41}$$

or from the marginal pdf (39) of $X(t)$ as

$$\begin{aligned}\boldsymbol{E}\left[X^k(t)\right] &= \int_0^\infty x^k f(x,t)\,\mathrm{d}x \\ &= \frac{x_0^{\frac{(1-\nu)\theta}{2}}\ \mathrm{e}^{-\frac{\nu^2\tau}{2}+\frac{\pi^2}{2\tau}}}{D\sqrt{2\pi^3\tau}} \int_0^\infty \mathrm{d}x\, x^{\theta+k-1} \int_0^\infty \mathrm{d}u \frac{(u+x^\theta)^{\frac{\nu+1}{2}}}{u^2} \\ &\quad\times \int_0^\infty \mathrm{d}s\ \mathrm{e}^{-\frac{x^\theta}{D\,\theta u}\left(x_0^\theta+2x_0^{\theta/2}\sqrt{u+x^\theta}\cosh\ s+u+x^\theta\right)} \mathrm{e}^{-\frac{s^2}{2\tau}}\ \sinh\ s\ \sin\ \frac{\pi s}{\tau}\end{aligned} \tag{42}$$

In particular the first moment (expectation) of the simple logistic ($\theta = 1$) in the two formulations is

$$\begin{aligned}\boldsymbol{E}\left[X(t)\right] &= \int_0^\infty \mathrm{d}a \int_{-\infty}^\infty \mathrm{d}b \frac{D\, x_0\ \mathrm{e}^{\,2b}}{D + 2x_0 a}\ \frac{\mathrm{e}^{-\frac{\nu^2\tau}{2}+\frac{\pi^2}{2\tau}}\ \mathrm{e}^{\,(\nu+1)\,b\,-\frac{1+e^{2b}}{2a}}}{a^2\sqrt{2\pi^3\tau}} \\ &\quad\times \int_0^\infty \mathrm{e}^{-\frac{e^b}{a}\cosh\ s}\ \mathrm{e}^{-\frac{s^2}{2\tau}}\ \sinh\ s\ \sin\ \frac{\pi s}{\tau}\ \mathrm{d}s\end{aligned} \tag{43}$$

$$\begin{aligned}&= \frac{x_0^{\frac{1-\nu}{2}}\ \mathrm{e}^{-\frac{\nu^2\tau}{2}+\frac{\pi^2}{2\tau}}}{D\sqrt{2\pi^3\tau}} \int_0^\infty \mathrm{d}x\, x \int_0^\infty \mathrm{d}u \frac{(u+x)^{\frac{\nu+1}{2}}}{u^2} \\ &\quad\times \int_0^\infty \mathrm{d}s\ \mathrm{e}^{-\frac{x}{D\,u}\left(x_0+2\sqrt{x_0(u+x)}\cosh\ s+u+x\right)} \mathrm{e}^{-\frac{s^2}{2\tau}}\ \sinh\ s\ \sin\ \frac{\pi s}{\tau}\end{aligned} \tag{44}$$

The multiple integrals listed in the present section cannot apparently be performed analytically and should therefore be computed numerically. This integration is rather tricky due to the presence of the inner oscillating integral (35). Even with spartan computational tools however it is possible to check that a number of available preliminary results are fully consistent with the previous theoretical forecasts. Taking for instance the non-stationary simple logistic process (27) with $\theta = 1$, $D = {}^1\!/_2$ and degenerate initial condition $x_0 = 0.1$, a numerical evaluation of

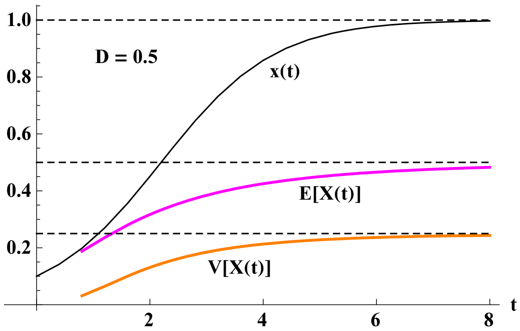

Figure 8. Time-dependent behavior of the expectation (magenta) and variance (orange) of a non stationary, simple logistic process with $D = {}^1\!/_2$ and degenerate initial condition $x_0 = 0.1$ as computed from (41). The two moments ergodically tend to their asymptotic stationary values (dashed lines respectively at 0.50 and 0.25) and are here compared to the noiseless growth $x(t)$ with the same initial condition.

the first two moments in a time interval from 0.8 to 8.0 leads to the time depending behavior of expectation and variance displayed in the figure 8. By ideally extrapolating the plots to $t = 0$ it is easy to see then that $\boldsymbol{E}\,[X(t)]$ and $\boldsymbol{V}[X(t)]$ steadily and monotonically grow from their initial values (respectively 0.1 and 0.0) toward their asymptotic, stationary values 0.50 and 0.25, so that in particular the asymptotic average level of the process stays well below the deterministic curve $x(t)$ of (11) as already anticipated in the section 3.1. The consistency of these simple result hints therefore to the fact that the exact, closed formulas presented in the present section can be now confidently adopted for every calculation regarding the non stationary logistic and θ logistic processes if one can master a few routine difficulties in the integration procedure.

3.2.5. The logistic transition pdf. Also the computation of the logistic transition pdf's is a demanding task that stimulated numerical investigations too [22, 41]. By exploiting a further general formula known in the literature [44] we will provide here another closed expressions for the transition pdf's of the SDE (22) whose finalization however again requires the calculation of some particular expectation: for more details about the derivation procedure see [42]. For the simple logistic and the θ-logistic processes we indeed respectively have

$$f(x, t|y, s) = g(x, t; y, s)\,\boldsymbol{E}\,[G(x, t; y, s)] \tag{45}$$

$$f_\theta(x, t|y, s) = g_\theta(x, t; y, s)\,\boldsymbol{E}\,[G_\theta(x, t; y, s)] \tag{46}$$

where, by taking advantage of the following Brownian bridge between $\overline{W}_{st}(0) = 0$ and $\overline{W}_{st}(1) = 0$

$$\overline{W}_{st}(r) = W(s + (t - s)r) - [rW(t) + (1 - r)W(s)] \quad 0 \leqslant r \leqslant 1 \tag{47}$$

we have defined

$$g(x, t; y, s) = \frac{\mathrm{e}^{-\frac{x-y}{2D} - \frac{1}{4D(t-s)}\left[(1-D)(t-s)-\ln\frac{x}{y}\right]^2}}{x\sqrt{4\pi D(t-s)}} \tag{48}$$

$$g_\theta(x, t; y, s) = \frac{\mathrm{e}^{-\frac{x^\theta-y^\theta}{2D\theta} - \frac{1}{4D(t-s)}\left[(1-D)(t-s)-\ln\frac{x}{y}\right]^2}}{x\sqrt{4\pi D(t-s)}} \tag{49}$$

$$G(x,t;y,s) = \mathrm{e}^{-\frac{t-s}{4D}\,H(x,t;y,s)} \qquad G_\theta(x,t;y,s) = \mathrm{e}^{-\frac{t-s}{4D}\,H_\theta(x,t;y,s)} \tag{50}$$

$$H(x,t;y,s) \;=\; y^2\int_0^1 \mathrm{d}r\left(\frac{x}{y}\right)^{2r}\,\mathrm{e}^{2\overline{W}_{st}(r)} - 2y\int_0^1 \mathrm{d}r\left(\frac{x}{y}\right)^{r}\,\mathrm{e}^{\overline{W}_{st}(r)} \tag{51}$$

$$H_\theta(x,t;y,s) \;=\; y^{2\theta}\int_0^1 \mathrm{d}r\left(\frac{x}{y}\right)^{2\theta r}\,\mathrm{e}^{2\theta\overline{W}_{st}(r)} - 2\left[1+(\theta-1)D\right]y^\theta\int_0^1 \mathrm{d}r\left(\frac{x}{y}\right)^{\theta r}\,\mathrm{e}^{\theta\overline{W}_{st}(r)} \tag{52}$$

The expected values contained in the above formulas can again be computed exactly by following the same steps presented in the previous sections because apparently they are once more expressed in terms of particular EFBM's and their distributions can therefore be traced back to the pdf (34). We will neglect however an explicit calculation for the sake of brevity.

4. Conclusions and outlook

In the present paper we presented several exact results referring to the stochastic logistic and θ-logistic models. Before dealing with these random instances, however, we preliminarily performed a careful analysis of the deterministic, noiseless logistic and θ-logistic growths, showing that they can be discussed in an unified context where the dynamics emerges from the proportionality between the relative increment of the number of elementary individuals and the percentage of resources exceeding the needs for the simple subsistence. The parameter θ is moreover interpreted as characterizing the level of correlation (classical coherence) among the individuals present in a system: in particular the correlation increases as θ decreases. In this framework, the Gompertz model—retrieved when θ goes to zero in a suitable sense—is placed by an anomalous scaling at the top of the hierarchy as *the more coherent one*.

Looking at the outlooks of the previous scheme, the most ambitious goal to consider (as it also happens in several different scientific fields) would be to assume the suggestions provided in our simplified formulation as the starting point of more sophisticated developments and of more detailed models, with the introduction of different forms of *interaction* leading to different levels of coherence in order to induce the different θ-dependencies. On the other hand, if suitably refined, this explicit formulation in terms of energies, besides clarifying the exchange mechanism between the system and the surrounding environment, can offer new perspectives also in more applicative contexts. One of such potential fields could be an inquiry about the theoretical bases of the effectiveness of the electromagnetic radiation (in particular of the ELF magnetic fields) in curtailing the growth of the biological systems. In fact, a simple modification of the equation (13) would show that the external devices which cause a partial wasting of the total energy available for the system (as it is the case for an electromagnetic source) can automatically reduce its carrying capacity, that is, its maximum size.

In the second part of the article, we went on to deal with stochastic logistic and θ-logistic models. After introducing the random fluctuations in agreement with our previous principles, we summarized the known results about the stochastic logistic and θ-logistic SDE's, i.e. their stationary distributions and their path-wise solutions. We performed next a few trajectories simulations whose inspection turns out to be instrumental to show that—whereas at a reasonably low level of noise the random paths fluctuate close to the deterministic curve, and then asymptotically stabilize around their ergodic expectation—a sensible increase of the noise intensity effectively destabilizes the process, making its behavior on the one hand more and more unpredictable, and on the other asymptotically vanishing in average as predicted in the stationary solutions.

We provided next our main results, i.e. the exact expressions (in an integral closed form) of the probability distributions and moments of the stochastic logistic and θ-logistic processes, deducing—with a suitable change of variable and a marginalization—their pdf from the joint distribution of a Brownian process and its associated EFBM already known in the literature [30–32]. In the simple logistic case ($\theta = 1$) a numerical computation of the time-behavior of expectation and variance was performed for a given noise intensity, showing that their values monotonically grow in time, and that they ergodically tend to their asymptotic, stationary values. In addition, we also provided a semi-explicit closed form for the transition pdf of the logistic SDE's, from which a fully explicit expression can be obtained by taking advantage of the same distributions previously exploited. We preferred however to postpone this computation to a possible forthcoming publication for the sake of brevity: we look forward indeed to extend these methods to obtain further exact or approximate results for other complex stochastic models describing more specific systems, and to deal with several unanswered questions.

Among the open problems, in particular, that of finding a suitable coarse-grained version of the logistic SDE's certainly is outstanding. We have shown in the previous sections that for $D \to 0$ the trajectories and the moments of a θ-logistic process apparently inch closer and closer to the deterministic behavior of a noiseless growth. This is a feature that the logistic models share with the Gompertz one, and of course it is what we were looking for in a stochastic model correctly generalizing a deterministic one. At least in the Gompertz case, however, it was proved in a previous paper [29] that there is something more: it is possible indeed to *coarse-grain* the model SDE's by finding a global quantity obeying a deterministic equation of the same type as the noiseless ODE's (ordinary differential equations) of the model. For stochastic systems that are either outright Gaussians (as for instance an Ornstein–Uhlenbeck process), or that can be traced back to some other Gaussian process (as a geometric Wiener process), this is simple enough to accomplish because of both the linearity of the involved SDE's and the symmetry of the distributions.

Take for instance the Gompertz stochastic model (for details see in particular [42]) satisfying the non-linear SDE

$$\mathrm{d}X(t) = [X(t) - \alpha X(t)\ln\ X(t)]\ \mathrm{d}t + X(t)\mathrm{d}W(t) \tag{53}$$

It is easy to see then that the transformed process $Y(t) = \ln X(t)$ satisfies the new, linear SDE

$$\mathrm{d}Y(t) = (1 - D - \alpha Y(t)) + \mathrm{d}W(t) \tag{54}$$

namely a modified Ornstein–Uhlenbeck equation with Gaussian solutions: therefore the original process $X(t)$ has a log–normal distribution. By taking the expectation of the linear SDE (54) it is easy to see moreover that the averaged quantity $\boldsymbol{E}\left[Y(t)\right]$ satisfies the ODE

$$\frac{\mathrm{d}\boldsymbol{E}\left[Y(t)\right]}{\mathrm{d}t} = (1 - D - \alpha\,\boldsymbol{E}\left[Y(t)\right]) \tag{55}$$

Remark that it would not be expedient to directly take the expectation of the SDE (53) because of its non linearity. If instead we now consider the *median* $\boldsymbol{M}[X(t)]$ of our process it is possible to show that, because of the symmetry of the Gaussian distribution of $Y(t)$, from the properties of the medians we have

$$\boldsymbol{M}\left[X(t)\right] = \boldsymbol{M}\left[\mathrm{e}^{Y(t)}\right] = \mathrm{e}^{\boldsymbol{M}[Y(t)]} = \mathrm{e}^{\boldsymbol{E}[Y(t)]}$$

and hence from (55) it is easy to check that the median satisfies the ODE

$$\frac{\mathrm{d}\boldsymbol{M}[X(t)]}{\mathrm{d}t} = \boldsymbol{M}[X(t)]\,(1 - D - \alpha\ \ln\ \boldsymbol{M}[X(t)]) \tag{56}$$

that plays here the role of a coarse-grained ODE coinciding with a slightly generalized Gomperts ODE

$$\dot{x}(t) = x(t)\,[\beta - \alpha\ \ln\ x(t)] \qquad \beta = 1 - D$$

and going back to its standard form (4) for $D \to 0$. This of course also explains why the Gomperts process $X(t)$ (its trajectories, distributions and moments) tends to its deterministic behavior $x(t)$ when the noise is switched off.

Not so, instead, for the stochastic logistic instance because—as we have shown in the previous sections—the distributions of the solutions are much more tangled. We know indeed that its trajectories, distributions and moments rightly show the bent to converge toward their deterministic behavior for vanishing noise, but in this case we are unable to recover a coarse-grained form of the SDE by proceeding along the same way trod in the case the Gompertz process. As a matter of fact the θ-logistic SDE (22) can be reduced to linear coefficients (see [42]): with the transformation $Y(t) = X^{-\theta}(t)$ we would in fact find

$$\mathrm{d}Y(t) = \theta\,[1 + ((1+\theta)D - 1)Y(t)]\ \mathrm{d}t - \theta Y(t)\mathrm{d}W(t) \tag{57}$$

but, albeit possible, it would be useless to take its expectation $\boldsymbol{E}\,[Y(t)]$. We know indeed that the path-wise solution of the SDE (57) is

$$Y(t) = \mathrm{e}^{-\theta Z(t)}\left[Y_0 + \theta\int_0^t \mathrm{e}^{\theta Z(u)}\mathrm{d}u\right]$$

where $Z(t)$ is defined in (25), and that its distributions discussed in the section 3.2 are especially intricate, confined on the positive half-axis and far from symmetric. As a consequence, even if we can easily find an equation for $\boldsymbol{E}\,[Y(t)]$, it would not be easy to manage a way to find a coarse grained quantity of the process $X(t)$ obeying some form of its noiseless equation as we did with the median in the Gompertz case, and we plan to tackle this problem in our future inquiries.

Looking finally to possible further developments within the stochastic framework, the link, here pointed out, between the logistic or θ-logistic processes, and the EFBM could lead to additional exact results in different or modified evolution stochastic models, and to a fruitful connection between very different fields as the growth processes and the mathematical finance.

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