IL NUOVO CIMENTO VOL. 40 B, N. 1 11 Luglio 1977

On the Observable Differences between Proper and Improper Mixtures. - I.

N. CUFARO PETRONI

Istituto di Fisica dell'Università - Bari
Istituto Nazionale di Fisica Nucleare - Sezione di Bari

(ricevuto il 7 Marzo 1977)

Summary. — A theorem is proved which allows one to construct the most general known class of sensitive observables, namely of those observables whose expectation value is measurably different according to whether the system is described quantum-mechanically with a state vector of the first type (proper mixture) or with a state vector of the second type (improper mixture). Our result should be of interest for the experimental observation of the hypothetical spontaneous reduction of nonfactorizable state vectors.

1. – Introduction.

Many surprising features of quantum mechanics (QM) are a consequence of the nonseparability of systems also when the interaction has ceased. This was firstly remarked by EINSTEIN, PODOLSKY and ROSEN [1], which, in a famous paper, have emphasized some paradoxical consequences of such nonseparability: if two quantum systems 1, 2 interact during a time interval $0 < t < T$, QM describes the whole system Σ by a wave function $\Psi(1, 2)$, but it does not assign independent wave functions to the individual systems 1 and 2 (see ref. [2], pp. 422-437). Such a description is valid also for $t > T$, namely when any type of connection between 1 and 2 has ceased. This introduces, for the systems

[1] A. EINSTEIN, B. PODOLSKY and N. ROSEN: *Phys. Rev.*, **47**, 777 (1935).
[2] J. VON NEUMANN: *Mathematical Foundations of Quantum Mechanics* (Princeton, N. J., 1955).

1 and 2, correlations not supported by physical interactions and hence paradoxical features for the results of measurements on 1 and 2.

Although BOHR thought to solve the question on the ground of his analysis based on « complementarity » (3), in the years attempts and analyses followed, among others, by FURRY (4), BOHM and AHARONOV (5), JAUCH (6), SELLERI (7), BARACCA, BERGIA, BIGONI and CECCHINI (8), PIRON (9), BEDFORD (10), either to « explain » this features of the Nature or to change the quantum-mechanical description of a compound system. A guiding idea for these attempts, recently reconsidered by DE BROGLIE (11), consists in the hypothesis that, during the evolution of the system Σ, after the end of all types of interaction some change of the wave function takes place. Generally one makes the hypothesis of a transformation of $\Psi(1, 2)$ (nonfactorizable or of II type, see sect. **2**) into a wave function $\Phi(1, 2) = \varphi(1) \cdot \psi(2)$ (factorizable or of I type, see sect. **2**), so that every subsystem regains its freedom and lies in a definite state.

Now it is meaningful to consider the experimental problem of a check, by suitable measurements, either of the presence or of the absence of II-type state vectors $\Psi(1, 2)$ also when the interaction has ceased. Proposals in this sense come out, firstly, from Bell's work (12) (that can be considered not only as a demonstration of a contradiction between local hidden variables and QM, but also as an indication of a first particular sensitive observable, that is an observable capable to distinguish between situations described by state vectors of I or of II type) and afterwards from the articles of Capasso, Fortunato, Garuccio and Selleri (13-15). On this ground we have recorded, mostly in recent years, numerous experimental works (16-19), but none of them, for theoretical and experimental reasons (20,21), can be considered definitive.

---

(3) N. BOHR: *Phys. Rev.*, **48**, 696 (1935).
(4) W. H. FURRY: *Phys. Rev.*, **49**, 393 (1936).
(5) D. BOHM and Y. AHARONOV: *Phys. Rev.*, **108**, 1070 (1957)
(6) J. M. JAUCH: *Rendiconti S.I.F.*, Course IL (New York, N. Y., 1971).
(7) F. SELLERI: *Rendiconti S.I.F.*, Course IL (New York, N. Y., 1971).
(8) A. BARACCA, B. BIGONI, S. BERGIA and A. CECCHINI: *Riv. Nuovo Cimento*, **4**, 169 (1974).
(9) C. PIRON: in *Quantum Mechanics, Determinism, Causality and Particles* (Dordrecht, 1976).
(10) D. BEDFORD: preprint.
(11) L. DE BROGLIE: *Compt. Rend.*, **278**, 721 (1974).
(12) J. S. BELL: *Physics*, **1**, 195 (1965).
(13) V. CAPASSO, D. FORTUNATO and F. SELLERI: *Int. Journ. Theor. Phys.*, **7**, 319 (1973).
(14) D. FORTUNATO and F. SELLERI: *Int. Journ. Theor. Phys.*, **15**, 333 (1976).
(15) D. FORTUNATO, A. GARUCCIO and F. SELLERI: *Int. Journ. Theor. Phys.*, to be published.
(16) J. F. CLAUSER, R. A. HOLT, M. A. HORNE and A. SHIMONY: *Phys. Rev. Lett.*, **23**, 880 (1969).
(17) S. J. FREEDMAN and J. F. CLAUSER: *Phys. Rev. Lett.*, **28**, 938 (1972).
(18) G. FARACI, D. GUTKOWSKY, S. NOTARRIGO and A. R. PENNISI: *Lett. Nuovo Cimento*,

The present paper, therefore, is a contribution to the definition of some tools for the construction of sensitive observables, so that the experimental research can test without hesitation the existence of the factorization mechanism for states of II type.

2. – Sensitive observables.

We recall here, for greater convenience, some well-known concepts ([13-15]): In QM the states of a system Σ, compound by two subsystems 1, 2, are described by elements of a Hilbert space H_Σ given by the tensorial product $H_1 \otimes H_2$ of the Hilbert spaces H_1, H_2 which describe the states of the subsystems 1, 2. We call state vector of I type an element $|y\rangle$ of H_Σ if it is possible to find two vectors $|u\rangle$ of H_1 and $|v\rangle$ of H_2 (with $\langle u|u\rangle = \langle v|v\rangle = 1$) such that $|y\rangle = |u\rangle|v\rangle$, whereas we call state vectors of II type the remaining vectors (namely, the nonfactorizable vectors).

Now let us consider a statistical ensemble S of identical systems Σ: we say that S is a I-type mixture if it exists

$$\{S_k\}_{k=1...m}\,, \qquad S_k \subset S \quad (k = 1 \dots m)\,, \qquad \bigcup_k S_k = S\,,$$

such that every S_k contains only systems described by a single state vector of I type $|y_k\rangle$. S, on the contrary, is a II-type mixture if all the systems of S are described by a single state vector of II type $|x\rangle$.

These types of mixtures are distinguishable by measurements, but one can show ([13]) that not all the observables definable on Σ can bring out this difference. Hence we shall say that Γ is a sensitive observable for a mixture of II type S if, for every mixture of I type S', we have (if $\langle\Gamma\rangle_S$ is the expectation value of Γ for S)

$$\langle\Gamma\rangle_S \neq \langle\Gamma\rangle_{S'}\,. \tag{1}$$

Therefore, it is obvious that is of great importance from the theoretical and mostly from the experimental point of view to have some theorems which allow one to construct suitable sensitive observables on the ground of a given mixture of II type S.

9, 607 (1974).
([19]) R. A. Holt and F. M. Pipkin: Harvard University, preprint.
([20]) J. P. Vigier: *Compt. Rend.*, **279**, 1 (1974).
([21]) A. Baracca, D. Bohm, B. J. Hiley and A. E. G. Stuart: *Nuovo Cimento*, **28** B, 453 (1975).

3. – Construction of sensitive observables.

The theorem, which we shall prove, can be considered as a generalization of the well-known ([14]) methods for the construction of sensitive observables. It has, furthermore, the advantage that we can adjust the sensitivity of the observable in question in such a way that the distinguishability of mixtures of I and II type be as large as possible.

Theorem. – Let us consider a given II-type vector $|\Psi\rangle$, an arbitrary complete orthonormal system $\{|\psi_k\rangle\}$ of elements of H_Σ such that $|\psi_N\rangle = |\Psi\rangle$ and a sequence of real numbers $\{\lambda_n\}$ such that $\lambda_N > \lambda_n$, $\forall\, n \neq N$ (or $\lambda_N < \lambda_n$, $\forall\, n \neq N$). We shall prove that the observable

$$\Gamma = \sum_n \lambda_n |\psi_n\rangle\langle\psi_n|$$

is a sensitive observable for the mixture of II type S_0 described by $|\Psi\rangle$.

Proof: It is easy to show that

$$\langle\Gamma\rangle_{S_0} = \langle\Psi|\Gamma|\Psi\rangle = \lambda_N\,. \tag{2}$$

But, if $|y\rangle$ is a I-type vector, we can also show that, for the bilinear functional

$$\langle y|\Gamma|y\rangle = \sum_n \lambda_n |\langle y|\psi_n\rangle|^2\,,$$

the inequality

$$\langle y|\Gamma|y\rangle < \lambda_N \qquad (\text{or} > \lambda_N)\,, \tag{3}$$

is always valid for every I-type $|y\rangle$. Indeed, it is impossible that $\langle y|\Gamma|y\rangle = \lambda_N$ because, if there exists a I-type $|y\rangle$ such that $\langle y|\Gamma|y\rangle - \lambda_N = 0$, we should have also

$$\sum_n (\lambda_n - \lambda_N)|\langle y|\psi_n\rangle|^2 = 0$$

$\left(\text{because, for the normalization, it is } \sum_n |\langle y|\psi_n\rangle|^2 = 1\right)$ and, since by hypothesis $\lambda_n - \lambda_N < 0$ (or > 0), $\forall\, n \neq N$, $|y\rangle$ should be orthogonal to all the $|\psi_n\rangle$, $\forall\, n \neq N$; we should have, therefore, $|y\rangle = k|\psi_N\rangle = K\,|\Psi\rangle(|K| = 1)$, that is, clearly, absurd because $|y\rangle$ is a I-type vector and $|\Psi\rangle$ is a II-type vector. Furthermore, we have, by hypothesis,

$$\langle y|\Gamma|y\rangle = \sum_n \lambda_n |\langle y|\psi_n\rangle|^2 < \lambda_N \sum_n |\langle y|\psi_n\rangle|^2 = \lambda_N$$

(and *vice versa* if $\lambda_N < \lambda_n$), which definitely proves (3).

If now S' is an arbitrary mixture of I type, we can show that

$$(4) \qquad \langle \Gamma \rangle_{S'} < \lambda_N \qquad (\text{or} > \lambda_N).$$

Indeed from (3) we have (if n_k is the number of system of S'_k and $\sum n_k = M$)

$$\langle \Gamma \rangle_{S'} = \sum_k \frac{n_k}{M} \langle y_k | \Gamma | y_k \rangle < \lambda_N \sum_k \frac{n_k}{M} = \lambda_N$$

(and *vice versa* if $\lambda_N < \lambda_n$). This proves that Γ is a sensitive observable for S_0.

Moreover, we shall show in appendix that only with the addition of the hypothesis that $\{\lambda_n\}$ is a bounded sequence it is possible to keep the difference between the expectation values in question finite (and hence experimentally observable), namely that

$$(5) \qquad \sup_{S'} \langle \Gamma \rangle_{S'} < \langle \Gamma \rangle_{S_0} \qquad (\text{or } \inf_{S'} \langle \Gamma \rangle_{S'} > \langle \Gamma \rangle_{S_0}).$$

4. – Concluding remarks.

From a first point of view, our theorem is an instrument which allows us to construct an entire class of sensitive observables Γ for a mixture S_0 (of the II type) described by a given state vector $|\Psi\rangle$ (of the II type). But, from another point of view (since Γ of our theorem describes all the possible observables for Σ with $|\Psi\rangle$ as an eigenstate relative to the maximum and the minimum eigenvalue), the theorem provides information also about the following « inverse problem »: let us consider a given observable Γ; we will determine the states $|\Psi\rangle$ of II type, such that Γ is sensitive for the mixtures S_0 described by these $|\Psi\rangle$.

If we consider the question in this second perspective, we can easily show that the theorem not only states that a given Γ is a sensitive observable for the mixture S_0 described by $|\Psi\rangle$, but says also that, for every mixture described by a vector

$$|\Phi\rangle \neq k|\Psi\rangle \qquad (|k| = 1),$$

Γ is less sensitive than for S_0. Of course $|\Psi\rangle$ must be an eigenstate of Γ for $\lambda_N = \max_n \{\lambda_n\}$ or $\lambda_N = \min_n \{\lambda_n\}$. Indeed we have, from the previous discussion,

$$\begin{cases} \max_S \langle \Gamma \rangle_S = \langle \Gamma \rangle_{S_0} = \lambda_N \\ \quad (\text{or } \min_S \langle \Gamma \rangle_S = \langle \Gamma \rangle_{S_0} = \lambda_N), \\ \langle \Gamma \rangle_S < \lambda_N \\ \quad (\text{or} > \lambda_N) \text{ for the } S \text{ mixture of I type or the mixture of II type described} \\ \qquad \text{by } |\Phi\rangle \neq k|\Psi\rangle (|k| = 1). \end{cases}$$

Hence $\left(\text{since for I-type mixtures } \sup_{S'} \langle\Gamma\rangle_{S'} < \lambda_N \text{ or } \inf_{S'} \langle\Gamma\rangle_{S'} > \lambda_N\right)$ we have that mixtures of the II type $S \neq S_0$ can exist such that Γ is sensitive, but it is

$$\sup_{S'} \langle\Gamma\rangle_{S'} < \langle\Gamma\rangle_S < \langle\Gamma\rangle_{S_0} = \lambda_N$$

(or *vice versa*), namely that Γ shows a smaller sensitivity for S.

Moreover, since for the continuity (see the appendix) of the functional $\langle y|\Gamma|y\rangle$ and the connectedness (see ref. [22], p. 72, and ref. [23], p. 145) of the set of I-type vectors the value of $\langle\Gamma\rangle_{S'}$ describes all the values between $\inf_{S'} \langle\Gamma\rangle_S$ and $\sup_{S} \langle\Gamma\rangle_{S'}$ (see ref. [22], p. 66, Bolzano's theorem), it is easy to show that Γ in no case is sensitive (that is for every II-type mixture!) if it exists even one eigenstate of the I type of Γ for the eigenvalue λ_N. Indeed, in this case, it is possible to find a $|y\rangle$ of the I type, such that $\langle y|\Gamma|y\rangle = \lambda_N$.

In conclusion we can say that observables of the type Γ, as they are defined in our theorem, are the best sensitive observables for the state $|\Psi\rangle$, as we shall show in detail in a forthcoming paper.

Appendix

We will prove here statement (5). If $\{\lambda_n\}$ is a bounded sequence, namely if it exists a number $M \in R^+$ such that $|\lambda_n| \leqslant M$ for every $n \in N$, it is easy to prove that $\langle y|\Gamma|y\rangle$ is a bounded, and hence continuous (*), bilinear functional in H_Σ; indeed, for every $|y\rangle$ of H_Σ,

$$|\langle y|\Gamma|y\rangle| \leqslant \sum_n |\lambda_n| \cdot |\langle y|\varphi_n\rangle|^2 \leqslant M \sum_n |\langle y|\psi_n\rangle|^2 = M\langle y|y\rangle\,.$$

Now, if H_Σ is a finite-dimensional space, it is obviously true that $\{\lambda_n\}$ is a bounded sequence. Moreover, it is possible to demonstrate that the set of I-type vectors is a compact set [14]. Thus we deduce from the continuity (Weierstrass theorem) that there exists a $|y_0\rangle$ of the I type such that (if the « sup » is taken for $|y\rangle$ of I type)

$$\sup_{|y\rangle} \langle y|\Gamma|y\rangle = \langle y_0|\Gamma|y_0\rangle < \lambda_N$$

(or *vice versa* if $\lambda_N = \min_n \{\lambda_n\}$). From this inequality it is easy to prove our statement.

([22]) J. Dieudonné: *Foundations of Modern Analysis* (New York, N. Y., 1960).

([23]) G. F. Simmons: *Introduction to Topology and Modern Analysis* (New York, N. Y., 1963).

(*) The boundedness of a multilinear mapping is a necessary and sufficient condition for the continuity of the mapping. (See ref. [22], p. 99.)

If H_Σ is an infinite-dimensional space, the boundedness of $\{\lambda_n\}$ becomes a necessary hypothesis for the continuity. To prove (5) in this case, let us define the convex envelope of the set of the I-type vectors

$$E = \{|x\rangle \in H_\Sigma \,|\, |x\rangle = \sum_i \mu_i |y_i\rangle;\ \mu_i \geqslant 0,\ \sum_i \mu_i = 1,\ |y_i\rangle \text{ of I type}\}$$

and its closure $\bar{E}$ (*). We will show that (5) is valid for $|\Psi\rangle \in H_\Sigma - \bar{E}$. Indeed $\langle y|\Gamma|y\rangle < \lambda_N$ for every $|y\rangle \in \bar{E}$ (**), because, if $\langle y|\Gamma|y\rangle = \lambda_N$, we must have $|y\rangle = K|\Psi\rangle$ ($|K| = 1$) and this is impossible because $|y\rangle \in \bar{E}$. Furthermore, from the Weierstrass theorem, $\bar{E}$ being a compact set ([14]) and $\langle y|\Gamma|y\rangle$ a bilinear continuous functional, there exists a $|y_0\rangle \in \bar{E}$ such that

$$\sup_{|y\rangle \in \bar{E}} \langle y|\Gamma|y\rangle = \langle y_0|\Gamma|y_0\rangle < \lambda_N \qquad \Big(\text{or } \inf_{|y\rangle \in \bar{E}} \langle y|\Gamma|y\rangle > \lambda_N\Big).$$

This inequality proves also our statement (5). In a recent paper ([24]), furthermore, it has been proved that, in a particular case, statement (5) holds without the compactness condition on the set of I-type vectors (or on $\bar{E}$ for infinite-dimensional spaces).

(*) We define this closure as $\bar{E} = \{|x\rangle \in H_\Sigma \,|\, \exists\, (|x_n\rangle)_{n\in N},\ |x_n\rangle \in E \ni' |x\rangle = \lim_n |x_n\rangle\}$.

(**) Since $\langle y|y\rangle \leqslant 1$ for $|y\rangle \in E$, we have

$$\langle y|\Gamma|y\rangle = \sum_n \lambda_n |\langle y|\psi_n\rangle|^2 \leqslant \lambda_N \langle y|y\rangle \leqslant \lambda_N\,, \qquad \forall\, |y\rangle \in E\,.$$

It is easy to prove this inequality also for $\bar{E}$.

([24]) D. FORTUNATO: *Lett. Nuovo Cimento*, **15**, 289 (1976).

● RIASSUNTO

Si dimostra un teorema che permette di costruire la più generale classe conosciuta di osservabili sensibili, cioé di quelle osservabili il cui valore d'attesa è misurabilmente diverso secondo che il sistema sia quantomeccanicamente descritto da un vettore di stato del primo tipo (miscela propria) o da un vettore di stato del secondo tipo (miscela impropria). Il risultato è interessante per l'osservazione sperimentale dell'ipotetica riduzione spontanea di vettori di stato nonfattorizzabili.

О наблюдаемых различиях между собственным и несобственным смешиваниями. - I

Резюме (*). — Доказывается теорема, которая позволяет сконструировать наиболее общий известный класс таких наблюдаемых величин, для которых ожидаемые величины существенно отличаются, в зависимости от того, описывается ли система квантовомеханически с вектором состояния первого типа (собственное смешивание) или с вектором состояния второго типа (несобственное смешивание). Наш результат является интересным для экспериментального наблюдения гипотетического спонтанного приведения нефакторизуемых векторов состояний.

(*) *Переведено редакцией.*