Controlled quantum evolutions and stochastic mechanics

Nicola Cufaro Petroni

INFN Sezione di Bari and Dipartimento di Fisica dell'Università di Bari,
via Amendola 173, 70126 Bari (Italy)
CUFARO@BARI.INFN.IT

Salvatore De Martino, Silvio De Siena and Fabrizio Illuminati

INFN Sezione di Napoli - Gruppo collegato di Salerno and
Dipartimento di Fisica dell'Università di Salerno
via S.Allende, 84081 Baronissi, Salerno (Italy)
DEMARTINO@PHYSICS.UNISA.IT, DESIENA@PHYSICS.UNISA.IT
ILLUMINATI@PHYSICS.UNISA.IT

ABSTRACT: *We perform a detailed analysis of the non stationary solutions of the evolution (Fokker-Planck) equations associated to either stationary or non stationary quantum states by the stochastic mechanics. For the excited stationary states of quantum systems with singular velocity fields we explicitely discuss the exact solutions for the HO case. Moreover the possibility of modifying the original potentials in order to implement arbitrary evolutions ruled by these equations is discussed with respect to both possible models for quantum measurements and applications to the control of particle beams in accelerators.*

1. Introduction

In a few papers [1] the analogy between diffusive classical systems and quantum systems has been reconsidered from the standpoint of the stochastic mechanics (SM) [2], [3], and particular attention was devoted there to the evolution of the classical systems associated to a quantum wave function when the conditions imposed by the stochastic variational principle are not satisfied (non extremal processes). The hypothesis that the evolving distribution converges in time toward the quantum distribution, constituted several years ago an important point in the answer by Bohm and Vigier to some criticisms to the assumptions of the Causal Interpretation of the Quantum Mechanics

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(CIQM) [4]. In the quoted papers [1] it was pointed out that, while the right convergence was in fact achieved for a few quantum examples, these results could not be considered general as shown in some counterexamples: in fact not only for particular non stationary wave functions (as for a minimal uncertainty packet), but also for stationary states with nodes (namely with zeros) we do not seem to get the right asymptotic behaviour. For stationary states with nodes the problem is that the corresponding velocity field to consider in the Fokker-Planck equation shows singularities in the locations of the nodes of the wave function. These singularities effectively separate the available interval of the space variables into (probabilistically) non communicating sections which trap any amount of probability initially attributed and make the system non ergodic.

In a more recent paper [5] it has been shown first of all that for transitive systems with stationary velocity fields (as, for example, a stationary state without nodes) we always have an exponential convergence to the right quantum probability distribution associated to the extremal process, even if we initially start from an arbitrary non extremal process. These results can also be extended to an arbitrary stationary state if we separately consider the process as confined in every configuration space region between two subsequent nodes. Moreover it has been remarked there that while the non extremal processes should be considered virtual, as trajectories in the classical Lagrangian mechanics, they can also be turned real if we modify the potential in a suitable way. The interest of this remark lies not only in the fact that non extremal processes are exactly what is lacking in quantum mechanics in order to interpret it as a totally classical stochastic process theory (for example in order to have a classical picture of a double slit experiment [6]), but also in the possibility of engineering some new controlled real evolutions of quantum states. In fact this could be useful to study (a) transitions between stationary states (b) possible models for measure theory [3] and (c) control of the particle beam dynamics in accelerators [7]. In a sense the SM is also a theory, independent from quantum mechanics, with applications in several physical fields, in particular for systems not perfectly described by the quantum formalism, but whose evolution is correctly controlled by quantum fluctuation: the so called mesoscopic or quantum-like systems. This behaviour characterizes, for example, the beam dynamics in particle accelerators and there is evidence that it could be described by the stochastic formalism of Nelson diffusions [1], [7]. Of course in this model trajectories and transition probabilities always are perfectly meaningful and, to study in detail the evolution of the probability distributions, and in particular to try to understand if and how it is possible to realize controlled evolutions, it is necessary to determine the fundamental solutions (transition probability densities) associated by SM to every quantum state in consideration: a problem dealt with in the following sections.

2. Fokker-Planck equations for stochastic mechanics

SM is a generalization of classical mechanics based on the theory of classical stochastic processes [2]. The variational principles of Lagrangian type provide a foundation for it, as for the classical

mechanics or the field theory [3]. In this scheme the deterministic trajectories of classical mechanics are replaced by the random trajectories of diffusion processes in the configuration space. The surprisig feature is that programming equations derived from the stochastic version of the lagrangian principle are formally identical to the equations of a Madelung fluid [8], the hydrodynamical equivalent of the Schrödinger equation in the Stochastic Interpretation of the Quantum Mechanics (SIQM) [9]. On this basis, it is possible to develop an interpretative scheme where the phenomenological predictions of SM coincide with that of quantum mechanics for all the experimentally measurable quantities. Within this interpretative code the SM is nothing but a quantization procedure, different from the ordinary ones only formally, but completely equivalent from the point of view of the physical consequences. Hence we consider here the SM as a probabilistic simulation of quantum mechanics, providing a bridge between this fundamental section of physics and the stochastic differential calculus. However it is well known that the most peculiar features of the involved stochastic processes, namely the transition probability densities, seem not always enter into this code scheme: in fact, if we want to check experimentally if the transition probabilities are the right ones for a given quantum state, we are obliged to perform repeated position measurements on the quantum system; but, according to quantum theory, the quantum state changes at every measurement (wave packet reduction), and since our transition probabilities are associated to a well defined wave function it will be in general practically impossible to experimentally observe a well defined transition probability. Several ways out of these difficulties have been explored: for example stochastic mechanic scheme could be modified by means of non constant diffusion coefficients [1]; or alternatively it would be possible to modify the stochastic evolution during the measurement [10]. Here we will rather assume that the processes which do not satisfy the stochastic variational principle still keep a physical meaning and that tey will rapidly converge (in time) toward the processes associated to quantum states. Indeed on the one hand any departure from the distributions of quantum mechanics will quickly be reabsorbed in the time evolution, at least in many meaningful cases; and on the other hand the non standard evolving distributions could be realized by suitable quantum systems for modified, time dependent potentials which may asymptotically in time rejoin the usual potentials.

SM is a model intended to achieve a connection between quantum mechanics and classical random phenomena: here we will recall a few notions in order to fix the notation. The position of a classical particle is promoted to a vector Markov process $\xi(t)$ defined on some probabilistic space $(\Omega, \mathcal{F}, \mathbf{P})$ and taking values in $\mathbf{R}^3$. We suppose that this process is characterized by a pdf $f(\mathbf{r}, t)$ and a transition pdf $p(\mathbf{r}, t|\, \mathbf{r}', t')$ and satisfies an Itô stochastic differential equation of the form

$$d\xi_j(t) = v_j\big(\xi(t), t\big)dt + d\eta_j(t) \tag{2.1}$$

where v_j are the components of the forward velocity field. However here v_j are not given a priori, but play the role of dynamical variables and are subsequently determined on the basis of a variational principle, namely on the basis of a dynamics. On the other hand $\eta(t)$ is a Brownian

process independent of $\xi(t)$ and such that

$$\mathbf{E}_t\big(d\eta_j(t)\big) = 0\,, \qquad \mathbf{E}_t\big(d\eta_j(t)\, d\eta_k(t)\big) = 2\,D\,\delta_{jk}\,dt \tag{2.2}$$

where $d\eta(t) = \eta(t+dt) - \eta(t)$ (for $dt > 0$), D is a diffusion coefficient, and $\mathbf{E}_t$ are the conditional expectations with respect to $\xi(t)$. In what follows we will limit ourselves to the case of the one dimensional trajectories, so that the Markov processes $\xi(t)$ considered will always take values in $\mathbf{R}$. Moreover we will suppose for the time being that the forces acting on the particle will be defined by means of a time-independent potential $V(x)$. A suitable definition of the Lagrangian and of the stochastic action functional for the system described by the dynamical variables f and v allows us to select, by means of the principle of stationarity of the action, the processes which reproduce the quantum mechanics [2], [3]. In fact, while the pdf $f(x,t)$ of the process satisfies, as usual, the Forward Fokker-Planck (FP) equation associated to (2.1)

$$\partial_t f = D\partial_x^2 f - \partial_x(vf) = \partial_x(D\partial_x f - vf)\,, \tag{2.3}$$

the following choice for the Lagrangian field

$$\mathcal{L}(x,t) = \frac{m}{2}v^2(x,t) + mD\partial_x v(x,t) - V(x) \tag{2.4}$$

enables us to define a stochastic action funcional

$$\mathcal{A} = \int_{t_0}^{t_1} \mathbf{E}\mathcal{L}\big(\xi(t),t\big)\,dt \tag{2.5}$$

which leads, through the stationarity condition $\delta\mathcal{A} = 0$, to the equation

$$\partial_t S + \frac{(\partial_x S)^2}{2m} + V\, - 2mD^2\,\frac{\partial_x^2\sqrt{f}}{\sqrt{f}} = 0 \tag{2.6}$$

involving a field $S(x,t)$ defined as

$$S(x,t) = -\int_t^{t_1} \mathbf{E}\left(\mathcal{L}\big(\xi(s),s\big)\,\big|\,\xi(t) = x\right)\,ds + \mathbf{E}\left(S_1\big(\xi(t_1)\big)\,\big|\,\xi(t) = x\right) \tag{2.7}$$

where $S_1(\,\cdot\,) = S(\,\cdot\,,t_1)$ is an arbitrary final condition. Now the relevant remark is that if $R(x,t) = \sqrt{f(x,t)}$, and if we define

$$\psi(x,t) = R(x,t)\,\mathrm{e}^{iS(x,t)/\hbar} \tag{2.8}$$

the equation (2.6) takes the form

$$\partial_t S + \frac{(\partial_x S)^2}{2m} + V\, -\, \frac{\hbar^2}{2m}\,\frac{\partial_x^2 R}{R} = 0\,, \tag{2.9}$$

and the complex wave function ψ will satisfy the Schrödinger equation

$$i\hbar\partial_t\psi = \hat{H}\psi = -\,\frac{\hbar^2}{2m}\,\partial_x^2\psi + V\psi\,, \tag{2.10}$$

provided that the diffusione coefficient be connected to the Planck constant by the relation

$$D = \frac{\hbar}{2m}\,. \tag{2.11}$$

This trail leading from classical stochastic processes (plus a dynamics) to quantum mechanics can also be trod in the reverse way following the line of reasoning of the SIQM which, as it is well known, is formally ruled by the same differential equations as the SM. If we start from the (one dimensional) Schrödinger equation (2.10) with the *Ansatz* (2.8), and if we separate the real and the imaginary parts as usual in SIQM [8], the function $f = R^2 = |\psi|^2$ comes out to be a particular solution of a FP equation of the form (2.3) with constant diffusion coefficient (2.11) and forward velocity field

$$v(x,t) = \frac{1}{m}\,\partial_x S + \frac{\hbar}{2m}\,\partial_x(\ln R^2)\,. \tag{2.12}$$

On the other hand the explicit dependence of v on the form of R clearly indicates that to have a solution of (2.3) which makes quantum sense we must pick-up just one, suitable, particular solution. In fact the system is ruled not only by the FP equation (2.3), but also by the second, dynamical equation (2.9), the so-called Hamilton-Jacobi-Madelung (HJM) equation, deduced by separating the real and imaginary parts of (2.10) (see [8]). The analogy between (2.3) and a FP equation, which looks rather accidental in a purely SIQM context, is more than formal since, as we have briefly recalled, the SM shows how to recover both the equations (2.3) and (2.9) (and hence the Schrödinger equation (2.10)) in a purely classical, dynamical stochastic context.

3. The eigenvalue problem for the FP equation

Let us recall here (see for example [11]) a few generalities about the pdf's (probability density functions) $f(x,t)$ solutions of a one-dimensional FP equation of the form

$$\partial_t f = \partial_x^2(Df) - \partial_x(vf) = \partial_x\big[\partial_x(Df) - vf\big] \tag{3.1}$$

defined for $x \in [a,b]$ and $t \geq t_0$, when $D(x)$ and $v(x)$ are two time independent functions such that $D(x) > 0$, $v(x)$ has no singularities in (a,b), and both are continuous and differentiable functions. The conditions imposed on the probabilistic solutions are of course

$$\begin{aligned} f(x,t) &\geq 0\,, \qquad\qquad a < x < b\,,\ t_0 \leq t\,, \\ \int_a^b f(x,t)\,dx &= 1\,, \qquad\qquad t_0 \leq t\,, \end{aligned} \tag{3.2}$$

and from the form of (3.1) the second condition also takes the form

$$\big[\partial_x(Df) - vf\big]_{a,b} = 0\,, \qquad t_0 \leq t\,. \tag{3.3}$$

Suitable initial conditions will be added to produce the required evolution: for example the transition pdf $p(x,t|x_0,t_0)$ will be selected by the initial condition

$$\lim_{t\to t_0^+} f(x,t) = f(x,t_0^+) = \delta(x - x_0)\,. \tag{3.4}$$

It is also possible to show by direct calculation that

$$h(x) = N^{-1}\,\mathrm{e}^{-\int [D'(x)-v(x)]/D(x)\,dx}\,, \qquad N = \int_a^b \mathrm{e}^{-\int [D'(x)-v(x)]/D(x)\,dx}\,dx \tag{3.5}$$

is an invariant (time independent) solution of (3.1) satisfying the conditions (3.2). Remark however that (3.1) is not in the standard self-adjoint form [12]; but if we define the function $g(x,t)$ by means of

$$f(x,t) = \sqrt{h(x)}\,g(x,t) \tag{3.6}$$

it would be easy to show that $g(x,t)$ obeys now an equation of the form

$$\partial_t g = \mathcal{L} g \tag{3.7}$$

where the operator $\mathcal{L}$ defined by

$$\mathcal{L}\varphi = \frac{d}{dx}\left[p(x)\,\frac{d\varphi(x)}{dx}\right] - q(x)\varphi(x)\,, \tag{3.8}$$

with

$$\begin{aligned} p(x) &= D(x) > 0\,, \\ q(x) &= \frac{\big[D'(x)-v(x)\big]^2}{4D(x)} - \frac{\big[D'(x)-v(x)\big]'}{2}\,, \end{aligned} \tag{3.9}$$

is now self-adjoint. Then, by separating the variables by means of $g(x,t) = \gamma(t)G(x)$ we have $\gamma(t) = \mathrm{e}^{-\lambda t}$ while G must be solution of a typical Sturm-Liouville problem associated to the equation

$$\mathcal{L}G(x) + \lambda G(x) = 0 \tag{3.10}$$

with the boundary conditions

$$\begin{aligned} \big[D'(a)-v(a)\big]G(a) + 2D(a)G'(a) &= 0\,, \\ \big[D'(b)-v(b)\big]G(b) + 2D(b)G'(b) &= 0\,. \end{aligned} \tag{3.11}$$

It easy to see that $\lambda = 0$ is always an eigenvalue for the problem (3.10) with (3.11), and that the corresponding eigenfunction is $\sqrt{h(x)}$ as defined from (3.5).

For the differential problem (3.10) with (3.11) we have that [12] the simple eigenvalues λ_n will constitute an infinite, increasing sequence and the corresponding eigenfunction $G_n(x)$ will have n simple zeros in (a,b). For us this means that $\lambda_0 = 0$, corresponding to the eigenfunction $G_0(x) = \sqrt{h(x)}$ which never vanishes in (a,b), is the lowest eigenvalue and that all other eigenvalues are strictly positive. Moreover the eigenfunctions will constitute a complete orthonormal set of functions in $L^2\big([a,b]\big)$ [13]. As a consequence the general solution of (3.1) with (3.2) will have the form

$$f(x,t) = \sum_{n=0}^{\infty} c_n \mathrm{e}^{-\lambda_n t}\sqrt{h(x)}G_n(x) \tag{3.12}$$

with $c_0 = 1$ for normalization (remember that $\lambda_0 = 0$). The coefficients c_n for a particular solution are selected by an initial condition

$$f(x,t_0^+) = f_0(x) \tag{3.13}$$

and are then calculated from the orthonormality relations as

$$c_n = \int_a^b f_0(x)\,\frac{G_n(x)}{\sqrt{h(x)}}\,dx\,. \tag{3.14}$$

In particular for the transition pdf we have from (3.4) that

$$c_n = \frac{G_n(x_0)}{\sqrt{h(x_0)}}\,. \tag{3.15}$$

Since $\lambda_0 = 0$ and $\lambda_n > 0$ for $n \geq 1$, the general solution (3.12) of (3.1) has a precise time evolution: all the exponential factors in (3.12) vanish with $t \to +\infty$ with the only exception of the term $n = 0$ which is constant, so that exponentially fast we will always have

$$\lim_{t\to+\infty} f(x,t) = c_0\sqrt{h(x)}G_0(x) = h(x)\,, \tag{3.16}$$

namely the general solution will always relax in time toward the invariant solution $h(x)$.

4. Stationary quantum states

Let us consider now a Schrödinger equation (2.10) with a time-independent potential $V(x)$ which gives rise to a purely discrete spectrum and bound, normalizable states, and let us use the following notations for stationary states, eigenvalues and eigenfunctions:

$$\begin{aligned} \psi_n(x,t) &= \phi_n(x)\,\mathrm{e}^{-iE_nt/\hbar} \\ \hat{H}\phi_n &= -\frac{\hbar^2}{2m}\,\phi_n'' + V\phi_n = E_n\phi_n\,. \end{aligned} \tag{4.1}$$

Taking into account the relation (2.11) the previous eigenvalue equation can also be recast in the following form

$$D\phi_n'' = \frac{V - E_n}{\hbar}\,\phi_n\,. \tag{4.2}$$

For these stationary states the pdf is the time independent, real function

$$f_n(x) = |\psi_n(x,t)|^2 = \phi_n^2(x)\,, \tag{4.3}$$

and

$$S(x,t) = -E_nt\,, \qquad R(x,t) = \phi_n(x)\,, \tag{4.4}$$

so that for our state the velocity field is

$$v_n(x) = 2D\,\frac{\phi_n'(x)}{\phi_n(x)}\,. \tag{4.5}$$

This means that now v_n is time-independent and it presents singularities in the zeros (nodes) of the eigenfunction. Since the n-th eigenfunction of a quantum system with bound states has exactly n simple nodes [12] that we will indicate with $x_1, \ldots, x_n$, the coefficients of the FP equation (2.3) are

not defined in these n points and we will be obliged to solve it in separate intervals by imposing the right boundary conditions connecting the different sections. In fact these singularities effectively separate the real axis in $n+1$ sub-intervals with walls impenetrable to the probability current. Hence the process will not have an unique invariant measure and will never cross the boundaries fixed by the singularities of $v(x)$: if we start in one of the intervals in which the axis is so divided we will always remain there [14].

As a consequence we must think the normalization integral (3.2) (with $a = -\infty$ and $b = +\infty$) as the sum of $n+1$ integrals over the sub-intervals $[x_k, x_{k+1}]$ with $k = 0, 1, \ldots, n$ (where we understand, to unificate the notation, that $x_0 = -\infty$ and $x_{n+1} = +\infty$). Hence for $n \geq 1$ we will be obliged to solve the equation (2.3) in every interval $[x_k, x_{k+1}]$ by requiring that the integrals

$$\int_{x_k}^{x_{k+1}} f(x,t)\, dx \tag{4.6}$$

be kept at a constant value for $t \geq t_0$: this value is not, in general, equal to one (only the sum of these $n+1$ integrals amounts to one) and, since the separate intervals can not communicate, it will be fixed by the choice of the initial conditions. Hence the boundary conditions associated to (2.3) require the conservation of the probability in $[x_k, x_{k+1}]$, namely the vanishing of the probability current at the end points of the interval:

$$\left[D\partial_x f - vf\right]_{x_k, x_{k+1}} = 0\,, \qquad t \geq t_0\,. \tag{4.7}$$

To have a particular solution we must moreover specify the initial conditions: in particular we will be interested in the transition pdf $p(x,t|x_0,t_0)$, which is singled out by the initial condition (3.4), since [1] the asymptotic approximation in L^1 among solutions of (2.3) is ruled by the asymptotic behavior of $p(x,t|x_0,t_0)$ through the Chapman-Kolmogorov equation

$$f(x,t) = \int_{-\infty}^{+\infty} p(x,t|y,t_0) f(y,t_0^+)\, dy\,. \tag{4.8}$$

It is clear at this point that in every interval $[x_k, x_{k+1}]$ (both finite or infinite) we can solve the equation (2.3) along the guidelines sketched in the section 3 by keeping in mind that in $[x_k, x_{k+1}]$ we already know the invariant, time-independent solution $\phi_n^2(x)$ (or, more precisely, its restriction to the said interval) which is never zero in this interval with the exception of the extremes x_k and x_{k+1}. Hence, as we have seen in the general case, with the position

$$f(x,t) = \phi_n(x) g(x,t) \tag{4.9}$$

we can reduce (2.3) to the form

$$\partial_t g = \mathcal{L}_n g \tag{4.10}$$

where $\mathcal{L}_n$ is now the self-adjoint operator defined on $[x_k, x_{k+1}]$ by

$$\mathcal{L}_n \varphi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}\varphi(x)}{\mathrm{d}x}\right] - q_n(x)\varphi(x) \tag{4.11}$$

where we have now

$$p(x) = D > 0\,; \qquad q_n(x) = \frac{v_n^2(x)}{4D} + \frac{v_n'(x)}{2}\,. \tag{4.12}$$

To solve (4.10) it is in general advisable to separate the variables, so that we immediately have $\gamma(t) = \mathrm{e}^{-\lambda t}$ while G must be solution of the Sturm-Liouville problem associated to the equation

$$\mathcal{L}_n G(x) + \lambda G(x) = 0 \tag{4.13}$$

with the boundary conditions

$$\left[2DG'(x) - v_n(x)G(x)\right]_{x_k, x_{k+1}} = 0\,. \tag{4.14}$$

The general behaviour of the solutions obtained as expansions in the system of the eigenfunctions of (4.13) has already been discussed in section 3. In particular we deduce from (3.12) that for the stationary quantum states (more precisely, in every subinterval defined by two subsequent nodes) all the solutions of (2.3) always converge in time toward the right quantum solution $|\phi_n|^2$: a general result not contained in the previous papers [1]. As a further consequence a quantum solution ϕ_n^2 defined on the entire interval $(-\infty, +\infty)$ will be stable under deviations from its initial condition.

5. Harmonic oscillator

To see in an explicit way how the pdf's of SM evolve, let us consider now in detail the particular example of a quantum harmonic oscillator (HO) characterized by the potential

$$V(x) = \frac{m}{2}\,\omega^2 x^2\,. \tag{5.1}$$

It is well-known that its eigenvalues are

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right)\,; \qquad n = 0, 1, 2\ldots \tag{5.2}$$

while, with the notation

$$\sigma_0^2 = \frac{\hbar}{2m\omega}\,, \tag{5.3}$$

the eigenfuncions are

$$\phi_n(x) = \frac{1}{\sqrt{\sigma_0\sqrt{2\pi}2^n n!}}\,\mathrm{e}^{-x^2/4\sigma_0^2}\,H_n\left(\frac{x}{\sigma_0\sqrt{2}}\right) \tag{5.4}$$

where H_n are the Hermite polynomials. The corresponding velocity fields are easily calculated and are for example

$$\begin{aligned} v_0(x) &= -\omega x\,, \\ v_1(x) &= 2\,\frac{\omega\sigma_0^2}{x} - \omega x\,, \\ v_2(x) &= 4\omega\sigma_0^2\,\frac{x}{x^2 - \sigma_0^2} - \omega x\,, \end{aligned} \tag{5.6}$$

with singularities in the zeros x_k of the Hermite polynomials. If we now keep the form of the velocity fields fixed we can consider (2.3) as an ordinary FP equation for a diffusion process and solve it to see the approach to the equilibrium of the general solutions. When $n = 0$ the equation (2.3) takes the form

$$\partial_t f = \omega\sigma_0^2\partial_x^2 f + \omega x\partial_x f + \omega f \tag{5.7}$$

and the fundamental solution comes out to be the Ornstein-Uhlenbeck transition pdf

$$p(x,t|x_0,t_0) = \frac{1}{\sigma(t)\sqrt{2\pi}}\,\mathrm{e}^{-[x-\alpha(t)]^2/2\sigma^2(t)}\,, \qquad (t \geq t_0) \tag{5.8}$$

where we used the notation

$$\alpha(t) = x_0\mathrm{e}^{-\omega(t-t_0)}\,, \qquad \sigma^2(t) = \sigma_0^2\big[1-\mathrm{e}^{-2\omega(t-t_0)}\big]\,, \qquad (t \geq t_0)\,. \tag{5.9}$$

The stationary Markov process associated to the transition pdf (5.8) is selected by the initial, invariant pdf

$$f(x) = \frac{1}{\sigma_0\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2\sigma_0^2} \tag{5.10}$$

which is also the asymptotic pdf for every other initial condition when the evolution is ruled by (5.7) (see [1]) so that the invariant distribution plays also the role of the limit distribution. Since this invariant pdf also coincides with the quantum stationary pdf $\phi_0^2 = |\psi_0|^2$ the process associated by the SM to the ground state of a quantum HO is nothing but the stationary Ornstein-Uhlenbeck process.

For $n \geq 1$ the solutions of (2.3) are no more so easy to find and, as discussed in the previus section, we will have to solve the eigenvalue problem (4.13) which, with $\epsilon = \hbar\lambda$, can be written as

$$-\frac{\hbar^2}{2m}\,G''(x) + \left(\frac{m}{2}\omega^2x^2 - \hbar\omega\,\frac{2n+1}{2}\right)G(x) = \epsilon G(x)\,, \tag{5.11}$$

in every interval $[x_k, x_{k+1}]$, with $k = 0, 1, \ldots, n$, between two subsequent singularities of the v_n field. The boundary conditions at the endpoints of these intervals, deduced from (4.7) through (4.9), are

$$[\phi_n G' - \phi_n' G]_{x_k, x_{k+1}} = 0 \tag{5.12}$$

and since ϕ_n (but not ϕ_n') vanishes in x_k, x_{k+1}, the conditions to impose are

$$G(x_k) = G(x_{k+1}) = 0 \tag{5.13}$$

where it is understood that for x_0 and x_{n+1} we respectively mean

$$\lim_{x\to-\infty} G(x) = 0\,, \qquad \lim_{x\to+\infty} G(x) = 0\,. \tag{5.14}$$

It is also useful at this point to give the eigenvalue problem in an adimensional form by using the new adimensional variable x/σ_0 (which will still be called x) and the eigenvalue $\mu = \lambda/\omega = \epsilon/\hbar\omega$. In this way the equation (5.11) with the conditions (5.13) becomes

$$\begin{gathered} y''(x) - \left(\frac{x^2}{4} - \frac{2n+1}{2} - \mu\right)y(x) = 0 \\ y(x_k) = y(x_{k+1}) = 0 \end{gathered} \tag{5.15}$$

where x, x_k, x_{k+1} are now adimensional variables. If μ_m and $y_m(x)$ are the eigenvalues and eigenfunctions of (5.15), the general solution of the corresponding FP equation (2.3) will be

$$f(x,t) = \sum_{m=0}^{\infty} c_m \mathrm{e}^{-\mu_m \omega t} \phi_n(x) y_m\left(\frac{x}{\sigma_0}\right) . \tag{5.16}$$

Of course the values of the coefficients c_m will be fixed by the initial conditions and by the obvious requirements that $f(x,t)$ must be non negative and normalized (on the whole x axis) along all its evolution. Two linearly independent solutions of (5.15) are

$$y^{(1)} = \mathrm{e}^{-x^2/4} M\left(-\frac{\mu+n}{2}, \frac{1}{2}; \frac{x^2}{2}\right) , \qquad y^{(2)} = x\mathrm{e}^{-x^2/4} M\left(-\frac{\mu+n-1}{2}, \frac{3}{2}; \frac{x^2}{2}\right) , \tag{5.17}$$

where $M(a,b;z)$ are the confluent hypergeometric functions.

We consider first the case $n = 1$ ($x_0 = -\infty$, $x_1 = 0$ and $x_2 = +\infty$) so that (5.15) will have to be solved separately for $x \le 0$ and for $x \ge 0$ with the boundary conditions $y(0) = 0$ and

$$\lim_{x\to-\infty} y(x) = \lim_{x\to+\infty} y(x) = 0 . \tag{5.18}$$

A long calculation [5] shows that the transition pdf is now

$$p(x,t|x_0,t_0) = \frac{x}{\alpha(t)} \frac{\mathrm{e}^{-[x-\alpha(t)]^2/2\sigma^2(t)} - \mathrm{e}^{-[x+\alpha(t)]^2/2\sigma^2(t)}}{\sigma(t)\sqrt{2\pi}} \tag{5.19}$$

where $\alpha(t)$ and $\sigma^2(t)$ are defined in (5.9). It must be remarked however that (5.19) must be considered as restricted to $x \ge 0$ when $x_0 > 0$ and to $x \le 0$ when $x_0 < 0$, and that only on these intervals it is suitably normalized. In order to take into account at once both these possibilities we can also introduce the Heavyside function $\Theta(x)$ so that for every $x_0 \neq 0$ we will have

$$p(x,t|x_0,t_0) = \Theta(xx_0) \frac{x}{\alpha(t)} \frac{\mathrm{e}^{-[x-\alpha(t)]^2/2\sigma^2(t)} - \mathrm{e}^{-[x+\alpha(t)]^2/2\sigma^2(t)}}{\sigma(t)\sqrt{2\pi}} . \tag{5.20}$$

This completely solves the problem for $n = 1$ since from (4.8) we can now deduce also the evolution of every other initial pdf. In particular it can be shown that

$$\lim_{t\to+\infty} p(x,t|x_0,t_0) = 2\Theta(xx_0) \frac{x^2 \mathrm{e}^{-x^2/2\sigma_0^2}}{\sigma_0^3\sqrt{2\pi}} = 2\Theta(xx_0)\phi_1^2(x) , \tag{5.21}$$

and hence, if $f(x,t_0^+) = f_0(x)$ is the initial pdf, we have for $t > t_0$

$$\begin{aligned} \lim_{t\to+\infty} f(x,t) &= \lim_{t\to+\infty} \int_{-\infty}^{+\infty} p(x,t|y,t_0) f_0(y)\, dy \\ &= 2\phi_1^2(x) \int_{-\infty}^{+\infty} \Theta(xy) f_0(y)\, dy = \Gamma(q;x)\phi_1^2(x) , \end{aligned} \tag{5.22}$$

where we have defined the function

$$\Gamma(q;x) = q\Theta(x) + (2-q)\Theta(-x) ; \qquad q = 2\int_0^{+\infty} f_0(y)\, dy . \tag{5.23}$$

Remark that when $q = 1$ (namely when the initial probability is equally shared on the two real semi-axis) we have $\Gamma(1; x) = 1$ and the asymptotical pdf coincides with the quantum stationary pdf $\phi_1^2(x)$; if on the other hand $q \neq 1$ the asymptotical pdf has the same shape of $\phi_1^2(x)$ but with different weights on the two semi-axis.

If then $n = 2$ we have $x_0 = -\infty$, $x_1 = -1$, $x_2 = 1$ and $x_3 = +\infty$, and the equation (5.15) must be solved in the three intervals $(-\infty, -1]$, $[-1, 1]$ and $[1, +\infty)$, but the eigenvalues and eigenfunctions are now not easy to find so that a complete analysis of this case (and of every other case with $n > 2$) has still to be elaborated. At present only a few indications can be obtained numerically [5]: for example it can be shown that, beyond $\mu_0 = 0$, the first eigenvalues in the interval $[-1, 1]$ can be calculated as the first values such that

$$M\left(-\frac{\mu+1}{2}, \frac{3}{2}; \frac{1}{2}\right) = 0 \tag{5.24}$$

and are $\mu_1 \sim 7.44$, $\mu_2 \sim 37.06$, $\mu_3 \sim 86.41$. Also for the unbounded interval $[1, +\infty)$ (the analysis is similar for $(-\infty, -1]$) the eigenvalues are derivable only numerically.

6. Controlled evolutions

It is important to remark now that solutions of the type (5.8) and (5.20), and any other solution different from $|\phi_n|^2$, are not associated to quantum mechanical states solutions of (2.10); in other words, they define processes that satisfy neither the stochastic variational principle [3] nor the Nelson dynamical equation [2]. That notwithstanding these processes still keep an interesting relation with the quantum mechanics. In fact to every solution $f(x, t)$ of a FP equation (3.1), with a given $v(x, t)$ and the constant diffusion coefficient (2.11), we can always associate the wave function of a quantum system if we take a suitable time-dependent potential. This means in practice that even the *virtual* (non optimal) processes discussed in this paper can be associated to proper quantum states, namely can be made optimal provided that the potential $V(x)$ of (2.10) be modified in a new $V(x, t)$ in order to control the evolution.

Let us take a solution $f(x, t)$ of the FP equation (3.1), with a given $v(x, t)$ and a constant diffusion coefficient (3.3): if we define the functions $R(x, t)$ and $W(x, t)$ from

$$f(x, t) = R^2(x, t)\,, \qquad v(x, t) = \partial_x W(x, t)\,, \tag{6.1}$$

if we remember from (2.12) that the following relation must hold

$$mv = \partial_x S + \hbar\,\frac{\partial_x R}{R} = \partial_x S + \frac{\hbar}{2}\,\frac{\partial_x f}{f} = \partial_x\left(S + \frac{\hbar}{2}\ln\tilde{f}\right) \tag{6.2}$$

where $\tilde{f}$ is an adimensional pdf (it is the argument of a logarithm) obtained by means of a suitable and arbitrary multiplicative constant, and if $S(x, t)$ is supposed to be the phase of a wave function as in (2.8), we immediately get the equation

$$S(x, t) = mW(x, t) - \frac{\hbar}{2}\ln\tilde{f}(x, t) - \theta(t) \tag{6.3}$$

which allows us to determine S from f and v (namely W) up to an additive arbitrary function of the time $\theta(t)$. However, in order that the wave function (2.8) with the said R and S be a solution of a Schrödinger equation, we must also be sure that the HJM equation (2.9) is satisfied. Since S and R are now fixed, the equation (2.9) must be considered as a relation defining a potential which, after a short calculation, becomes

$$V(x,t) = \frac{\hbar^2}{4m}\,\partial_x^2 \ln \tilde{f} + \frac{\hbar}{2}\left(\partial_t \ln \tilde{f} + v\partial_x \ln \tilde{f}\right) - \frac{mv^2}{2} - m\partial_t W + \dot{\theta}\,. \tag{6.4}$$

Of course if we start with a quantum wave function for a given potential and if we pick up as a solution of (2.3) exactly $f = R^2$, the formula (6.4) will correctly give back the initial potential, as can be seen for both the ground state and the first excited state of the HO which (by choosing respectively $\theta(t) = \hbar\omega t/2$ and $\theta(t) = 3\hbar\omega t/2$, which amounts to suitably fix the zero of the potential energy) give as result the usual harmonic potential (5.1).

If on the other hand we consider for example the (non stationary) fundamental solution (5.8) associated to the velocity field $v_0(x)$ of (5.6) for the case $n = 0$ of the HO (we put $t_0 = 0$ to simplify the notation) we have already remarked that it does not correspond to a quantum wave function whatsoever. However a short calculation shows that, by choosing

$$\dot{\theta}(t) = \frac{\hbar\omega}{2}\left(\frac{2\sigma_0^2}{\sigma^2(t)} - 1\right) = \frac{\hbar\omega}{2}\,\frac{1}{\tanh \omega t} \to \frac{\hbar\omega}{2}\,, \qquad (t \to +\infty)\,, \tag{6.5}$$

and the time-dependent controlling potential

$$V(x,t) = \frac{\hbar\omega}{2}\left[\frac{x-\alpha(t)}{\sigma(t)}\right]^2 \frac{\sigma_0^2}{\sigma^2(t)} - \frac{m\omega^2 x^2}{2} \to \frac{m\omega^2 x^2}{2}\,, \qquad (t \to +\infty) \tag{6.6}$$

we can define a quantum state (a wave function solution of a Schrödinger equation) which realizes the required evolution (5.8). Of course the fact that for $t \to +\infty$ we recover the harmonic potential is associated to the fact, aleady remarked, that the usual quantum pdf $\phi_0^2(x)$ is also the limit distribution for every initial condition and in particular also for the pdf (5.8). In the case $n = 1$, with $v_1(x)$ from (5.6) and the transition probability (5.20) as given non-stationary solution, the calculations are lenghtier. However if we define

$$F(x,t) = \frac{\mathrm{e}^{-[x-\alpha(t)]^2/2\sigma^2(t)}}{\sigma(t)\sqrt{2\pi}}\,, \qquad G(x,t) = \frac{\mathrm{e}^{-[x+\alpha(t)]^2/2\sigma^2(t)}}{\sigma(t)\sqrt{2\pi}}\,, \tag{6.7}$$

$$T\left[\frac{x\alpha(t)}{\sigma^2(t)}\right] = \frac{x\alpha(t)}{\sigma^2(t)}\,\frac{F(x,t)+G(x,t)}{F(x,t)-G(x,t)}\,, \qquad T(x) = \frac{x}{\tanh x}\,, \tag{6.8}$$

and if we choose

$$\dot{\theta}(t) = \frac{\hbar\omega}{2}\left(\frac{4\sigma_0^2}{\sigma^2(t)} - \frac{2\sigma_0^2\alpha^2(t)}{\sigma^4(t)} - 1\right) \to \frac{3}{2}\,\hbar\omega\,, \qquad (t \to +\infty) \tag{6.9}$$

we have as time dependent potential for every $x \neq 0$

$$\begin{aligned} V(x,t) &= \frac{m\omega^2 x^2}{2}\left(\frac{2\sigma_0^4}{\sigma^4} - 1\right) + \hbar\omega\left[1 - \frac{\sigma_0^2}{\sigma^2}\,T\left(\frac{x\alpha}{\sigma^2}\right)\right] - \frac{\hbar^2}{4mx^2}\,\left[1 - T\left(\frac{x\alpha}{\sigma^2}\right)\right] \\ &\to \frac{m\omega^2 x^2}{2}\,, \qquad (t \to +\infty)\,. \end{aligned} \tag{6.10}$$

In this case the asymptotic potential is the usual harmonic potential, but we must consider it separately on the positive and negative x semi-axis since in the point $x = 0$ a singular behaviour would show up when $t \to 0$. This means that, also if asymptotically we recover the right potential, this will be associated with new boundary conditions in $x = 0$ since we will be obliged to keep the system bounded on the positive (for example) semi-axis.

7. Modelling transitions

The explicit knowledge of the transition pdf of the type (5.8) and (5.20), And the possibility of turning optimal any suitable (f, v) state by a right choice of $V(x, t)$ enable us also to explore the possibility of modelling evolutions leading, for example, from the pdf of a given stationary state to another (decays and excitations). In fact a spontaneuous generalization of this idea hints to the possibility of modelling evolutions ffrom a given, arbitrary pdf and the pdf of an eigenfuncition of some observable: something which could become an element for very simple models of quantum measurements where we try to dynamically describe the wave packet collapse. As a first example let us consider the transition between the invariant pdf's

$$\begin{aligned} f_0(x) = \phi_0^2(x) &= \frac{1}{\sigma_0\sqrt{2\pi}}\, \mathrm{e}^{-x^2/2\sigma_0^2}\,, \\ f_1(x) = \phi_1^2(x) &= \frac{x^2}{\sigma_0^3\sqrt{2\pi}}\, \mathrm{e}^{-x^2/2\sigma_0^2}\,. \end{aligned} \tag{7.1}$$

If for instance we choose to describe the decay $1 \to 0$ we should just use the Chapman-Kolmogorov equation (4.8) with (5.8) as transition pdf and $f_1(x)$ as initial pdf ($t_0 = 0$). An elementary integration will show in this case that the resulting evolution takes the form

$$f_{1\to 0}(x, t) = \beta^2(t) f_0(x) + \gamma^2(t) f_1(x) \tag{7.2}$$

where we used the notation

$$\beta^2(t) = 1 - \mathrm{e}^{-2\omega t}\,, \qquad \gamma(t) = \mathrm{e}^{-\omega t}\,. \tag{7.3}$$

Taking now $v_0(x)$ from (5.6) and the evolving pdf from (7.2) and putting them in (6.4) (remark that, since v_0 is stationary, $\partial_t W = 0$) we get the following form of the controlling potential:

$$V(x, t) = \frac{m\omega^2 x^2}{2} - 2\hbar\omega U(x/\sigma_0; \beta/\gamma) \tag{7.4}$$

where

$$U(x; b) = \frac{x^4 + b^2x^2 - b^2}{(b^2 + x^2)^2}\,. \tag{7.5}$$

In our example the parameter

$$b^2(t) = \frac{\beta^2(t)}{\gamma^2(t)} = \mathrm{e}^{2\omega t} - 1 \tag{7.6}$$

is such that $b^2(0^+) = 0$ and $b^2(+\infty) = +\infty$ and hence U goes everywhere to zero for $t \to +\infty$, but is everywhere 1 with a negative singularity in $x = 0$ for $t \to 0^+$. As a consequence, while for $t \to +\infty$ the controlling potential (7.4) behaves like the HO potential (5.1), for $t \to 0^+$ it presents an unessential shift of $-2\hbar\omega$ in the zero level, but shows also a deep negative singularity in $x = 0$.

Apart from this singular behaviour of the controlling potential, a problem arises from the form of the phase funcion S. In fact from (6.3) we easily have for our decay

$$S(x,t) = -\frac{\hbar}{2} \ln \left[\beta^2(x,t) + \frac{x^2}{\sigma_0^2} \gamma^2(x,t) \right] - \frac{\hbar\omega}{2}\, t \tag{7.7}$$

so that in particular we have

$$S(x,0^+) = -\frac{\hbar}{2}\, \ln \frac{x^2}{\sigma_0^2}\,, \tag{7.8}$$

while we would have expected that initially our phase function be independent from x as for every stationary wave function: this means that in our supposed evolution the phase function presents a discontinuous behaviour for $t \to 0^+$. The problem arises here from the fact that in our simple model we initially have a stationary state characterized by a ddp $f_1(x)$ and a velocity field $v_1(x)$, and then suddenly, in order to start the decay, we suppose the same f_1 embedded in a different velocity field $v_0(x)$ which drags it toward a new stationary $f_0(x)$. This discontinuous change from v_1 to v_0 is of course responsible for the remarked discontinuous change in the phase of the wave function. Hence a more realistic model for a controlled transition must take into account a continuous and smooth (albeit widely arbitrary) modification of the initial velocity field into the final one, a requirement which compels us to consider a new class of FP equations with time-dependent velocity field $v(x,t)$. In particular to achieve the proposed controlled decay between two stationary states we should solve an evolution equation with a velocity field $v(x,t)$ continuously, and possibly smoothingly, going from $v_1(x)$ to $v_0(x)$; but this seems at present beyond the reach of our possibilities since every reasonable such $v(x,t)$ field has proven intractable from the point of view of the solution of the FP equation (2.3). However we can show the results for another meaningful example which does not present the same technical difficulties of the decay between two stationary states: namely the controlled evolution from a coherent oscillating packet in a HO, and the ground state of the same HO.

To do this we will recall a simple result [1] which indicates how to find the solutions of a particular class of evolution equations (2.3) which contains the situation of our proposed example. If the velocity field of the evolution equation (2.3) has the linear form

$$v(x,t) = A(t) + B(t)x \tag{7.9}$$

with $A(t)$ and $B(t)$ continuous functions of time, then there are always solutions of the form $\mathcal{N}\big(\mu(t), \nu(t)\big)$ where $\mu(t)$ and $\nu(t)$ are calculated from the differential equations

$$\mu'(t) - B(t)\mu(t) = A(t)\,; \qquad \nu'(t) - 2B(t)\nu(t) = 2D \tag{7.10}$$

with suitable initial conditions. On the other hand the (non stationary) wave function of the oscillating coherent wave packet with initial displacement a is

$$\psi_c(x,t) = \left(\frac{1}{2\pi\sigma_0^2}\right)^{1/4} \exp\left[-\frac{(x-a\cos\omega t)^2}{4\sigma_0^2} - i\left(\frac{4ax\sin\omega t - a^2\sin 2\omega t}{8\sigma_0^2} + \frac{\omega t}{2}\right)\right] \tag{7.11}$$

so that the corresponding forward velocity field will be

$$v_c(x,t) = a\omega(\cos\omega t - \sin\omega t) - \omega x\,, \tag{7.12}$$

namely it will have the required form (7.9) with $A(t) = a\omega(\cos\omega t - \sin\omega t)$ and $B(t) = -\omega$, while the position pdf will be

$$f_c(x,t) = |\psi_c(x,t)|^2 = f_0(x - a\cos\omega t)\,. \tag{7.13}$$

Now it is very easy to show that when $B(t) = -\omega$, as in the case of our wave packet, there are stable, coherent (non dispersive) solutions with $\nu(t) = \sigma_0^2$ of the form $\mathcal{N}\big(\mu(t), \sigma_0^2\big)$, namely of the form

$$f(x,t) = f_0\big(x - \mu(t)\big)\,. \tag{7.14}$$

Of course the time evolution of such coherent solutions can be determined in one step, without implementing the two steps procedure of first calculating the transition pdf and then, through the Chapman-Kolmogorov equation, the evolution of an arbitrary initial pdf. On the other hand if we compare (5.6) and (7.12) we see that the difference between v_0 and v_c consists in the first, time dependent term of the second one; hence it is natural to consider the problem of solving the evolution equation (2.3) with a velocity field of the type

$$\begin{aligned} v(x,t) &= A(t) - \omega x \\ A(t) &= a\omega(\cos\omega t - \sin\omega t)F(t) \end{aligned} \tag{7.15}$$

where $F(t)$ is an arbitrary function varying smoothly between 1 and 0, or vice verssa. In this case the evolution equation (2.3) still has stable, coherent (non dispersive) solutions of the form (7.14) with a $\mu(t)$ dependent on our choice of $F(t)$ through (7.10).

A completely smooth transition from the coherent, oscillating wave function (7.11) to the ground state ϕ_0 (5.4) of the HO can now be achieved for example by means of the following choice of the function $F(t)$:

$$F(t) = 1 - \big(1 - \mathrm{e}^{-\Omega t}\big)^N = \sum_{k=1}^{N} (-1)^{k+1} \binom{N}{k} \mathrm{e}^{-\omega_k t} \tag{7.16}$$

where

$$\Omega = \frac{\ln N}{\tau}\,, \qquad \omega_k = k\Omega\,; \qquad \tau > 0\,, \qquad N \geq 2\,. \tag{7.17}$$

In fact this $F(t)$ goes monotonically from $F(0) = 1$ to $F(+\infty) = 0$ with a flex point in τ (which can be considered as the arbitrary instant of the transition) where its derivative $F'(\tau)$ is negative and grows, in absolute value, logarithmically with N. The condition $N \geq 2$ also guarantees that

$F'(0) = 0$, and hence that the controlling potential $V(x,t)$ of (6.4) will continuously start at $t = 0$ from the HO potential (5.1), and eventually come back to it for $t \to +\infty$. Finally the phase function $S(x,t)$ too will change continuously from that of ψ_c to that of the HO ground state. A long but simple calculation will now show that the explicit form of the controlling potential is

$$V(x,t) = m\omega^2 \frac{x^2}{2} - m\omega a x \sum_{k=1}^{N} (-1)^{k+1} \binom{N}{k} \left[U_k(t) \omega_k \mathrm{e}^{-\omega_k t} - W_k \omega \mathrm{e}^{-\omega t} \right] \tag{7.18}$$

where

$$\begin{aligned} U_k(t) &= \sin \omega t + \frac{2\omega^2 \sin \omega t - \omega_k^2 \cos \omega t}{(\omega_k - \omega)^2 + \omega^2}\,, \\ W_k &= 1 + \frac{2\omega^2 - \omega_k^2}{(\omega_k - \omega)^2 + \omega^2} = \sqrt{2} U_k \left(\frac{\pi}{4\omega} \right)\,. \end{aligned} \tag{7.19}$$

The parameters τ and N, with the limitations (7.17), are free and connected to the particular form of the transition that we want to implement. We conclude this section by remarking that, in a HO, the transition between a coherent, oscillating wave packet and the ground state is a transition between a (Poisson) superposition of all the energy eigenstates to just one energy eigenstate: an outcome which is similar to that of an energy measurement, but for the important fact that here the result (the energy eigenstate) is deterministically controlled by a time dependent potential. In fact our controlled transition does not produce mixtures, but pure states (eigenstates) and in some way realizes a dynamical model for one of the branches of a measurement leading to an eigenvalue and an eigenstate.

8. Beam dynamics in particle accelerators

As a model which tries to put in evidence the classical aspects of the quantum physics, the SM seems especially suitable to the description of systems whose nature in some sense lies between classical and quantum: the so called mesoscopic or quantum-like systems [15]. We will propose now a few preliminary remarks about the possibility of making use of this characteristic in a particular physical domain [7]. The dynamical evolution of beams in particles accelerators is a typical example of mesoscopic behaviour. Since they are governed by external electromagnetic forces and by the interaction of the beam particles among themselves and with the environment, charged beams are higly nonlinear dynamical systems, and most of the studies on colliding beams rely either on classical phenomena such as nonlinear resonances, or on isolated sources of unstable behaviors as building blocks of more complicated chaotic instabilities. This line of inquiry has produced a general qualitative picture of dynamical processes in particle accelerators at the classical level. However, the coherent oscillations of the beam density and profile require, to be explained, some mechanism of local correlation and loss of statistical independence. This fundamental observation points towards the need to take into account all the interactions as a whole. Moreover, the overall interactions between charged particles and machine elements are really nonclassical in the sense that of the many sources of noise that are present, almost all are mediated by fundamental quantum

processes of emission and absorbtion of photons. Therefore the equations describing these processes must be, in principle, quantum.

Starting from the above considerations, two different approaches to the classical collective dynamics of charged beams have been developed, one relying on the FP equation [16] for the beam density, another based on a mathematical coarse graining of Vlasov equation leading to a quantum-like Schrödinger equation, with a thermal unit of emittance playing the role of Planck constant [17]. The study of statistical effects on the dynamics of electron (positron) colliding beams by the FP equation has led to several interesting results, and has become an established reference in treating the sources of noise and dissipation in particle accelerators by standard classical probabilistic techniques [18]. Concerning the relevance of the quantum-like approach, at this stage we only want to point out that some recent experiments on confined classical systems subject to particular phase-space boundary conditions seem to to be well explained by a quantum-like (Schrödinger equation) formalism [19]. In this approach [20] the (one dimensional) transverse density profile of the beam is described in terms of a complex function, called beam wave function, whose squared modulus give the transverse density profile of the beam. This beam wave function satisfies a Schrödinger-like equation where $\hbar$ is replaced by the transverse beam emittance ϵ:

$$i\epsilon\frac{\partial\psi(x,z)}{\partial z} = -\frac{\epsilon^2}{2}\frac{\partial^2\psi(x,z)}{\partial x^2} + U(x,z)\psi(x,z)\,. \tag{8.1}$$

On the other hand a recently proposed model for the description of collective beam dynamics in the semiclassical regime [21] relies on the idea of simulating semiclassical corrections to classical dynamics by suitable classical stochastic fluctuations with long range coherent correlations, whose scale is ruled by Planck constant. This elaborates a hypothesis first proposed by Calogero [22] in his attempt to prove that quantum mechanics might be interpreted as a tiny chaotic component of the individual particles' motion in a gravitationally interacting universe. The virtue of the proposed semiclassical model is twofold: on the one hand it can be formulated both in a probabilistic FP fashion and in a quantum-like (Schrödinger) setting, thus bridging the formal gap between the two approaches. On the other hand it goes further by describing collective effects beyond the classical regime due to the semiclassical quantum corrections.

Since we are interested in the description of the stability regime, when thermal dissipative effects are balanced on average by the RF energy pumping, and the overall dynamics is conservative and time-reversal invariant in the mean, the choice to model the random kinematics with the Nelson diffusions, that are nondissipative and time-reversal invariant, is particularly natural. The diffusion process describes the effective motion at the mesoscopic level (interplay of thermal equilibrium, classical mechanical stability, and fundamental quantum noise) and therefore the diffusion coefficient is set to be the semiclassical unit of emittance provided by qualitative dimensional analysis. In other words, we simulate the quantum corrections to classical deterministic motion (at leading order in Planck constant) with a suitably defined random kinematics replacing the classical deterministic trajectories. Therefore, apart from the different objects involved (beam spatial density versus Born probability density; Planck constant versus emittance), the dynamical

equations of our model formally reproduce the equations of the Madelung fluid (hydrodynamic) representation of quantum mechanics. In this sense, the present scheme allows for a quantum-like formulation equivalent to the probabilistic one.

With a few changes in the notation we can now reproduce, for the beam dynamics, the SM approach sketched in section 2. Let $q(t)$ be the process representing some collective degree of freedom of the beam with a pdf $\rho(x,t)$. Then, in suitable units, the basic stochastic kinematical relation is an Itô stochastic differential equation of the type (2.1) where the emittance ϵ of the beam plays the role of a diffusion coefficient. Since we are interested in the stability regime of the bunch oscillations, the bunch itself can be considered in a quasi-stationary state, during which the energy lost by dissipation is regained in the RF cavities. In such a quasi-stationary regime the dynamics is, on average, invariant for time-reversal and we can define a classical effective Lagrangian $L(q,\dot{q})$ of the system, where the classical deterministic kinematics is replaced by the random diffusive kinematics (2.1). The equations for the dynamics can then be obtained from the classical Lagrangian by means of the stochastic variational principles.

Introducing now the time-like coordinate $s = ct$ we get now the analog of the equations (2.3) and (2.6) in the form of a HJM equation

$$\partial_s S + \frac{v^2}{2} - 2\epsilon^2 \frac{\partial_x^2 \sqrt{\rho}}{\sqrt{\rho}} + V(x,s) = 0\,, \tag{8.2}$$

and of a continuity equation

$$\partial_s \rho = -\partial_x(\rho v)\,. \tag{8.3}$$

Remark that now the symbol v no more represents the forward velocity fields, but rather the drift velocity connected to the forward and backward velocities by the relation $2v = v_{(+)} + v_{(-)}$, and to the phase function by the relation $v = \partial_x S$. The observable structure is now quite clear: $\mathbf{E}v$ is the average velocity of the bunch center oscillating along the transverse direction; $\mathbf{E}q$ gives the average coordinate of the bunch center; finally the second moment $(\Delta q)^2 = E\big(q - E(q)\big)^2$ determines the dispersion (spreading) of the bunch. The coupled equations of dynamics may now be used to achieve a controlled coherence: given a desired state (ρ, v) the equations of motion (8.2) and (8.3) can be solved to calculate the external controlling potential $V(x,s)$ that realizes this state.

General techniques to obtain localized quantum wavepackets as dynamically controlled systems in SM have already been introduced [23]. In this way one can construct for general systems either coherent packets following the classical trajectories with constant dispersion, or coherent packets following the classical trajectories with time-dependent, but at any time bounded dispersion. These results can now be extended also to the quantum-like description of the transverse dynamics of a particle beam and hence it will be possible to select a current velocity, by fixing the characteristics of the motion of the packet center, to determine the corresponding solutions of the FP (continuity) equation and finally to use the HJM equation as a constraint giving us the controlling device. The formal details of this program will be developed in a subsequent paper.

9. Concluding remarks

It has been observed that the inverse problem of determining a controlling potential for a given quantum evolution in fact does not need to be formulated in terms of SM. Given two quantum wave function ψ_1 and ψ_2 we could indeed design a new wave function $\psi(x,t)$, evolving from ψ_1 to ψ_2 plug it, as required evolution, directly in the Schrödinger equation (2.10) and eventually deduce from that the form of the controlling potential. At first glance this seems to completely circumwent the need for a model like the SM: given an arbitrary evolving state we can always calculate the potential producing it. However about that two remarks are in order.

First of all, from a purely technical point of view, the simplification introduced by this procedure shows up to be elusive. In fact we must remember that a quantum wave function has complex values and hence, if we simply take an arbitrary evolution, the resulting potential calculated from the Schrödinger equation (2.10) will also be complex. This means that, to have a real valued potential, we must impose some conditions on the supposed evolution. These conditions of course depend on the hypothesized form of ψ. For example, if we fix the evolution of its modulus, the said condition will materialize in a partial differential equation on the phase function S of the wave function. On the other hand the use of the HJM equation (2.9) as the tool to solve the inverse problem always give a real valued potential as a result.

However both the two proposed procedures are possible and, to identically posed questions, they will give identical answers. Given this obvius equivalence, the second remark is that our choice of the procedure will be operated on the basis of opportunity considerations. In both cases the result will be influenced by the starting hypothesis on the supposed evolution of the state ψ modelling the transition from ψ_1 to ψ_2. But, since the observable part of the wave function is its square modulus, namely the position pdf, the relevant hypothesis will be on its evolution. The phase function, or, equivalently, the velocity fields, are not directly observable, and hence are at first sight of secondary concern. Their importance become apparent only when we require that the potential be real or that the transitions show a realistic, smooth behaviour. Hence, depending on the specific problem we are dealing with, it could be more suitable to approach it in terms of a state given through a wave function ψ, or in terms of a state given through the couple (f, v). The two approaches are certainly equivalent, but one may prove to be more suggestive. In particular that based on the SM equations seems to be better for the treatment of systems, like as the mesoscopic, quantum-like ones, which are well described by classical probabilistic models in terms of real space-time trajectories.

REFERENCES

1. N.Cufaro Petroni and F.Guerra: *Found.Phys.* **25** (1995) 297;
   N.Cufaro Petroni: Asymptotic behaviour of densities for Nelson processes, in *Quantum communications and measurement*, V.P.Belavkin et al. Eds., Plenum Press, New York, 1995, p. 43;

N.Cufaro Petroni, S.De Martino and S.De Siena: Non equilibrium densities of Nelson processes, in *New perspectives in the physics of mesoscopic systems*, S.De Martino et al. Eds., World Scientific, Singapore, 1997, p. 59.

2. E.Nelson: *Phys.Rev.* **150** (1966) 1079;
   E.Nelson: *Dynamical Theories of Brownian Motion* (Princeton U.P.; Princeton, 1967);
   E.Nelson: *Quantum Fluctuations* (Princeton U.P.; Princeton, 1985);
   F.Guerra: *Phys.Rep.* **77** (1981) 263.
3. F.Guerra and L.Morato: *Phys.Rev.* **D 27** (1983) 1774;
   F.Guerra and R.Marra: *Phys.Rev.* **D 28** (1983) 1916;
   F.Guerra and R.Marra: *Phys.Rev.* **D 29** (1984) 1647.
4. D.Bohm and J.P.Vigier: *Phys.Rev.* **96** (1954) 208.
5. N.Cufaro Petroni, S.De Martino and S.De Siena: Exact solutions of Fokker-Planck equations associated to quantum wave functions; in press on Phys.Lett. **A**.
6. N.Cufaro Petroni: *Phys.Lett.* **A141** (1989) 370;
   N.Cufaro Petroni: *Phys.Lett.* **A160** (1991) 107;
   N.Cufaro Petroni and J.P.Vigier: Found.Phys. **22** (1992) 1.
7. N.Cufaro Petroni, S.De Martino, S.De Siena and F.Illuminati: A stochastic model for the semiclassical collective dynamics of charged beams in partcle accelerators; contribution to the 15th ICFA Advanced Beam Dynamics Workshop, Monterey (California, US), Jan 98.
   N.Cufaro Petroni, S.De Martino, S.De Siena, R. Fedele, F.Illuminati and S. Tzenov: Sochastic control of beam dynamics; contribution to the EPAC'98 Conference, Stockholm (Sweden), Jun 98.
8. E.Madelung: *Z.Physik* **40** (1926) 332;
   L.de Broglie: *C.R.Acad.Sci.Paris* **183** (1926) 447;
   L.de Broglie: *C.R.Acad.Sci.Paris* **184** (1927) 273;
   L.de Broglie: *C.R.Acad.Sci.Paris* **185** (1927) 380;
   D.Bohm: *Phys.Rev.* **85** (1952) 166, 180.
9. L.de la Peña and A.M.Cetto: *Found.Phys.* **5** (1975) 355;
   N.Cufaro Petroni and J.P.Vigier: *Phys.Lett.* **A73** (1979) 289;
   N.Cufaro Petroni and J.P.Vigier: *Phys.Lett.* **A81** (1981) 12;
   N.Cufaro Petroni and J.P.Vigier: *Phys.Lett.* **A101** (1984) 4;
   N.Cufaro Petroni, C.Dewdney, P.Holland, T.Kyprianidis and J.P.Vigier: *Phys.Lett.* **A106** (1984) 368;
   N.Cufaro Petroni, C.Dewdney, P.Holland, T.Kyprianidis and J.P.Vigier: *Phys.Rev.* **D32** (1985) 1375.
10. F.Guerra: The problem of the physical interpretation of Nelson stochastic mechanics as a model for quantum mechanics, in *New perspectives in the physics of mesoscopic systems*, S.De Martino et al. Eds., World Scientific, Singapore, 1997, p. 133.
11. H.Risken: *The Fokker-Planck equation* (Springer, Berlin, 1989).
12. F.Tricomi: *Equazioni differenziali* (Einaudi, Torino, 1948).
13. F.Tricomi: *Integral equations* (Dover, New York, 1985).
14. S.Albeverio and R.Høgh-Krohn: *J.Math.Phys.* **15** (1974) 1745.
15. *New perspectives in the physics of mesoscopic systems*, S.De Martino et al. Eds., World Scientific, Singapore, 1997.
16. F. Ruggiero, *Ann.Phys.* (N.Y.) **153**, (1984) 122;
    J. F. Schonfeld, *Ann.Phys.* (N.Y.) **160**, (1985) 149.
17. R. Fedele, G. Miele and L. Palumbo, *Phys.Lett.* **A194**, (1994) 113.
18. S. Chattopadhyay, AIP Conf. Proc. **127**, 444 (1983);
    F. Ruggiero, E. Picasso and L. A. Radicati, *Ann.Phys.* (N. Y.) **197**, (1990) 396.
19. R. K. Varma, in *Quantum-like Models and Coherence Effects*, R. Fedele et al. Eds. World Scientific, Singapore, 1996.

20. S.De Nicola, R.Fedele, G.Miele and V.Man'ko, in *New perspectives in the physics of mesoscopic systems*, S.De Martino et al. Eds., World Scientific, Singapore, 1997, p. 89.
21. S. De Martino, S. De Siena and F. Illuminati, *Mod.Phys.Lett.* **B12** (1998), in press.
22. F. Calogero, *Phys.Lett.* **A228**, (1997) 335.
23. S.De Martino, S. De Siena and F. Illuminati, *J.Phys.* **A30** (1997) 4117.