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Phase ordering in chaotic map lattices with additive noise

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Abstract

We present some result about phase separation in coupled map lattices with additive noise. We show that additive noise acts as an ordering agent in this class of systems. In particular, in the weak coupling region, a suitable quantity of noise leads to complete phase separation. Extrapolating our results at small coupling, we deduce that this phenomenon could take place also in the limit of zero coupling. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The role of noise as an ordering agent has been broadly studied in recent years in the context of both temporal and spatiotemporal dynamics. In the temporal case, that was first considered, external fluctuations were found to produce and control transitions (known as *noise-induced transitions*) from monostable to bistable stationary distributions in a large variety of physical, chemical and biological systems [1]; the phenomenon of *noise-induced order* in chaotic systems has also been analyzed (see, for example, [2]). As far as spatiotemporal systems are concerned, the combined effects of the spatial coupling and noise may produce an ergodicity breaking of a bistable state, leading to phase transitions between spatially homogeneous and heterogeneous phases. Results obtained in this field include critical-point shifts in standard models of phase transitions [3], pure *noise-induced phase transitions* [4], stabilization of propagating fronts [5], and noise-driven structures in pattern-formation processes [6]. In all these cases, the qualitative (and somewhat counterintuitive) effect of noise is to enlarge the domain of existence of the ordered phase in the parameter space.

It is the purpose of this Letter to analyse the role of additive noise in the phase separation of multiphase coupled map lattices (CMLs) [7,8]. Coupled map lattices are networks of chaotic elements introduced to investigate complex dynamical phenomena in spatially extended systems. They consists of chaotic maps locally coupled diffusively with some coupling strength g. In these systems one observes multistability that is the remainder, for small couplings, of the completely uncoupled case [9]. For large enough couplings one observes non-trivial collective behavior (NTCB) [10]: collective quantities, such as spatial averages, display the onset of long-range order in spite

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of local chaotic fluctuations. Moreover, the temporal evolution of these quantities is “non-trivial”, i.e., not asymptotically stationary.

Recently [11], phase separation mechanisms have been investigated in a coupled map lattice model where the one-body probability distribution functions of local (continuous) variables has two disjoint supports. By introducing Ising spin variables, the phase ordering process following uncorrelated initial conditions was numerically studied and complete phase separation was observed for large coupling values. The characteristic length of domains $R(t)$ (evaluated as the width at midheigth of the two-point correlation function) showed a slow crossover from a short time behaviour to an asymptotic regime of *normal* curvature-driven domain growth. The short time behaviour was characterized by an effective growth exponent z (defined by the scaling law $R(t) \sim t^z$) continuously varying with the coupling g; at larger times the normal growth $R(t) \sim At^{1/2}$ (peculiar to the class of universality of the time dependent Ginzburg–Landau equation [12]) was observed, the prefactor A being dependent on the coupling. This study of the phase ordering properties allowed to determine the limit value g_c beyond which multistability disappears and NTCB is observed. Indeed, the following relations were used, the first related to the early stage of the dynamics, the second dealing with the asymptotic scaling regime:

$$z \sim (g - g_c)^{w_1}, \tag{1}$$

$$A \sim (g - g_c)^{w_2}. \tag{2}$$

Fitting early times data by (1) or asymptotic data by (2) lead to similar estimates for g_c (0.169 and 0.171, respectively, for the case studied in [11]). The persistence exponent θ (defined by $p(t) \sim t^{-\theta}$, where the persistence probability $p(t)$ is the proportion of spins that has not changed sign up to time t) was found to be universal in the asymptotic regime and equal to 0.204.

A similar crossover phenomenon was observed in a lattice model of chaotic maps where the corresponding Ising spin model conserves the order parameter [13]. This model is equivalent to a conserved Ising model with couplings that fluctuate over the same time scale as spin moves, in contact with a thermal bath at temperature T. The short time scaling exponents θ and z were found to vary with temperature; in particular, the effective growth exponent z was observed to increase with temperature. In the long time regime z assumes the value $1/3$, corresponding to the universality class of a Langevin equation known as model B [14], that describes the standard conserved Ising model (when bulk diffusion dominate over surface diffusion [15]). The duration of the transient decreases with temperature, becoming negligible for $T \gtrsim 1/3T_c$, where T_c is the temperature beyond which no phase ordering occurs. As a matter of fact one can conclude that, in this class of models, a proper amount of thermal noise speeds up the phase ordering process.

In this Letter we investigate the effect of additive noise on the phase ordering properties of a lattice of coupled chaotic maps, where the corresponding Ising order parameter is not conserved. It will be shown that external noise can induce complete phase ordering for coupling values not leading to phase separation in the absence of the noise term. Furthermore this dynamical transition is reentrant: phase separation appears at a critical value of the noise intensity but disappears again at one higher value of the noise strength.

The Letter is organized as follows. In the next section the coupled map lattice model here considered is introduced. In Section 3 we present our numerical results. Section 4 summarizes our conclusions.

2. The model

Let us consider a two-dimensional square lattice of coupled identical maps f acting on real variables x_i, whose evolution is governed by the difference equation

$$x_i(t+1) = (1-4g) f\left[x_i(t)\right] + g \sum_{j \in \mathcal{N}_i} f\left[x_j(t)\right] + \xi_i(t), \tag{3}$$

where $\mathcal{N}_i$ is the set of the nearest neighbors of site i, ξ_i is a random number uniformly distributed in $[-\sigma/2, \sigma/2]$, g is the coupling strength and periodic boundary conditions are assumed. We have chosen the following map:

$$f(x) = \begin{cases} -\frac{\mu}{3} \exp\left[\alpha\left(x + \frac{1}{3}\right)\right] & \text{if } x \in \left[-\infty, -\frac{1}{3}\right], \\ \mu x & \text{if } x \in \left(-\frac{1}{3}, \frac{1}{3}\right), \\ \frac{\mu}{3} \exp\left(\alpha\left[\frac{1}{3} - x\right]\right) & \text{if } x \in \left[\frac{1}{3}, +\infty\right], \end{cases} \tag{4}$$

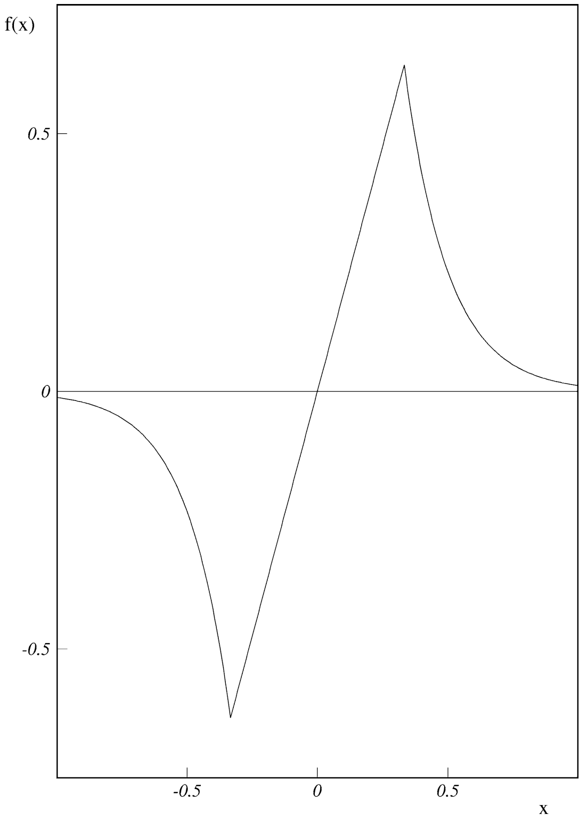

Fig. 1. The map $f(x)$ defined in (4).

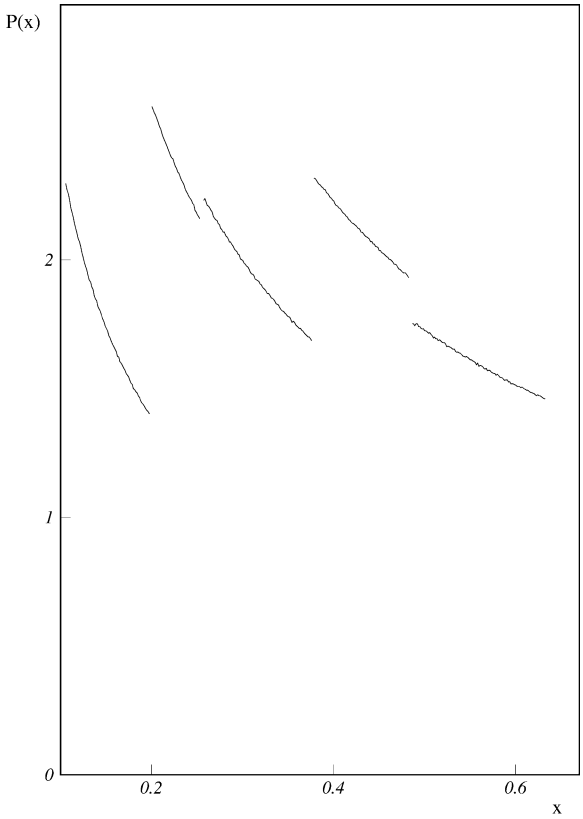

Fig. 2. Invariant probability distribution for the positive attractor of $f(x)$.

that is defined for every x in the real axis (see Fig. 1). The map here considered is a modified version of the map used in [11]; the modification is motivated by the fact that, due to the noise term ξ_i, variables $x_i(t)$ are not constrained to take value in $[-1, 1]$. Choosing $\mu = 1.9$ and $\alpha = 6$, f has two symmetric chaotic attractors, one with $x > 0$ and the other with $x < 0$. In Fig. 2 we show the invariant distribution of the attractor with positive x's: it is composed of smooth pieces. The Lyapunov exponent of the map was evaluated 0.558.

To study the phase ordering process, uncorrelated initial conditions were generated as follows: one half of the sites were chosen at random and the corresponding values of x were assigned according to the invariant distribution of the chaotic attractor with $x > 0$, while the other sites were similarly assigned values with $x < 0$. We associated an Ising spin configuration $s_i(t) = \mathrm{sgn}[x_i(t)]$ to each configuration of the x variable. Large lattices (up to 1000×1000) with periodic boundary conditions were used; the persistence $p(t)$ was measured as the proportion of sites that has not changed s the initial value. The average domain size $R(t)$ was measured by the relation $C[R(t), t] = 1/2$, where $C(r, t) = \langle s_{i+r}(t)s_i(t)\rangle$ is the two-point correlation function of the spin variables. Both $p(t)$ and $R(t)$ were averaged over many (up to thirty) different samples of initial conditions.

3. Results

As a first step we considered the noise-free case, putting $\sigma = 0$ in our model. For various values of g we measured the characteristic length R and the persistence p as functions of time; both these quantities saturate for small couplings and show scaling behaviour for large g values. Since the local map we used is slightly different from the one used in [11], we rede-

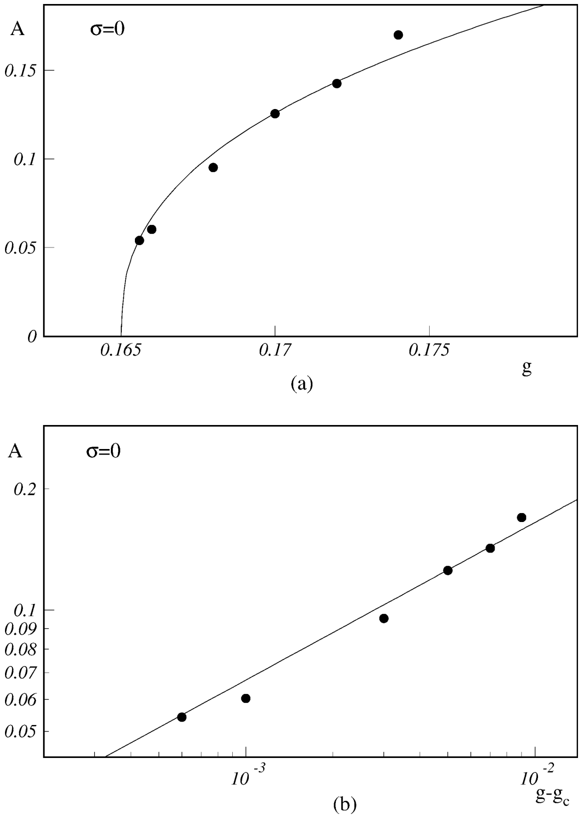

Fig. 3. The prefactor A calculated by $R(t) \sim A\sqrt{t}$ in the asymptotic time regime at $\sigma = 0$ in linear (a) and log–log (b) plot. Solid lines are best fits leading to the determination of g_c through Eq. (2).

termined the value of g_c by fitting both early times (by Eq. (1)) and asymptotic (by Eq. (2)) data (see Fig. 3). Our estimates are $g_c = 0.1652$ and $w_1 = 0.2260$ in the first case, and $g_c = 0.1650$ and $w_2 = 0.3918$ in the second one. While obviously, due to the difference between Eqs. (1) and (2), the growth exponent depends on the time scale, the estimate of the critical coupling is essentially the same, thus confirming the results of [11]. We also studied at $\sigma = 0$ the behaviour of persistence probability $p(t)$. At early times we found that the exponent θ depends on g according to the following law:

$$\theta \sim (g - g_c)^{w_3}, \tag{5}$$

with $g_c = 0.1654$. In the asymptotic regime θ is independent on g and equal to 0.209 (to be compared with the value $\theta = 0.204$ reported in [11]).

A similar behaviour is observed for not vanishing and small noise strength σ. For example, in Figs. 4(a) and (b) we show, respectively, the fit of z and θ versus g in the early times regime, while keeping σ fixed and equal to 0.1. As one can see, data are well fitted by the scaling forms (1) and (5), and the estimated values are $g_c = 0.1628$, $w = 0.2197$ for the z exponent, and $g_c = 0.1632$, $w = 0.2024$ for the θ exponent [16]. The ratio θ/z was estimated at 0.3838. Normal coarsening was recovered at late times; also in this case the prefactor A depended on g according to (2), giving the estimates $g_c = 0.1629$ and $w_2 = 0.3904$. We remark that our estimates of the critical coupling g_c, when non-vanishing and small noise is present, are all smaller than the noise-free critical value. This fact clearly shows that a proper amount of noise favours the phase separation process of the system.

The comparison between the values of g_c calculated by Eq. (1) and the ones calculated by Eq. (2) has been made for various values of σ. As already noticed in the cases of $\sigma = 0$ and $\sigma = 0.1$, the two estimates coincided within a reasonable degree of approximation. Therefore in the following we limit ourselves to report results evaluated in the early stages of the domain coarsening process.

Let us now consider the region $g < g_c(\sigma = 0) = 0.1652$. Here in the noise-free case the system evolves towards blocked configurations and no phase separation takes place. We checked, however, that this asymptotic regime was attained after very long evolution times: the system spent a lot of time in metastable states, so that the evolution curve for R and p displayed typical stairs structure. This structure (the times marking the steps of the curve) was very robust, in the sense that:

– it resisted to a change of the initial conditions (chosen following the particular prescription of Section 2),
– it did not depend on lattice dimension,
– a little noise (low σ) did not destroy it.

Nevertheless, when the amount of noise was increased, the life time of these metastable states became shorter and shorter, till they definitely disappeared for σ greater than a critical value $\sigma_c(g)$. For $\sigma > \sigma_c(g)$ we got again power laws for $R(t)$ and $p(t)$, showing that the system separates for large times. This behaviour is shown in Fig. 5.

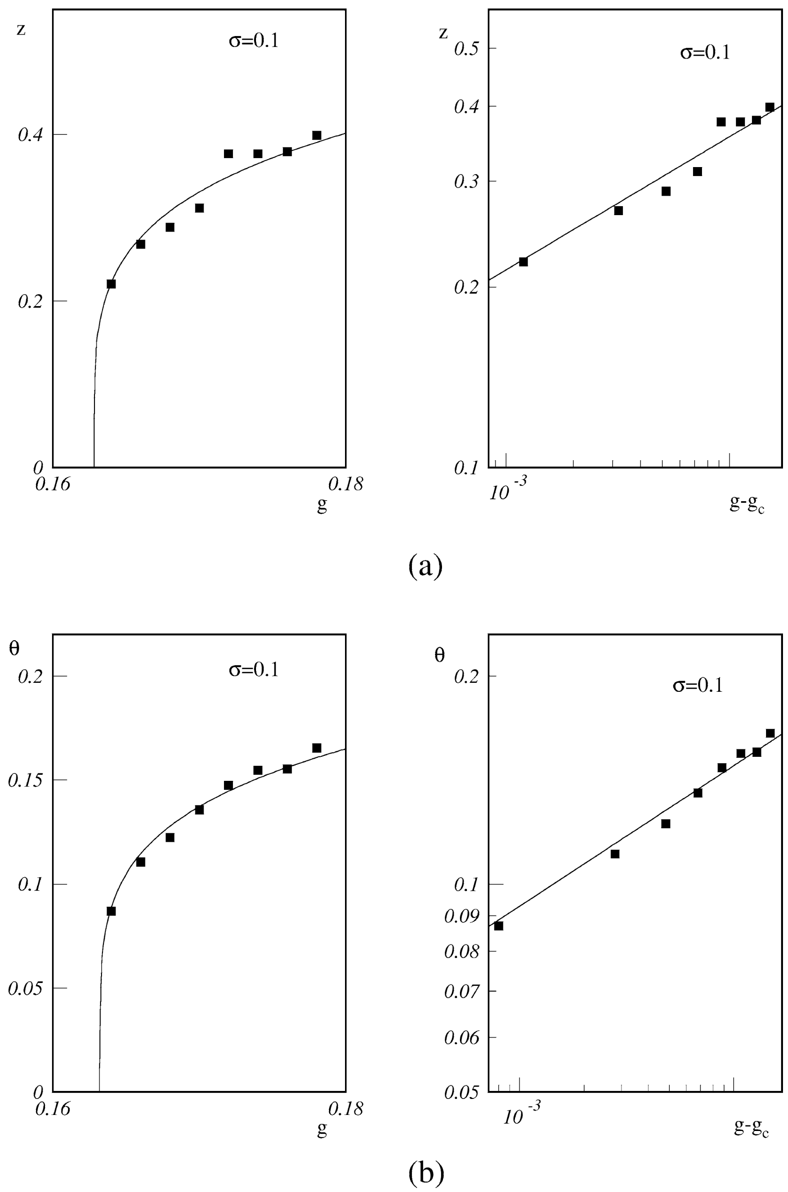

Fig. 4. The estimated scaling exponents at fixed noise $\sigma = 0.1$: (a) the dependence of the growth exponent z from g in linear and log–log plot, (b) the dependence of the persistence exponent θ from g in linear and log–log scale. Solid lines are best fits leading to the determination of g_c and w through the use of (1).

We estimated the critical value σ_c by fitting our data with the ansatz $z \sim (\sigma - \sigma_c)^w$. In Fig. 6 we show our data corresponding to $g = 0.16$: we evaluated $\sigma_c = 0.1094$ and $w = 0.3152$. The choice of this ansatz provided accurate fitting of data for a large interval of g letting us to give a precise measurement of σ_c. We were able to measure in such a way σ_c for g greater than 0.025; at smaller values of g the dynamics became very slow and we were not able to numerically extract the exponent z.

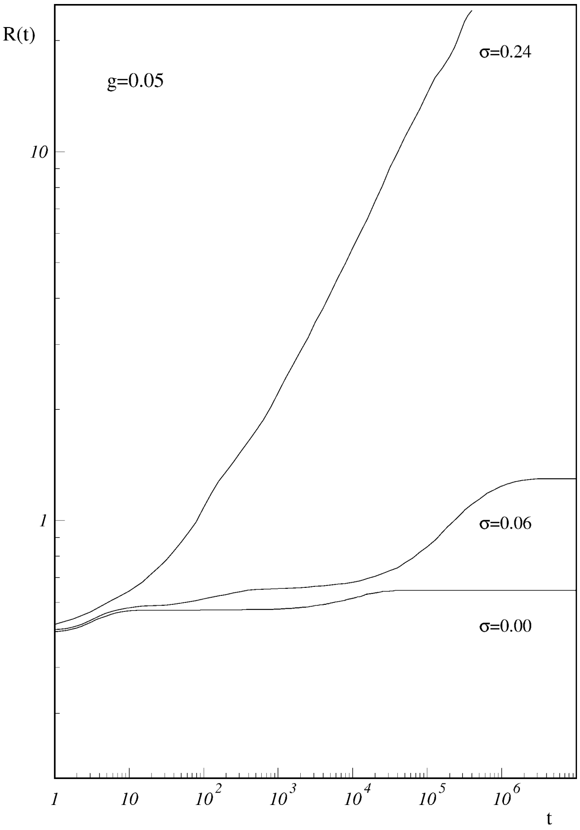

Fig. 5. The effect of additive noise on the time evolution of the domain size $R(t)$ at $g = 0.05$. The three curve are relative to $\sigma = 0$, $\sigma = 0.06$, $\sigma = 0.24$.

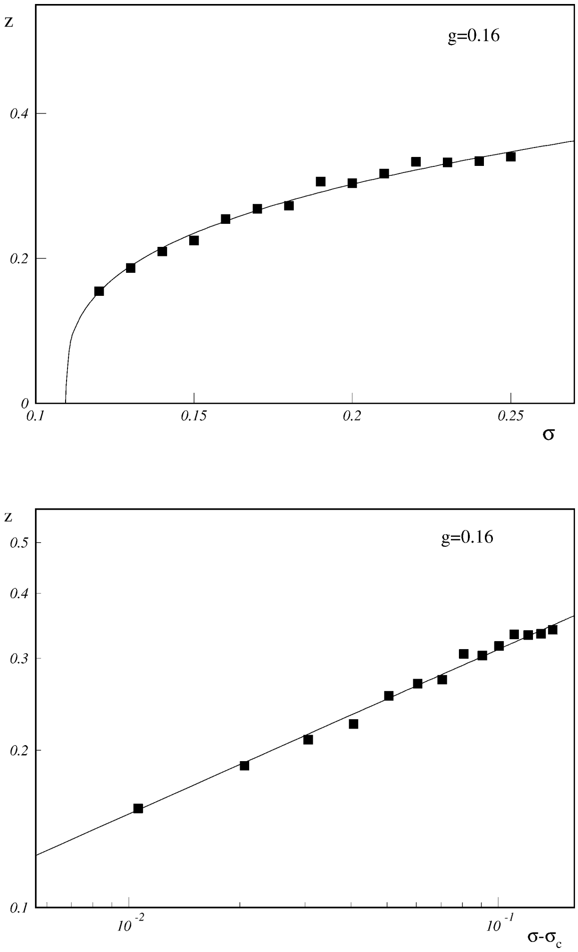

Fig. 6. The estimated growth exponent z versus σ at fixed coupling $g = 0.16$ in linear and log–log scale. Also shown is the best fit with the function $z \sim (\sigma - \sigma_c)^w$.

As σ was increased, we found a transition at another critical value of the noise strength showing that the system does not separate beyond this critical σ. As an example in Fig. 7 we show the exponent z versus σ for g fixed and equal to 0.17. The transition seems to be discontinuous.

We repeated this analysis for several values of g. Interpolating the above described data for the critical noise strength, we built the phase diagram for the model shown in Fig. 8. The system separates in the shaded area, that is it tends asymptotically to complete phase ordering. Points in the white area correspond to an asymptotic regime of the system where clusters of the two phases are dynamical but their mean size remains constant; only for $\sigma = 0$ one has blocked configurations with clusters fixed in time. Our data concern g greater than 0.025, however we extrapolated the two critical curves towards $g = 0$. We observe, interestingly, that the extrapolation of the two curves seem to meet at $g = 0$; further investigation is needed to clarify the behavior of the noisy system close to $g = 0$.

4. Conclusions

In this Letter we have shown that additive noise acts as an ordering agent in this class of systems, i.e., for a suitable amount of noise the system may order even for values of the coupling strength for which no

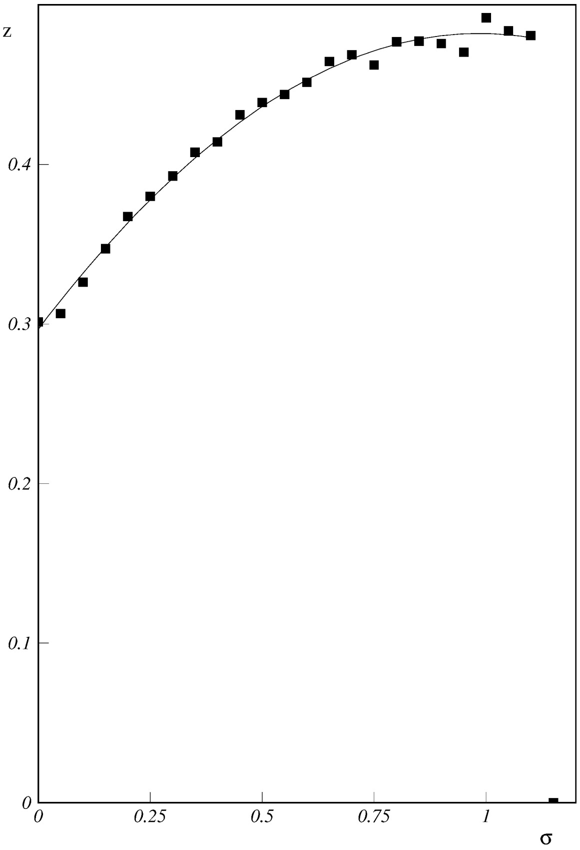

Fig. 7. The estimated growth exponent z versus σ at fixed coupling $g = 0.17$. z goes abruptly to zero at $\sigma = 1.2$ showing that the system does not separate beyond this threshold.

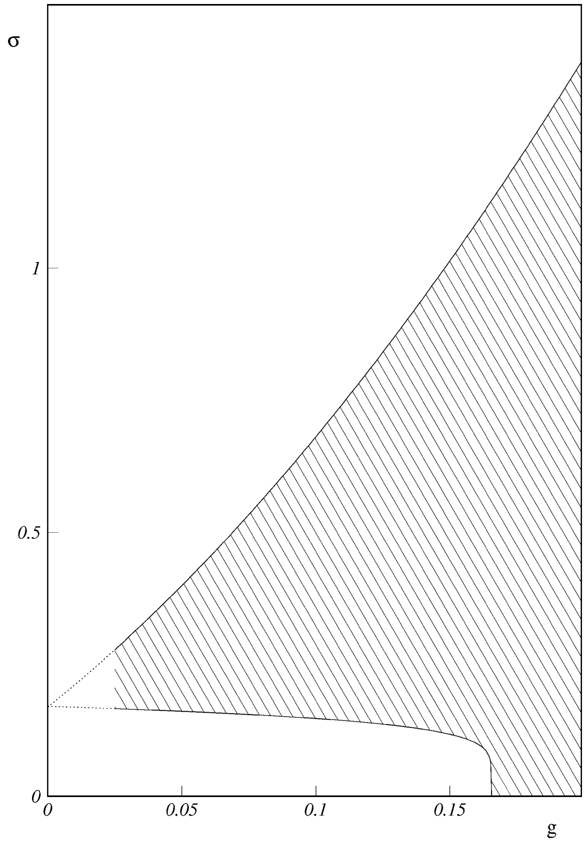

Fig. 8. The phase diagram in the plane σ–g. The shaded area represents the parameter region in which the system separates asymptotically.

separation is observed in the absence of the noise term. We have also reported some evidence that this might hold in the deep multistability region, i.e., g close to zero. A simple explanation for this behavior is the following. Small values of the spatial coupling lead, in the noise-free case, to spatially blocked configurations where interfaces between clusters of each phase are strictly pinned. A proper amount of noise makes the system cross these barriers thus leading to complete phase separation. We have numerically constructed a phase diagram describing this behavior. As already mentioned, a similar effect was observed in chaotic map lattices evolving with conserved dynamics, where we found that the growth process is favoured by temperature [13]; in the present case the additive noise plays the role of the thermal noise.

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