Form of a spin-dependent quantum potential

Nicola Cufaro Petroni
Istituto Nazionale di Fisica Nucleare ed Istituto di Fisica dell'Università, I-70100 Bari, Italy

Philippe Gueret
Institut de Mathématiques Pures et Appliées, Université Pierre et Marie Curie, 4 place Jussieu, F-75230 Paris, France

Jean Pierre Vigier
Institut H. Poincaré, Laboratoire de Physique Théorique, 11 rue Pierre et Marie Curie, F-75231 Paris, France

(Received 14 February 1984)

The form of the quantum potential for spin-$\frac{1}{2}$ particles obeying a second-order wave equation is derived and the perspectives of its future applications are briefly discussed.

In a recent paper[1] the authors presented a Lagrangian formalism for spin-$\frac{1}{2}$ particles based on the Lagrangian density

$$\mathcal{L} = m^2\bar{\psi}\psi - \overline{(i\not{\partial} - e\not{A})\psi}\,(i\not{\partial} - e\not{A})\psi \quad (\hbar = c = 1) \tag{1}$$

which leads to the second-order Feynman–Gell-Mann wave equation for the four-component spinor ψ:

$$\begin{aligned}&[(i\not{\partial} - e\not{A})(i\not{\partial} - e\not{A}) - m^2]\psi \\ &\quad = [(i\partial_\mu - eA_\mu)(i\partial^\mu - eA^\mu) - \tfrac{1}{2}eF_{\mu\nu}\sigma^{\mu\nu} - m^2]\psi = 0 \ ,\end{aligned} \tag{2}$$

where $\sigma_{\mu\nu} = \frac{1}{2}i[\gamma_\mu, \gamma_\nu]$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Then, as usual[2] in the causal fluidodynamical interpretation of the wave equations, we split (2) into its real and imaginary parts by defining a real positive scalar field density

$$Q^2 = |\bar{\psi}\psi| \tag{3}$$

and a unit spinor

$$w = \psi/Q \quad (\bar{w}w = \pm 1) \ . \tag{4}$$

For the real part we obtained the following equation:

$$\begin{aligned}&\bar{w}(i\partial_\mu - eA_\mu)w\bar{w}(i\partial^\mu - eA^\mu)w - \frac{\Box Q}{Q} + \bar{w}w\partial_\mu\bar{w}\partial^\mu w \\ &\qquad + \bar{w}\partial_\mu w\bar{w}\partial^\mu w - m^2 - \tfrac{1}{2}eF_{\mu\nu}\frac{\bar{w}\sigma^{\mu\nu}w}{\bar{w}w} = 0 \ .\end{aligned} \tag{5}$$

The aim of the present report is to put (5) in a suitable compact form in order to interpret it as a generalized Hamilton-Jacobi equation. In fact, if we consider (as in the scalar case) $\Box Q/Q$ as the quantum potential,[2] we find in (5) some extra uninterpreted terms which must be wiped out by a double gauge transformation[1] on ψ and A_μ if one wants a causal interpretation of (5).

If, on the contrary, we want to deal with the complete equation (5) we are obliged to regard as quantum potential the expression

$$U = \frac{\Box Q}{Q} - \bar{w}w\partial_\mu\bar{w}\partial^\mu w - \bar{w}\partial_\mu w\bar{w}\partial^\mu w \ , \tag{6}$$

which is explicitly spin dependent, in qualitative agreement with other preceding analyses.[3]

In order to put (6) in a more suggestive form we define the 4×4 matrix

$$P = \frac{w\bar{w}}{\bar{w}w} \quad \left(P_{\alpha\beta} = \frac{w_\alpha\bar{w}_\beta}{\bar{w}_\gamma w_\gamma}\right) \ . \tag{7}$$

We remark here that, for a given spinor ψ, the product $\psi\bar{\psi}$ (with $\bar{\psi}$ on the right side) represents a 4×4 matrix with elements $\psi_\alpha\bar{\psi}_\beta$, whereas the product $\bar{\psi}\psi$ (with $\bar{\psi}$ on the left side) is the number $\bar{\psi}_\alpha\psi_\alpha = \mathrm{Tr}(\psi\bar{\psi})$. It can be easily shown that

$$\begin{aligned}&\bar{P} = \gamma_0 P^\dagger\gamma_0 = P, \quad P^2 = P \quad (P^n = P) \ , \\ &\mathrm{Tr}(P) = 1, \quad Pw = w \ ,\end{aligned} \tag{8}$$

so that P can be considered as a pseudoprojector in the w direction (not an ordinary projector because it is not Hermitian in the ordinary sense, $P^\dagger = P$). We can deduce now, from the properties of the unit spinor w $[\partial_\mu(\bar{w}w) = \Box(\bar{w}w) = 0]$ that

$$\begin{aligned}\mathrm{Tr}(P\Box P) &= \bar{w}[\Box(w\bar{w})]w \\ &= -2(\bar{w}w\partial_\mu\bar{w}\partial^\mu w + \bar{w}\partial_\mu w\bar{w}\partial^\mu w)\end{aligned} \tag{9}$$

so that the quantum potential (6) becomes

$$U = \frac{\Box Q}{Q} + \tfrac{1}{2}\mathrm{Tr}(P\Box P) \ . \tag{10}$$

Furthermore, we remark that if we define the transverse pseudoprojector

$$T = I - P \quad (PT = TP = 0) \tag{11}$$

with

$$\begin{aligned}&\bar{T} = T, \quad T^2 = T \quad (T^n = T) \ , \\ &\mathrm{Tr}(T) = 3, \quad Tw = 0 \ ,\end{aligned} \tag{12}$$

and if we consider a function

$$F(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \tag{13}$$

we can immediately prove that

$$F(P) = F(1)P + F(0)T \ , \tag{14}$$

so that in general

$$f(Q;P)=\alpha(Q)P+\beta(Q)T \ . \tag{15}$$

It is possible to show now that, if we pose

$$\Omega=\alpha(Q)P+\beta(Q)T \ ,$$
$$\overline{\Omega}=\gamma_0\Omega^\dagger\gamma_0=\alpha^*P+\beta^*T \ , \tag{16}$$

the quantum potential (10) takes the final form

$$U=\frac{\Box Q}{Q}+\tfrac{1}{2}\mathrm{Tr}(P\Box P)=\frac{\mathrm{Tr}(\overline{\Omega}\Box\Omega)}{\mathrm{Tr}(\overline{\Omega}\Omega)} \tag{17}$$

if $\alpha(Q)$ and $\beta(Q)$ satisfy the conditions

$$\frac{\alpha^*\alpha''+3\beta^*\beta''}{|\alpha|^2+3|\beta|^2}=0 \ ,$$
$$\frac{\alpha^*\alpha'+3\beta^*\beta'}{|\alpha|^2+3|\beta|^2}=\frac{1}{Q} \ , \tag{18}$$
$$\frac{|\alpha-\beta|^2}{|\alpha|^2+3|\beta|^2}=\frac{1}{2} \ .$$

These conditions can be fulfilled by supposing, for example,

$$\alpha(Q)=aQ, \quad \beta(Q)=bQ \ ,$$
$$|a|^2-|b|^2-4\,\mathrm{Re}(a^*b)=0 \ . \tag{19}$$

A possible choice for a and b is $a=1$ and $b=i$ so that

$$\Omega=Q(P+iT)=QF(P) \ , \tag{20}$$

where $F(z)$ is a function such that $F(0)=i$ and $F(1)=1$; for example,

$$f(z)=\sqrt{2z-1} \tag{21}$$

so that

$$\Omega=Q\sqrt{P-T}=\sqrt{\mathscr{P}-\mathscr{T}} \ , \tag{22}$$

where

$$\mathscr{P}=Q^2P=\frac{\psi\overline{\psi}}{\overline{w}w} \ ,$$
$$\mathscr{T}=Q^2T=\frac{\overline{\psi}\psi-\psi\overline{\psi}}{\overline{w}w} \ . \tag{23}$$

Finally, by recalling that[1] the conserved current for the Lagrangian (1) is split into a drift and a spin part

$$J_\mu=j_\mu+j'_\mu \ ,$$
$$j_\mu=\frac{Q^2}{m}\overline{w}(i\partial_\mu-eA_\mu)w \ , \tag{24}$$
$$j'_\mu=\frac{Q^2}{m}\left(\frac{\partial^\nu Q}{Q}\overline{w}\sigma_{\mu\nu}w+\tfrac{1}{2}\partial^\nu(\overline{w}\sigma_{\mu\nu}w)\right) \ ,$$

and by defining a generalized drift momentum density

$$g^\mu=\frac{m}{\overline{\psi}\psi}j^\mu=\frac{\overline{w}(i\partial^\mu-eA^\mu)w}{\overline{w}w} \ , \tag{25}$$

Eq. (5) can be put in the form of a generalized Hamilton-Jacobi equation, namely,

$$g_\mu g^\mu-m^2-\frac{\mathrm{Tr}(\overline{\Omega}\Box\Omega)}{\mathrm{Tr}(\overline{\Omega}\Omega)}-\tfrac{1}{2}eF_{\mu\nu}\frac{\overline{w}\sigma^{\mu\nu}w}{\overline{w}w}=0 \tag{26}$$

Moreover, in the free case ($A^\mu=0$) Eq. (26) can be put in a form which generalizes the relation $p_\mu p^\mu=m^2$:

$$G_\mu G^\mu=M^2 \ , \tag{27}$$

where, of course,

$$G_\mu=\overline{w}i\partial_\mu w \ ,$$
$$M^2=\frac{\mathrm{Tr}[\overline{\Omega}(\Box+m^2)\Omega]}{\mathrm{Tr}(\overline{\Omega}\Omega)} \ . \tag{28}$$

We conclude with some remarks.

(a) The form (17) of the quantum potential constitutes an effective generalization of the corresponding expression for the scalar case; in fact (17) reduces to the usual form

$$U=\frac{\Box Q}{Q} \tag{29}$$

if ψ is a scalar function because, from (23), we would have $\mathscr{P}=Q^2$, $\mathscr{T}=0$, and hence $\Omega=Q=(|\psi|^2)^{1/2}$. Moreover, in the same way we can see that the term M^2 in (28) is the correct generalization of the "variable proper mass," first introduced by de Broglie,[4] to the case of spin-$\frac{1}{2}$ particles.

(b) The new generalized form (17) of the quantum potential poses the problem of its connection with a relativistic stochastic formalism[5] if one wants to interpret the quantum behavior of a spinning particle as the overall manifestation of the chaotic character of a random subquantum ether.[6] Indeed, until now, this connection was made[1,5] only for quantum potentials of the form (29), even for spinning particles.[7]

(c) The expression (17) should be used now to carry out careful calculations of the "trajectories" of a spinning particle subject to the corresponding "quantum forces." It is well known that these calculations have been performed[8] for the scalar case (29): the generalization to the spinning case could be very useful in interpreting in a causal nonlocal way the results of Aspect's experiments[9] on the Einstein-Podolsky-Rosen-Bohm paradox.[10] Of course, a fundamental step in doing it is the generalization of the proof, given for the scalar case,[11] that the action at a distance, carried by a spin-dependent quantum potential (17) for two correlated particles, satisfies the compatibility conditions required by the predictive mechanics.[12]

(d) Finally we must remark that our analysis is not an artifact of the choice of gauge quoted at the beginning of this paper with regard to Eq. (5). In fact we must consider the approach that considers (29) as the total quantum potential[1] as not entirely satisfactory for two main reasons.

(1) In order to get a generalized Hamilton-Jacobi equation with (29) as quantum potential it is necessary to impose[1] some extra conditions on the gauge phase in the form of a partial differential first-order equation on this gauge phase. The choice of the gauge was determined, in some unnatural way, by the particular spinor to which the transformation was applied.

(2) Moreover, the connection of the generalized Hamilton-Jacobi equation [with (29) as quantum potential] obtained by means of the gauge transformation with a stochastic process (that we consider as a fundamental step in our theory), needs a supplementary hypothesis:[1] the vector $\overline{w}\partial_\mu w$ must be gradient of a scalar function. That is obvi-

ously true in some particular cases (e.g., plane waves) but that does not hold for every unit spinor w. Therefore, we cannot consider the formulation based on (29) as a complete demonstration of the equivalence between a stochastic process and the Feynman–Gell-Mann equation.

In conclusion, for spin-$\frac{1}{2}$ fields, we consider the expression (29), obtained for the quantum potential by means of a gauge transformation, only as a reduction of the complete quantum potential (6) or (17): this reduction was useful for a first approach to the problem of the connection with stochastic processes but, of course, the definitive solution of this problem needs the most general form (6) or (17).

[1]N. Cufaro Petroni, Ph. Gueret, and J. P. Vigier, Nuovo Cimento (to be published).

[2]D. Bohm, Phys. Rev. **85**, 166 (1952); **85**, 180 (1952).

[3]D. Bohm, R. Schiller, and J. Tiomno, Nuovo Cimento Suppl. 1, 48 (1955); L. de la Peña-Auerbach, J. Math. Phys. **12**, 453 (1971).

[4]L. de Broglie, J. Phys. Rad. **8**, 225 (1927).

[5]W. Lehr and J. Park, J. Math. Phys. **18**, 1235 (1977); F. Guerra and P. Ruggiero, Lett. Nuovo Cimento **23**, 529 (1978); J. P. Vigier, *ibid.* **29**, 265 (1979).

[6]P. A. M. Dirac, Nature **168**, 906 (1951); J. P. Vigier, Lett. Nuovo Cimento **29**, 467 (1980); N. Cufaro Petroni and J. P. Vigier, Found. Phys. **13**, 253 (1983).

[7]N. Cufaro Petroni and J. P. Vigier, Phys. Lett. **73A**, 289 (1979); **81A**, 12 (1981).

[8]C. Philippidis, C. Dewdney, and B. J. Hiley, Nuovo Cimento **52B**, 15 (1979).

[9]A. Aspect, G. Grangier, and G. Roger, Phys. Rev. Lett. **47**, 460 (1981); **49**, 91 (1982); A. Aspect, J. Dalibard, and G. Roger, *ibid.* **49**, 1804 (1982).

[10]A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935); D. Bohm and Y. Aharonov, *ibid.* **108**, 1070 (1957).

[11]N. Cufaro Petroni, Ph. Droz-Vincent, and J. P. Vigier, Lett. Nuovo Cimento **31**, 415 (1981).

[12]Ph. Droz-Vincent, Phys. Scr. **2**, 129 (1970); L. Bel, Ann. Inst. Henri Poincaré **3**, 307 (1970); Ph. Droz-Vincent, *ibid.* **27**, 407 (1977); Phys. Rev. D **19**, 702 (1979).