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Antiferromagnetic effects in chaotic map lattices with a conservation law

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Abstract

Some results about phase separation in coupled map lattices satisfying a conservation law are presented. It is shown that this constraint is the origin of interesting antiferromagnetic effective couplings and allows transitions to antiferromagnetic and superantiferromagnetic phases. Similarities and differences between these models and statistical spin models are pointed out.

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1. Introduction

Coupled map lattices (CMLs) are spatially extended dynamical systems that have been considerably investigated recently on various grounds [1]. They are approximations to continuous systems particularly well suited for numerical and analytical calculations. From another point of view they can be considered as phenomenological models describing the dynamics of a large number of macroscopic chaotic subsystems. In both cases one is interested to study their behaviour at length scales larger than the scale where chaos is present.

Particular attention has been dedicated to the study of collective behaviour in CMLs in presence of conservation laws. This class of models are considered relevant to describe several physical phenomena like surface waves in a container or disturbances in fluids, where mass, momentum and energy are conserved. A motivation for their interest is that, as it can be easily argued, the existence of conservative quantities should play an important role in determining the long distance properties in such systems. Several authors [2–4] studied coupled map models with conservation laws as possible models for chaos in extended systems, examining phase transitions from ordered to chaotic states that are driven by the map parameters or by the value of the conserved quantity.

CMLs have been studied [5–8] also from another point of view. In this case authors try to transfer con-

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cepts and results from equilibrium statistical mechanics to systems, like CMLs, whose dynamics is microscopically irreversible and does not satisfy detailed balance. In particular, studying systems of chaotic maps that exhibit a Ising-like symmetry, it can be shown that they undergo, in particular conditions, a phase-ordering dynamics. At large scale lengths the evolution of these deterministic systems has coarsening properties similar to Ising models or their continuous versions, the time-dependent Ginzburg–Landau equations [9,10]; the local chaos plays the role of the stochastic noise present in a heat bath or in the noise term of the Langevin equation. One expects obviously that, in the conserved order parameter case, this kind of CMLs should show a behaviour similar to the so-called model B. This model is described by the Cahn–Hilliard [11] equation, leading to asymptotic state in which the phases occupy two large domains separated by a single boundary. Phase separation in CMLs with conserved dynamics was observed [12] in the case of a lattice of chaotic maps in contact with a thermal bath evolving with Kawasaki dynamics. In this case the temperature plays the role of an additive noise, while chaos produces chaotically fluctuating couplings. It was definitely shown by Kockelkoren and Chaté [13] that the coarsening process for this model is strictly similar to that of the corresponding Ising model. In the same article the authors proposed to study conserved dynamics of CMLs using the approach introduced by Oono and Puri [14]. In this approach the dynamics corresponds to the discretization of the Cahn–Hilliard equation, each map on a site of the lattice representing the effect of a coarse-grained free energy. It was also shown that this model undergoes a phase transition between a short range and a long range ordered phase.

The purpose of this Letter is to extend the analysis of this model and to study the effect on the phase separation process of the coordination number used to discretize Laplacian operators on the lattice. In particular, it will be shown that the effect of the second Laplacian derived from the Cahn–Hilliard equation, assuring the order parameter conservation, amounts to the introduction of an effective antiferromagnetic coupling. The competition between ferromagnetic and antiferromagnetic couplings in these systems plays an important role in the phase separation process.

2. The model

Following [13,14] we consider a two-dimensional square lattice of coupled identical maps f acting on real variables x_i . The discrete-time dynamics is governed by the following equations:

$$\mathcal{F}(x_i^t) = (1 - \mathcal{N}g)f(x_i^t) + g \sum_j^i f(x_j^t), \quad (1)$$

$$x_i^{t+1} = \mathcal{F}(x_i^t) - \frac{1}{\mathcal{N}} \sum_j^i (\mathcal{F}(x_j^t) - x_j^t), \quad (2)$$

where \mathcal{N} is the chosen number of neighbors of site i , the sum is over these neighbors and g is the coupling strength; periodic boundary conditions are assumed. The conservation of the order parameter

$$M = \sum_i x_i \quad (3)$$

is incorporated by Eq. (2), representing the second Laplacian in the Cahn–Hilliard equation.

The map used in the numerical simulations is the following:

$$f(x) = \begin{cases} -\frac{\mu}{3} \exp[\alpha(x + \frac{1}{3})] & \text{if } x \in [-\infty, -\frac{1}{3}], \\ \mu x & \text{if } x \in [-\frac{1}{3}, \frac{1}{3}], \\ \frac{\mu}{3} \exp[\alpha(\frac{1}{3} - x)] & \text{if } x \in [\frac{1}{3}, +\infty], \end{cases} \quad (4)$$

i.e., a modified version of the map used in [13], that was defined on the interval $[-1, 1]$. The map (4) is defined for every $x \in \mathbb{R}$; the modification is motivated by the fact (already stressed in [4] and verified by the author) that, due to the redistribution step of the Oono–Puri dynamics (2), variables $x_i(t)$ are not constrained to take value in $[-1, 1]$. Details on this map can be found in [15], where it was used for similar motivations. Choosing $\mu = 1.9$ and $\alpha = 6$, f has two symmetric chaotic attractors, one with $x > 0$ and the other with $x < 0$; this allows the unambiguous definition of Ising spin variables $\sigma_i^t = \text{sgn}[x_i^t]$ associated to each dynamical system.

To study the phase separation process, uncorrelated initial conditions were generated as follows: one half of the sites were chosen at random and the corresponding values of x were assigned according to the invariant distribution of the chaotic attractor with $x > 0$,

while to the other sites were similarly assigned values with $x < 0$. With a good approximation the order parameter M vanishes. We associated an Ising spin configuration $\{s_i(t)\} = \{\text{sgn}[x_i(t)]\}$ to each configuration of the x variable. Lattices from 256×256 up to 512×512 with periodic boundary conditions were used. The average domain size $R(t)$ was measured by the relation $C[R(t), t] = 1/2$, where $C(r, t) = \langle s_{i+r}(t)s_i(t) \rangle$ is the two-point correlation function of the spin variables. $R(t)$ was averaged over many different samples of initial conditions. More complicated correlation functions and related lengths, that will be introduced in the following, have been measured. Another variable is often considered in growth processes: the persistence $p(t)$ [16], defined as the fraction of sites that have not changed their initial s values. It has not been evaluated because, to get a reliable calculation of persistence, one should use very large lattices which is not compatible with a fine scan of the coupling variable.

One normally expects that, for large couplings, the conservation of the total “magnetization” (2) leads to equilibrium states where there are only two large domains with aligned spins. However, this conservation law is also compatible with more complicated phases. In statistical spin models, they are generated by the presence of repulsive couplings in the Hamiltonian. In the sequel it will be shown that Eq. (2), in addition to the conservation law, generates these new couplings.

3. Nearest neighbors simulations

As a first step we considered maps interacting with their nearest neighbors, corresponding to $\mathcal{N} = 4$ in Eqs. (1), (2). For various values of g the characteristic length R was measured as a function of time: R saturates for weak couplings at values small compared to the lattice size. For larger couplings it shows scaling behaviour and one gets complete phase separation in the spin variables. The value of g that discriminates between these two regimes is $g \simeq 0.03$; its precise evaluation is beyond the purposes of this paper and will be presented elsewhere. Figs. 1 and 2 illustrate these changes. Fitting late times growth in the phase separation region with the law $R(t) \sim At^z$ one gets the value $z = 0.34 \pm 0.01$, in agreement with the result found by [13] in the case of 8 neighbors and with the Model B class of universality expectation (Lifshitz–Slyozov law). For higher values of the coupling one could expect a faster transition to the coarsening regime; however, one finds a metastability region between $g = 0.07$ and $g = 0.125$. Studying R as a function of time one finds at small values of R a plateau whose extension grows with g . Between $g = 0.120$ and $g = 0.125$ the system gets trapped into blocked configurations with interfaces between equal phase domains completely pinned. At $g = 0.127$ the behaviour of $R(t)$ suddenly changes, the plateau being reduced to a flex point; eventually, for $g \gtrsim 0.128$ in few time-steps the lattice reaches a completely or-

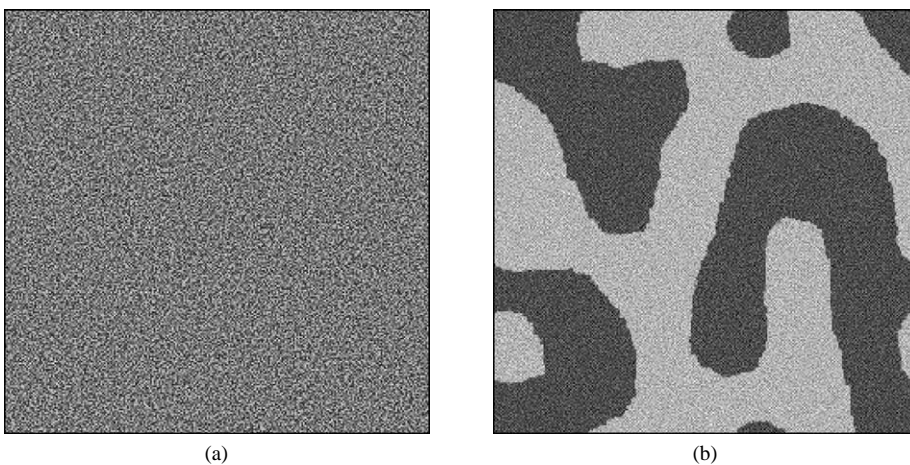


Fig. 1. Snapshots of a 256×256 sites CML with four neighbors for $g = 0.01$ (a) and $g = 0.05$ (b) after 10^6 time steps. The full range of maps' values has been colored with a 16 gray levels scale from black to white.

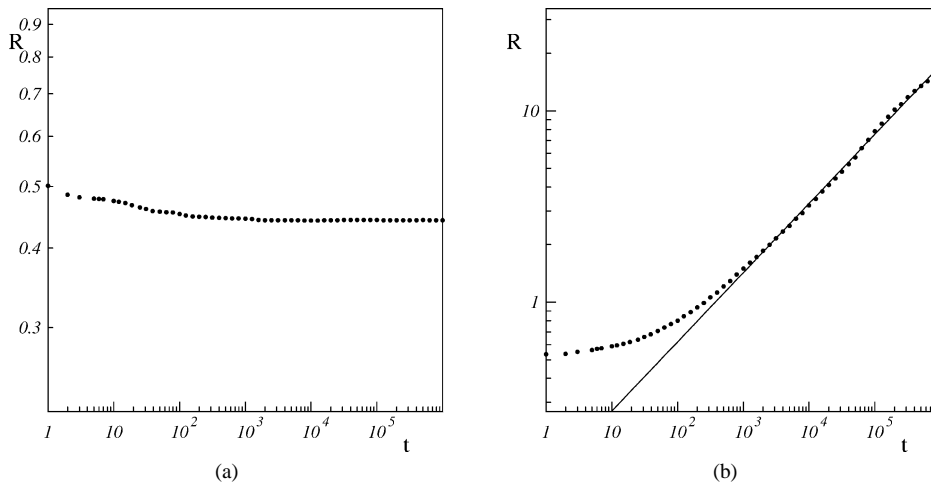


Fig. 2. Time evolution of the domain size $R(t)$ for $g = 0.01$ (a) and $g = 0.05$ (b). The solid lines is a best fit to the late time growth with power law $R(t) = At^z$ with $z = 0.341$.

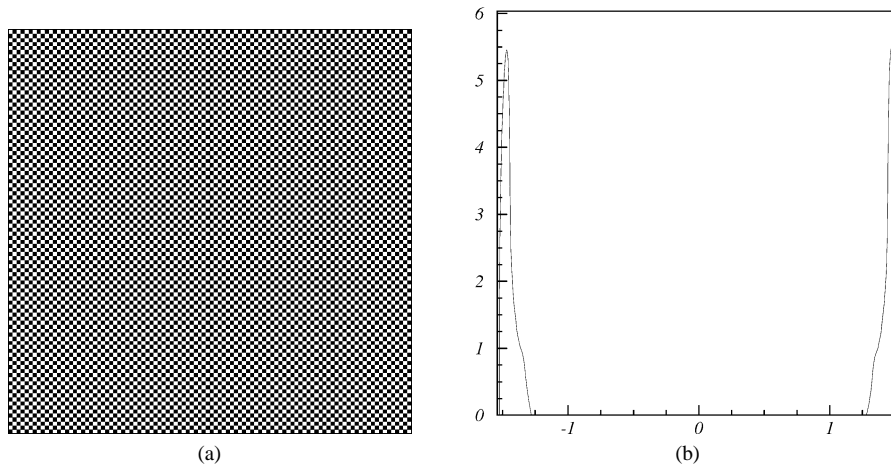


Fig. 3. The antiferromagnetic ordering at $g = 0.128$ for a CML with four neighbors: (a) a magnified portion (100×100) of a 512×512 lattice after 2000 time steps, (b) the probability distribution function for the same lattice. The full range of maps' values has been colored with a 16 gray levels scale from black to white.

dered antiferromagnetic state (see Fig. 3(a)). Antiferromagnetic (AF) ordering at large g is a normal phenomenon: for $g > 1/\mathcal{N}$ a map at time $t + 1$ gets a contribution with changed sign from its value at time t and configurations with aligned spins become unstable. This happens also in the case of CMLs with non conserved order parameter, i.e., in absence of the second step of the dynamics. However, in the present case the AF ordering is extremely precocious. The reason for this behaviour is the second redistributive step (2) of the conservative dynamics. Let us focus our atten-

tion on site i and let us suppose, for example, that the first step (1) increases, in average, the values of its neighbor maps. It is evident that, in the second step (2), this increment will result in a negative contribution to the x_i variable. This fact produces an effective antiferromagnetic coupling. As we said, at $g = 0.128$ the system quickly reaches antiferromagnetic ordering, and this phenomenon is preceded by a blocked phase and by metastability, which can be considered as pre-transitional effects. In fact, for lower values of g , one can see that the domains of aligned spins have a

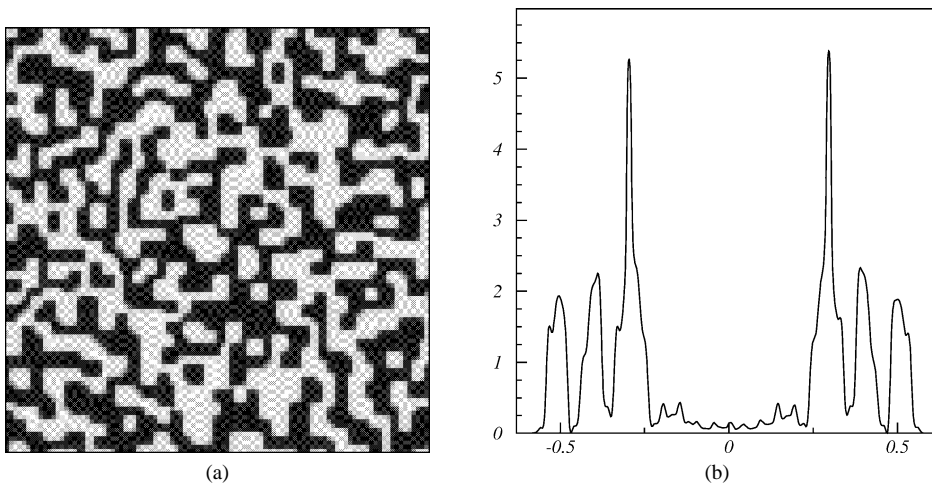


Fig. 4. Pre-transitional checkerboard structure at $g = 0.127$: (a) a magnified portion (100×100) of a 256×256 lattice after 10^6 time steps, (b) the probability distribution function for the same lattice. The full range of maps' values has been colored with a 16 gray levels scale from black to white.

checkerboard structure in which values of maps belonging to various peaks of the asymptotic probability distribution functions (PDFs) alternate (see Fig. 4). The transition to the AF ordering occurs because, for $g \gtrsim 0.128$, the asymptotic PDF reduces to two narrow peaks of opposite sign (see Fig. 3(b)). The AF ordering appears suddenly, the antiferromagnetic correlation function displaying a jump after few tens of time steps. This ordering process survives for higher values of g ; subsequently the dynamics slows down until, by increasing g , the system evolves towards blocked configurations. As many characteristics of this behaviour are similar to the case of $\mathcal{N} = 8$, we defer a more detailed analysis and discussion. Indeed, a new interesting phenomenology comes out from the study of this same system when “Laplacians” are discretized using a higher number of neighbors, and it confirms the role of the second step of the dynamics as generator of an effective AF coupling.

4. Nearest and next to nearest neighbors simulations

For small g the behaviour of the CML in the case of eight neighbors is not substantially different from the case of four neighbors, apart from the fact that the scale of g is smaller. For $g < 0.007$

no phase separation is attained and the system gets blocked when equal spin domains reach the dimension of few lattice spacings. For g between 0.007 and 0.110 the growth of opposite phases domains does not stop and one measures a growth exponent of $1/3$ at late times. However, for intermediate times scales, in the interior of equal spin domains one can note the formation of a new sub arrangement of the maps, which corresponds to stripe-ordering. The stripes in each domain correspond to peaks in PDFs belonging to the same attractor of the map (4). Fig. 5 shows for $g = 0.065$ a snapshot of the lattice and the sites PDF after 100 000 time steps. For $g \lesssim 0.110$ the competition between ferromagnetic and lamellar ordering processes is resolved in favor of the former, but this situation reverses for higher values of g . In this case the dynamics is essentially composed by two stages: starting from a random configuration (with zero average) of the map variables there is initially the formation of little striped domains oriented in the two directions of the lattice, then the motion of the domains walls produces a coarsening process.

To study the growth of these striped domains we have considered, following [17,18], a new order parameter and two related correlation functions. These articles study the growth processes of striped domains in the superantiferromagnetic (SAF) phase of the Ising model with nearest and next to nearest neighbors

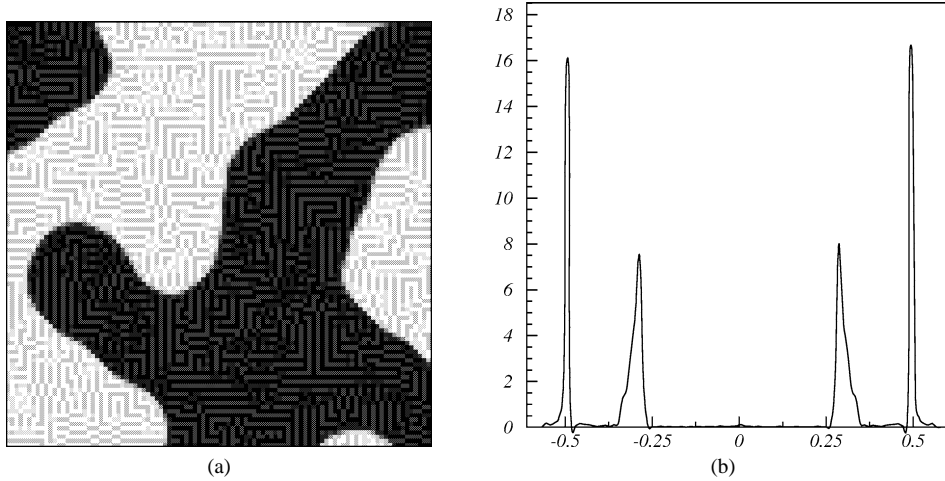


Fig. 5. The underlying striped structure for a CML with eight neighbors $g = 0.065$: (a) a magnified portion (100×100) of a 256×256 lattice after 10^5 time steps, (b) the probability distribution function for the same lattice. The full range of maps' values has been colored with a 16 gray levels scale from black to white.

interactions. We recall that the SAF phase, in which the ground state is four-fold degenerate and consists in alternate up and down spins rows or columns, is related to the existence of an antiferromagnetic coupling between next to nearest neighbors. In fact, if we call J_1 and J_2 the couplings between nearest neighbors and between next to nearest neighbors respectively, the SAF phase corresponds to $|J_1| < 2|J_2|$ and $J_2 > 0$. The existence of striped domains is, therefore, another corroboration for the existence of an effective antiferromagnetic coupling in the model here presented. Let us divide the lattice in 2×2 cells and consider a two-component local order parameter

$$\Psi^{\vec{\alpha}} = \begin{pmatrix} \psi_1^{\vec{\alpha}} \\ \psi_2^{\vec{\alpha}} \end{pmatrix} \quad (5)$$

defined in each cell $\vec{\alpha}$ in the following way:

$$\begin{aligned} \psi_1^{\vec{\alpha}} &= \sigma_1^{\vec{\alpha}} + \sigma_2^{\vec{\alpha}} - \sigma_3^{\vec{\alpha}} - \sigma_4^{\vec{\alpha}}, \\ \psi_2^{\vec{\alpha}} &= \sigma_1^{\vec{\alpha}} - \sigma_2^{\vec{\alpha}} - \sigma_3^{\vec{\alpha}} + \sigma_4^{\vec{\alpha}}, \end{aligned} \quad (6)$$

where $\sigma_i^{\vec{\alpha}}$ are the clockwise ordered spins of the cell $\vec{\alpha}$. This order parameter allows a univocal labelling of the four SAF phase ground states. Ψ allows the introduction of two correlation functions [18]:

$$\Gamma_\ell(r, t) = \frac{1}{2} \langle \psi_1^{\vec{\alpha}} \psi_1^{\vec{\alpha}+r\vec{x}} \rangle + \frac{1}{2} \langle \psi_2^{\vec{\alpha}} \psi_2^{\vec{\alpha}+r\vec{y}} \rangle, \quad (7)$$

$$\Gamma_t(r, t) = \frac{1}{2} \langle \psi_1^{\vec{\alpha}} \psi_1^{\vec{\alpha}+r\vec{y}} \rangle + \frac{1}{2} \langle \psi_2^{\vec{\alpha}} \psi_2^{\vec{\alpha}+r\vec{x}} \rangle, \quad (8)$$

where \vec{x} and \vec{y} are unit vectors in the x and y directions and $\langle \dots \rangle$ indicates the average over lattice cells and different initial conditions. Γ_ℓ and Γ_t measure, respectively, the correlation properties in the direction where the spins are aligned and in the direction where the spins are alternate.

Similarly to $C(r, t)$, it is possible to get from the measurement of $\Gamma_\ell(r, t)$ and $\Gamma_t(r, t)$ two characteristic lengths $R_\ell(t)$ and $R_t(t)$ and to point out the competition between the ordering processes. Fig. 6(a) shows R , R_ℓ and R_t as a function of time at $g = 0.105$. There is an initial increase of all these three lengths, then, after few dozens of time steps, the ferromagnetic ordering prevails, R_ℓ and R_t go to zero and the growth of R goes on, reaching the scaling regime power law with exponent $1/3$. For higher values of g the situation reverses. Fig. 6(b) shows what happens at $g = 0.111$; R_ℓ and R_t get quickly the Lifshitz–Slyozov law regime. These two evolutions are also represented in Fig. 7 and Fig. 8 through two snapshots of the lattice at $t = 100$ and $t = 50000$ for these two values of g . Another interesting feature is the presence of an anisotropic growth of domains in the SAF phase similar to that measured by [18]: R_ℓ is always greater than R_t , as long as we are far enough from the formation of a single domain covering the whole lattice. This happens, as reported in [18], when the coupling between nearest neighbors is ferromagnetic, while the reverse is true when this coupling changes sign.

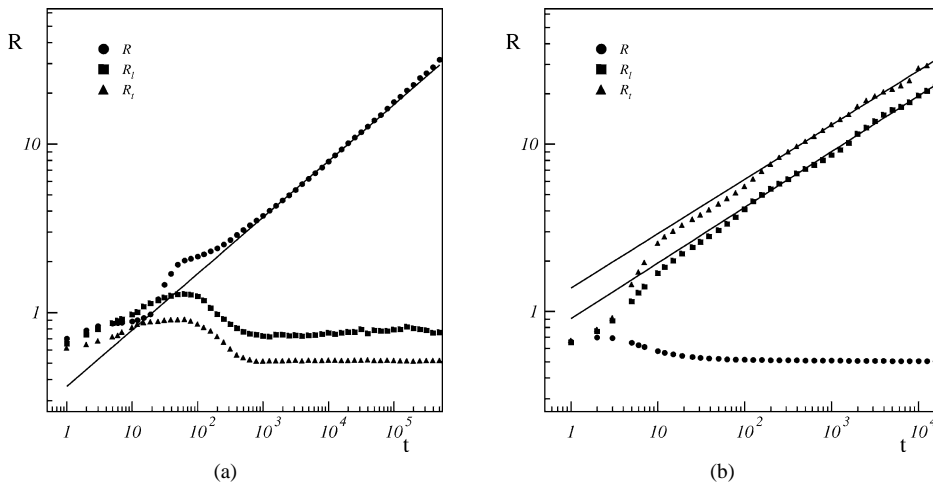


Fig. 6. (a) The three measured lengths R (circles), R_l (squares), R_r (triangles) as a function of time at $g = 0.105$; the solid lines is a best fit to the late time growth with power law $R(t) = At^z$ with $z = 0.334$. (b) The same quantities at $g = 0.111$; a power law fit gives in this case $z = 0.337$ for R_l and $z = 0.326$ for R_r .

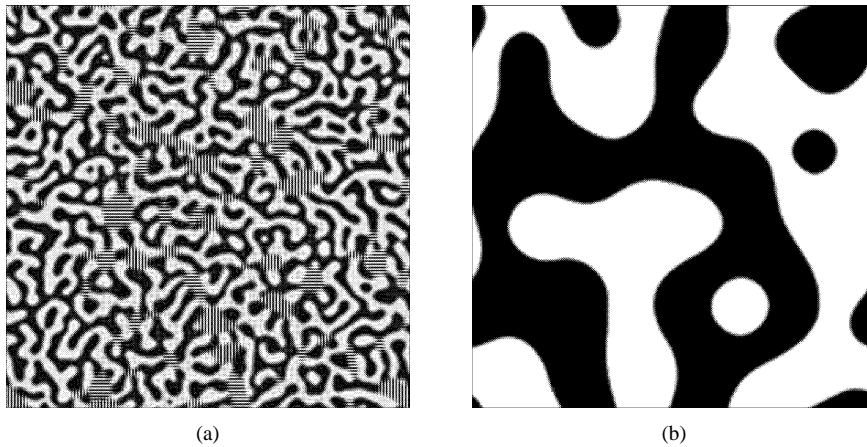


Fig. 7. Snapshots of a 256×256 sites CML with eight neighbors for $g = 0.105$ (a) after 100 and (b) after 50 000 time steps. The full range of maps' values has been colored with a 16 gray levels scale from black to white.

This situation, in which one has complete striped ordering, persists increasing g until $g \simeq 0.13$. For higher values the incapacity of the system to get complete phase separation couples to a behaviour of the maps' asymptotic PDFs that have characteristics similar to the case $\mathcal{N} = 4$. In fact, while at $g \simeq 0.13$, e.g., PDFs display two narrow peaks at opposite values of the x variable (Fig. 9(a)), increasing g they display first the broadening of these peaks, then the birth of two new peaks at symmetric x positions (Fig. 9(b)).

A further increase of g gives rise to an increasing number of peaks (Fig. 9(c)).

At this point, order parameters like those borrowed from the Ising model, the ferromagnetic, antiferromagnetic and lamellar ones, become inadequate to describe the dynamics of these systems. Even when the lengths related to these order parameters have a monotonous behaviour, one finds growth exponents different from the expected value of $1/3$. This exponent is related to the mechanism decreasing the cur-

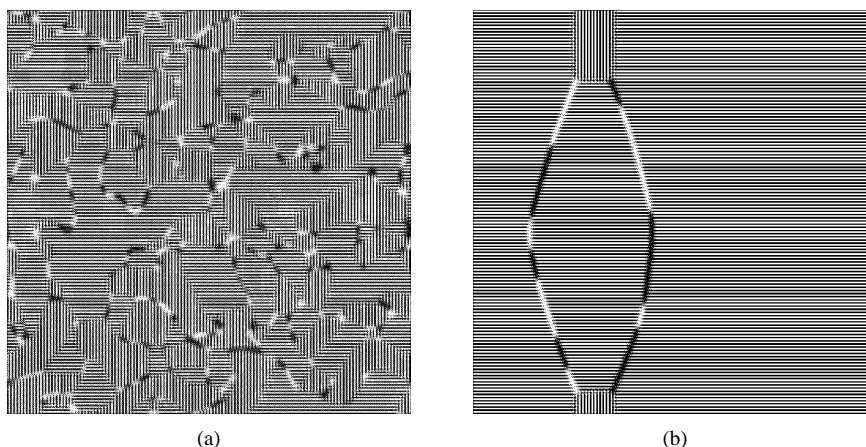


Fig. 8. Snapshots of a 256×256 sites CML with eight neighbors for $g = 0.111$ (a) after 100 and (b) after 50 000 time steps. The full range of maps' values has been colored with a 16 gray levels scale from black to white.

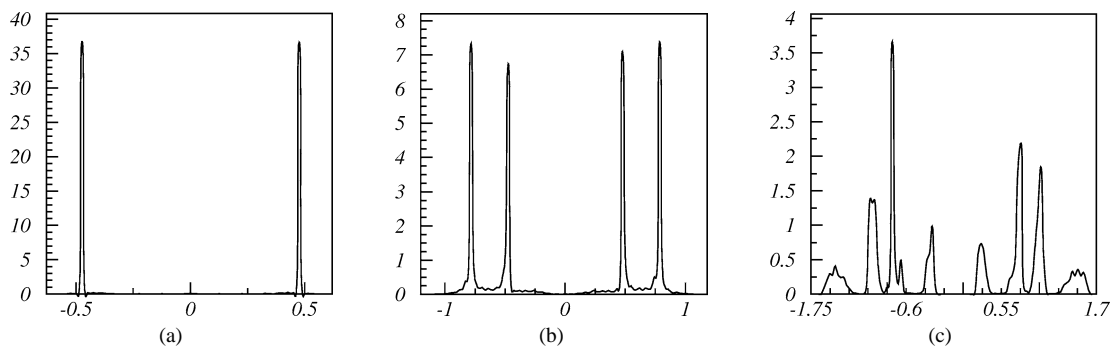


Fig. 9. Probability distribution functions for CMLs with eight neighbors after 10^5 time steps for $g = 0.13$ (a), $g = 0.20$ (b), $g = 0.30$ (c).

vature between domains in which the order parameter has an homogeneous value and to the conservation law. It is worthy to mention that the systems we are studying satisfy, in this regime, the conservation law of a quantity that is no longer the true order parameter for the coarsening process.

5. Conclusions

In this Letter some results on the time evolution of lattices of coupled maps (to which an Ising spin can be associated) in presence of a conservation law have been presented. Studying both the cases of nearest and next to nearest neighbors, intervals of the coupling have been located in which the coarsening process is

similar to the classical growth phenomena described by model B, confirming that also deterministic systems like CMLs are able to show similar processes of phase separation. It has been pointed out an interesting feature, i.e., the effective antiferromagnetic coupling rising from the conservation law constraint. Normally the study of these systems has been limited to low values of the coupling g and antiferromagnetic effects have not been considered. The present analysis shows that they give rise to an interesting phenomenology, including a scaling region and a phase diagram which resembles in some case the Ising model. For example, for $\mathcal{N} = 8$, starting from a ferromagnetic phase, one can go, increasing the coupling value, to a superantiferromagnetic phase through a paramagnetic one. However, particularly for strong couplings, we

have shown the greater complexity of these models. This complexity cannot be described by concepts and quantities transferred from the study of spin statistical models.

Furthermore, this study confirms the importance of theoretical investigations of the asymptotic probability distribution functions [19,20] with respect to the ordering process. With regard to this, one cannot exclude that, after a time longer than the one used in these simulations, eight neighbors CMLs evolve definitely towards SAF configurations and their PDFs towards two peaks structure also in the case of strong coupling.

Finally, in this analysis the parameter in (2), corresponding to the mobility coefficient in the Cahn–Hilliard model, has the fixed value $1/\mathcal{N}$, so that, with the subsequent sum, makes an average of the neighbors' value increments. Exploring the role of this parameter could be the subject of future investigations on these systems.

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