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Exact solutions of Fokker–Planck equations associated to quantum wave functions

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Abstract

We analyse the non-stationary solutions of the Fokker–Planck equations associated to quantum states by stochastic mechanics. In particular we study the exact solutions for the stationary states of the harmonic oscillator and the potentials which realize new possible evolutions ruled by the same equations. © 1998 Published by Elsevier Science B.V.

1. Introduction

In a few recent papers [1] the analogy between diffusive classical systems and quantum systems has been reconsidered from the standpoint of stochastic mechanics (SM) [2,3]. Particular attention was devoted there to the evolution of the classical systems associated to a quantum wave function when the conditions imposed by the stochastic variational principle are not satisfied (non-extremal processes) in order to check if and how the evolving distribution converges in time toward the quantum distribution. This hypothesis constitutes an important point in the discussion of Bohm and Vigier several years ago about some criticisms to the assumptions of the causal interpretation of quantum mechanics (CIQM) [4]. In Ref. [1] it was pointed out that the right convergence was achieved for a few quantum examples, but no general results were available to this effect. Moreover there were also a few counterexamples: in fact not only for non-stationary wave functions (as for a minimal uncertainty packet) there may be no convergence at all, but also in the case of stationary states with nodes (namely with zeros) we do not get the right asymptotic behaviour. The problem is that for stationary states with nodes the corresponding velocity field to consider in the Fokker–Planck equation shows singularities in the locations of the nodes of the wave function. These singularities effectively separate the available interval of the space variables into (probabilistically) non-communicating sections which trap any amount of probability initially attributed and make the system non-ergodic.

The first new result of this Letter is well understood in the light of these opening remarks: we show, by means of a general method (the eigenfunction expansion for the diffusion equations) that for transitive systems with stationary velocity fields (as, for example, a stationary ground state) we always have the right exponential convergence to the right quantum probability distribution associated to the extremal process,

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even if we initially start from an arbitrary non extremal process. These results can also be extended to an arbitrary stationary state if we consider separately the process as confined in every configuration space region between two subsequent nodes. However, this is not the unique path we trod along in the following pages: the second new feature of this article starts from the remark that non-extremal processes can be considered virtual, as trajectories in the classical Lagrangian mechanics, but that in the same way they can also be turned real if we modify the potential in a suitable way. The interest for that lies not only in the fact that non-extremal processes are exactly what is lacking in quantum mechanics in order to interpret it as a classical stochastic process theory (for example in order to have a classical picture of a double slit experiment [5]), but also to engineer some new controlled real evolutions of quantum states. In particular this could be useful to study (a) transitions between stationary states (b) possible models for measure theory [3] and (c) control of the particle beam dynamics in accelerators [6]. In fact it should be pointed out that stochastic mechanics is also a theory, independent from quantum mechanics, which has applications in several physical fields, in particular for systems not perfectly described by the quantum formalism, but whose evolution is correctly controlled by quantum fluctuation: the so called mesoscopic or quantum-like systems. This behaviour characterizes, for example, the beam dynamics in particle accelerators and there is evidence that it could be well described by the stochastic formalism of Nelson diffusions [6]. Of course in these systems trajectories and transition probabilities always are perfectly meaningful. Since to study in detail the evolution of the probability distributions in these cases, and in particular to try to understand if and how it is possible to make a controlled transition between two quantum states, it is necessary to determine the fundamental solutions (transition probability densities) associated by SM to every quantum state in consideration, we eventually devoted the second part of this Letter to a sketch of the way we could undertake this task.

SM is a generalization of classical mechanics based on the theory of classical stochastic processes [2]. The variational principles of Lagrangian type provide a solid foundation for it, as for the classical mechanics or the field theory [3]. In this scheme the deterministic trajectories of classical mechanics are replaced by the random trajectories of diffusion processes in the configuration space. The surprising feature is that programming equations derived from the stochastic version of the lagrangian principle are formally identical to the equations of a Madelung fluid [7], the hydrodynamical equivalent of the Schrödinger equation in the stochastic interpretation of quantum mechanics (SIQM) [8]. On this basis, it is possible to develop an interpretative scheme where the phenomenological predictions of SM coincide with that of quantum mechanics for all the experimentally measurable quantities. Within this interpretative code the SM is nothing but a quantization procedure, different from the ordinary ones only formally, but completely equivalent from the point of view of the physical consequences. Hence we interpret here the SM as a probabilistic simulation of quantum mechanics, providing a bridge between this fundamental section of physics and the stochastic differential calculus. However, it is well known that the most peculiar features of the involved stochastic processes, namely the transition probability densities, seem not always enter into this code scheme: in fact, if we want to check experimentally if the transition probabilities are the right ones for a given quantum state, we are obliged to perform repeated position measurements on the quantum system, so that, according to quantum theory, the quantum state changes in every measurement (wave packet reduction). On the other hand our transition probabilities are associated to a well defined wave function: hence it will be practically impossible in general to experimentally observe these transition probabilities.

Several ways out of these difficulties can be explored: for example stochastic mechanic scheme could be modified by means of non-constant diffusion coefficients [1], or alternatively it would be possible to modify the stochastic evolution during the measurement [9]. Here we rather assume the standpoint of the convergence (in time) of the processes which do not satisfy the stochastic variational principle toward the processes associated to quantum states. In this way any departure from the distributions of quantum mechanics is quickly reabsorbed in the time evolution [1], at least in many meaningful cases. However, the possibility is also considered here that the non-standard evolving distributions can be realized by suitable quantum systems for modified, time dependent potentials which, on the other hand, asymptotically in time rejoin the

usual potentials. Starting from Ref. [1] we will study the correspondence between stochastic and quantum mechanics in several cases of stationary states, and in Section 2 we will in particular recall the well known technique of the eigenfunction expansion for Fokker–Planck equations in order to approach these problems in a general way in Section 3. In Section 4 we will discuss a few explicit examples of Fokker–Planck equations with singular drifts associated to particular wave functions of the quantum harmonic oscillator. Finally in Section 5 we will briefly discuss how it is possible to consider any classical evolution as produced by a suitable quantum system: a step instrumental to point out that also non-extremal evolutions could be in principle produced by means of a suitable (time-dependent) deformation of the potential.

2. Eigenvalue problem for the Fokker–Planck equation

Let us recall here (see, for example, Ref. [10] p. 101) a few general facts about the pdf's (probability density functions) $f(x,t)$, solutions of a one-dimensional Fokker–Planck equation of the form

$$\partial_t f = \partial_x^2(Df) - \partial_x(vf) = \partial_x[\partial_x(Df) - vf] \quad (2.1)$$

defined for $x \in [a,b]$ and $t \geqslant t_0$. Here $D(x)$ and $v(x)$ are two time independent functions such that $D(x) > 0$ and $v(x)$ has no singularities in (a,b); moreover they are both continuous and differentiable functions. The conditions imposed on the probabilistic solutions are of course

$$f(x,t) \geqslant 0\,, \quad a < x < b\,, \quad t_0 \leqslant t\,,$$
$$\int_a^b f(x,t)\,\mathrm{d}x = 1\,, \quad t_0 \leqslant t\,, \quad (2.2)$$

and from the form of (2.1) the second condition also takes the form

$$[\partial_x(Df) - vf]_{a,b} = 0\,, \quad t_0 \leqslant t\,. \quad (2.3)$$

Suitable initial conditions will be added to produce the required evolution: for example the transition pdf $p(x,t|x_0,t_0)$ will be selected by the initial condition

$$\lim_{t\to t_0^+} f(x,t) = f(x,t_0^+) = \delta(x-x_0)\,. \quad (2.4)$$

It is also possible to show by direct calculation that

$$h(x) = N^{-1}\exp\left(-\int [D'(x) - v(x)]/D(x)\,\mathrm{d}x\right),$$
$$N = \int_a^b \exp\left(-\int [D'(x) - v(x)]/D(x)\,\mathrm{d}x\right)\mathrm{d}x \quad (2.5)$$

is an invariant (time independent) solution of (2.1) satisfying the conditions (2.2). Remark that (2.1) is not in the standard self-adjoint form (see Ref. [11] p. 114). However, if we define the function $g(x,t)$ by means of

$$f(x,t) = \sqrt{h(x)}\,g(x,t) \quad (2.6)$$

it would be easy to show that $g(x,t)$ obeys now an equation of the form

$$\partial_t g = \mathcal{L}g, \quad (2.7)$$

where $\mathcal{L}$ defined by

$$\mathcal{L}\varphi = \frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\,\frac{\mathrm{d}\varphi(x)}{\mathrm{d}x}\right] - q(x)\varphi(x)\,, \quad (2.8)$$

with

$$p(x) = D(x) > 0\,,$$
$$q(x) = \frac{[D'(x) - v(x)]^2}{4D(x)} - \frac{[D'(x) - v(x)]'}{2}\,, \quad (2.9)$$

is now self-adjoint. Then, by separating the variables by means of $g(x,t) = \gamma(t)G(x)$ we have $\gamma(t) = \mathrm{e}^{-\lambda t}$ while G must be a solution of a typical Sturm–Liouville problem associated to the equation

$$\mathcal{L}G(x) + \lambda G(x) = 0 \quad (2.10)$$

with the boundary conditions

$$[D'(a) - v(a)]G(a) + 2D(a)G'(a) = 0\,,$$
$$[D'(b) - v(b)]G(b) + 2D(b)G'(b) = 0\,. \quad (2.11)$$

It easy to see that $\lambda = 0$ is always an eigenvalue for the problem (2.10) with (2.11), and that the corresponding eigenfunction is $\sqrt{h(x)}$ defined in (2.5).

For the differential problem (2.10) with (2.11) we have that (see Ref. [11] p. 138) the simple eigenvalues λ_n will constitute an infinite, increasing sequence and the corresponding eigenfunction $G_n(x)$ will have n simple zeros in (a, b). For us this means that $\lambda_0 = 0$, corresponding to the eigenfunction $G_0(x) = \sqrt{h(x)}$ which never vanishes in (a, b), is the lowest eigenvalue so that all other eigenvalues are strictly positive. Moreover the eigenfunctions will constitute a complete orthonormal set of functions in $L^2([a, b])$ (see Ref. [12] p. 134). As a consequence the general solution of (2.1) with (2.2) will have the form

$$f(x,t) = \sum_{n=0}^{\infty} c_n \, \mathrm{e}^{-\lambda_n t} \sqrt{h(x)} G_n(x) \tag{2.12}$$

with $c_0 = 1$ for normalization (remember that $\lambda_0 = 0$). The coefficients c_n for a particular solution selected by an initial condition

$$f(x, t_0^+) = f_0(x) \tag{2.13}$$

are then calculated from the orthonormality relations as

$$c_n = \int_a^b f_0(x) \, \frac{G_n(x)}{\sqrt{h(x)}} \, \mathrm{d}x \,, \tag{2.14}$$

and in particular for the transition pdf we have from (2.4) that

$$c_n = \frac{G_n(x_0)}{\sqrt{h(x_0)}} \,. \tag{2.15}$$

Since $\lambda_0 = 0$ and $\lambda_n > 0$ for $n \geqslant 1$, the general solution (2.12) of (2.1) has a precise time evolution. In fact all the exponential factors in (2.12) vanish with $t \to +\infty$ with the only exception of the term $n = 0$ which is constant, so that exponentially fast we will always have

$$\lim_{t \to +\infty} f(x,t) = c_0 \sqrt{h(x)} G_0(x) = h(x) \,, \tag{2.16}$$

namely: the general solution will always relax in time toward the invariant solution $h(x)$.

3. Fokker–Planck equations for stochastic mechanics

We will now examine the consequences of the previous remarks on the evolution equations of the SM, and for the sake of brevity in the presentation of the notation we will take a shortcut borrowed from the SIQM which, as it is well known, is formally ruled by the same differential equations as the SM. Let us consider the (one dimensional) Schrödinger equation

$$\mathrm{i}\hbar \partial_t \psi = \hat{H}\psi = -\frac{\hbar^2}{2m} \partial_x^2 \psi + V\psi \tag{3.1}$$

for a time-independent potential $V(x)$ which gives rise to a purely discrete spectrum and bound, normalizable states, and let us use the following notations for stationary states, eigenvalues and eigenfunctions,

$$\psi_n(x,t) = \phi_n(x) \, \mathrm{e}^{-\mathrm{i}E_n t/\hbar},$$

$$\hat{H}\phi_n = -\frac{\hbar^2}{2m} \phi_n'' + V\phi_n = E_n \phi_n. \tag{3.2}$$

For later convenience we will also introduce the constant

$$D = \frac{\hbar}{2m} \tag{3.3}$$

which will play the role of a diffusion coefficient when we will associate a stochastic process to every quantum state. As a consequence the previous eigenvalue equation can be recast in the following form,

$$D\phi_n'' = \frac{V - E_n}{\hbar} \, \phi_n \,. \tag{3.4}$$

It is also well-known that for these stationary states the pdf is the time independent, real function

$$f_n(x) = |\psi_n(x,t)|^2 = \phi_n^2(x) \,. \tag{3.5}$$

With the ansatz usual in SIQM (see for example Ref. [7])

$$\psi(x,t) = R(x,t) \, \mathrm{e}^{\mathrm{i}S(x,t)/\hbar} \tag{3.6}$$

where R and S are real functions, the function $R^2 = |\psi|^2$ turns out to be a particular solution of a Fokker–Planck equation (with constant diffusion coefficient and time-independent forward velocity field) of the form

$$\partial_t f = D\partial_x^2 f - \partial_x(vf) = \partial_x(D\partial_x f - vf) \,, \tag{3.7}$$

where

$$v(x,t) = \frac{1}{m}\,\partial_x S + \frac{\hbar}{2m}\,\partial_x(\ln R^2)\,. \tag{3.8}$$

Remark that the explicit dependence of v on the form of R clearly indicates that to have a solution of (3.7) which makes quantum sense we must pick up only one, suitable, particular solution. In fact in the stochastic mechanical framework the system is ruled not only by the Fokker–Planck equation (3.7), but also by the second, dynamical equation

$$\partial_t S + \frac{(\partial_x S)^2}{2m} + V - \frac{\hbar^2}{2m}\,\frac{\partial_x^2 R}{R} = 0 \tag{3.9}$$

namely the so-called Hamilton–Jacobi–Madelung equation deduced by separating the real and imaginary parts of (3.1) (see Ref. [7]). The analogy between (3.7) and the Fokker–Planck equation (2.1) is more than formal in the sense that the SM (see Ref. [2]) shows how to recover both the Eqs. (3.7) and (3.9) (namely the Schrödinger equation (3.1)) in a classical, dynamical stochastic context. This dynamics can also be connected to a stochastic variational principle (see Ref. [3]) which, as usual in the dynamical theories, selects the actual trajectories of the system among all the virtually possible evolutions. In particular from this standpoint the velocity field v is not a given field, but becomes a dynamical variable to be determined by the equations of the system.

We will now fix our attention on a given stationary solution $\psi_n(x,t)$ and we will remark that in this case we have

$$S(x,t) = -E_n t\,, \quad R(x,t) = \phi_n(x)\,, \tag{3.10}$$

so that for our state the velocity field is

$$v_n(x) = 2D\,\frac{\phi_n'(x)}{\phi_n(x)}\,. \tag{3.11}$$

This means that now v_n is time-independent and it possibly presents singularities in the zeros (nodes) of the eigenfunction. Since the nth eigenfunction of a quantum system with bound states has exactly n simple nodes (see Ref. [11] p. 138) that we will indicate with $x_1,\ldots,x_n$, the coefficients of the Fokker–Planck equation (3.7) are not defined in these n points and we will be obliged to solve it in separate intervals by imposing the right boundary conditions connecting the different sections. In fact these singularities effectively separate the real axis in $n+1$ sub-intervals with impenetrable (to the probability current) walls. Hence the process will not have a unique invariant measure and will never cross the boundaries fixed by the singularities of $v(x)$: if we start in one of the intervals in which the axis is so divided we will always remain there (see Ref. [13]).

As a consequence we must think the normalization integral (2.2) (with $a = -\infty$ and $b = +\infty$) as the sum of $n+1$ integrals over the sub-intervals $[x_k, x_{k+1}]$ with $k = 0, 1, \ldots, n$ (where we understand, to unify the notation, that $x_0 = -\infty$ and $x_{n+1} = +\infty$). Hence for $n \geqslant 1$ we will be obliged to solve Eq. (3.7) in every interval $[x_k, x_{k+1}]$ by requiring that the integrals

$$\int_{x_k}^{x_{k+1}} f(x,t)\,\mathrm{d}x \tag{3.12}$$

be kept at a constant value for $t \geqslant t_0$: this value is not, in general, equal to one (only the sum of these $n+1$ integrals amounts to one) and, since the separate intervals can not communicate, it will be fixed by the choice of the initial conditions. The boundary conditions associated to (3.7) are hence imposed by the conservation of the probability in $[x_k, x_{k+1}]$ and that means the vanishing of the probability current at the end points of the interval,

$$[D\partial_x f - vf]_{x_k, x_{k+1}} = 0\,, \quad t \geqslant t_0\,. \tag{3.13}$$

To have a particular solution we must moreover specify the initial conditions: in particular we will be interested in the transition pdf $p(x,t|x_0,t_0)$, which is singled out by the initial condition (2.4), since (see Ref. [1]) the asymptotic approximation in L^1 among solutions of (3.7) is ruled by the asymptotic behaviour of $p(x,t|x_0,t_0)$ through the Chapman–Kolmogorov equation

$$f(x,t) = \int_{-\infty}^{+\infty} p(x,t|y,t_0)\,f(y,t_0^+)\,\mathrm{d}y\,. \tag{3.14}$$

It is clear at this point that in every interval $[x_k, x_{k+1}]$ (both finite or infinite) we can solve Eq. (3.7) along the guidelines sketched in Section 1 by keeping in mind that in $[x_k, x_{k+1}]$ we already know the invariant, time-independent solution $\phi_n^2(x)$ (or, more precisely,

its restriction to the said interval) which is never zero in this interval with the exception of the extremes x_k and x_{k+1}. Hence, as we have seen in the general case, with the position

$$f(x,t) = \phi_n(x)g(x,t) \tag{3.15}$$

we can reduce (3.7) to the form

$$\partial_t g = \mathcal{L}_n g, \tag{3.16}$$

where $\mathcal{L}_n$ is now the self-adjoint operator defined on $[x_k, x_{k+1}]$ by

$$\mathcal{L}_n\varphi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}\varphi(x)}{\mathrm{d}x}\right] - q_n(x)\varphi(x), \tag{3.17}$$

where we have now

$$p(x) = D > 0\,, \quad q_n(x) = \frac{v_n^2(x)}{4D} + \frac{v_n'(x)}{2}\,. \tag{3.18}$$

To solve (3.16) it is in general advisable to separate the variables, we then immediately have $\gamma(t) = \mathrm{e}^{-\lambda t}$ while G must be a solution of the Sturm–Liouville problem associated to the equation

$$\mathcal{L}_n G(x) + \lambda G(x) = 0 \tag{3.19}$$

with the boundary conditions

$$[2DG'(x) - v_n(x)G(x)]_{x_k,x_{k+1}} = 0\,. \tag{3.20}$$

The general behaviour of the solutions obtained as expansions in the system of the eigenfunctions of (3.19) has already been discussed in Section 1.

4. Harmonic oscillator states

To see in an explicit way how the pdf's of SM evolve, let us consider now in detail the particular example of a quantum harmonic oscillator (HO) characterized by the potential

$$V(x) = \frac{m}{2}\,\omega^2 x^2\,. \tag{4.1}$$

It is well-known that its eigenvalues are

$$E_n = \hbar\omega\left(n + \tfrac{1}{2}\right)\,, \quad n = 0, 1, 2\ldots \tag{4.2}$$

while, with the notation

$$\sigma_0^2 = \frac{\hbar}{2m\omega}\,, \tag{4.3}$$

the eigenfunctions are

$$\phi_n(x) = \frac{1}{\sqrt{\sigma_0\sqrt{2\pi}2^n n!}}\,\mathrm{e}^{-x^2/4\sigma_0^2}\,H_n\left(\frac{x}{\sigma_0\sqrt{2}}\right), \tag{4.4}$$

where H_n are the Hermite polynomials. The corresponding velocity fields are easily calculated and are for example

$$\begin{aligned} v_0(x) &= -\omega x\,,\\ v_1(x) &= 2\,\frac{\omega\sigma_0^2}{x} - \omega x\,,\\ v_2(x) &= 4\omega\sigma_0^2\,\frac{x}{x^2-\sigma_0^2} - \omega x\,, \end{aligned} \tag{4.5}$$

with singularities in the zeros x_k of the Hermite polynomials. If we now keep the form of the velocity fields fixed we can consider (3.7) as an ordinary Fokker–Planck equation for a diffusion process and solve it to see the approach to the equilibrium of the general solutions. When $n = 0$ Eq. (3.7) takes the form

$$\partial_t f = \omega\sigma_0^2\partial_x^2 f + \omega x\partial_x f + \omega f \tag{4.6}$$

and the fundamental solution turns out to be the Ornstein–Uhlenbeck transition pdf

$$p(x,t|x_0,t_0) = \frac{1}{\sigma(t)\sqrt{2\pi}}\,\mathrm{e}^{-[x-\alpha(t)]^2/2\sigma^2(t)}\,, \quad (t \geqslant t_0) \tag{4.7}$$

where we used the notation

$$\begin{aligned} \alpha(t) &= x_0\,\mathrm{e}^{-\omega(t-t_0)}\,,\\ \sigma^2(t) &= \sigma_0^2[1 - \mathrm{e}^{-2\omega(t-t_0)}] \quad (t \geqslant t_0)\,. \end{aligned} \tag{4.8}$$

The stationary Markov process associated to the transition pdf (3.1) is selected by the invariant pdf

$$f(x) = \frac{1}{\sigma_0\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2\sigma_0^2} \tag{4.9}$$

which is also the asymptotic pdf for every initial condition when the evolution is ruled by (4.6) (see for example Ref. [1]) so that the invariant distribution plays also the role of the limit distribution. It is remarkable that this invariant pdf also coincides with the quantum stationary pdf $\phi_0^2 = |\psi_0|^2$; in other words, the process associated by the SM to the ground state of a

quantum HO is nothing but the stationary Ornstein–Uhlenbeck process.

For $n \geqslant 1$ the solutions of (3.7) are no more so easy to find and, as discussed in the previous section, we will have to solve the eigenvalue problem (3.19) which, with $\epsilon = \hbar\lambda$, can be written as

$$-\frac{\hbar^2}{2m} G''(x) + \left(\frac{m}{2}\omega^2 x^2 - \hbar\omega \frac{2n+1}{2}\right) G(x) = \epsilon G(x) \tag{4.10}$$

in every interval $[x_k, x_{k+1}]$ with, $k = 0, 1, \ldots, n$, between two subsequent singularities of the v_n field. The boundary conditions at the endpoints of these intervals are deduced from (3.13)–(3.15) and are

$$[\phi_n G' - \phi_n' G]_{x_k, x_{k+1}} = 0 \tag{4.11}$$

and since ϕ_n (but not ϕ_n') vanishes in x_k, x_{k+1}, the boundary conditions to impose are

$$G(x_k) = G(x_{k+1}) = 0 \tag{4.12}$$

where it is understood that for the conditions in x_0 and x_{n+1} we respectively mean

$$\lim_{x\to-\infty} G(x) = 0\,, \qquad \lim_{x\to+\infty} G(x) = 0\,. \tag{4.13}$$

It is also useful at this point to give the eigenvalue problem in an nondimensional form by using the new nondimensional variable x/σ_0 (which will still be called x) and the eigenvalue $\mu = \lambda/\omega = \epsilon/\hbar\omega$. In this way Eq. (4.10) with the conditions (4.12) becomes

$$y''(x) - \left(\frac{x^2}{4} - \frac{2n+1}{2} - \mu\right) y(x) = 0, \qquad y(x_k) = y(x_{k+1}) = 0 \tag{4.14}$$

where x, x_k, x_{k+1} are now nondimensional variables. If μ_m and $y_m(x)$ are the eigenvalues and eigenfunctions of (4.14), the general solution of the corresponding Fokker–Planck equation (3.7) will be

$$f(x,t) = \sum_{m=0}^{\infty} c_m \,\mathrm{e}^{-\mu_m \omega t} \phi_n(x)\, y_m\left(\frac{x}{\sigma_0}\right)\,. \tag{4.15}$$

Of course the values of the coefficients c_m will be fixed by the initial conditions and by the obvious requirements that $f(x,t)$ must be non-negative and normalized (on the whole x axis) along all its evolution. Two linearly independent solutions of (4.14) are

$$y^{(1)} = \mathrm{e}^{-x^2/4} M\left(-\frac{\mu+n}{2}, \frac{1}{2}; \frac{x^2}{2}\right)\,,$$
$$y^{(2)} = x\,\mathrm{e}^{-x^2/4} M\left(-\frac{\mu+n-1}{2}, \frac{3}{2}; \frac{x^2}{2}\right)\,, \tag{4.16}$$

where $M(a,b;z)$ are the confluent hypergeometric functions.

We consider now the case $n = 1$ ($x_0 = -\infty$, $x_1 = 0$ and $x_2 = +\infty$) so that (4.14) will have to be solved separately for $x \leqslant 0$ and for $x \geqslant 0$ with the boundary conditions $y(0) = 0$ and

$$\lim_{x\to-\infty} y(x) = \lim_{x\to+\infty} y(x) = 0\,. \tag{4.17}$$

The eigenvalues are $\mu_m = 2m$ with $m = 0, 1, \ldots$ and the complete set of eigenfunctions is given by

$$y_m(x) = x\,\mathrm{e}^{-x^2/4} M\left(-m, \frac{3}{2}; \frac{x^2}{2}\right) = \frac{(-1)^m m!}{\sqrt{2}(2m+1)!}\,\mathrm{e}^{-x^2/4} H_{2m+1}\left(\frac{x}{\sqrt{2}}\right)\,. \tag{4.18}$$

In fact this is the form of the eigenfunctions for both $x \geqslant 0$ and $x \leqslant 0$. In particular it is easy to see that

$$y_0(x) = x\,\mathrm{e}^{-x^2/4} \tag{4.19}$$

and its relation with the quantum eigenfunction ϕ_1 is

$$\phi_1(x) = \frac{y_0(x/\sigma_0)}{\sqrt{\sigma_0\sqrt{2}}}\,, \tag{4.20}$$

and the general solution of (3.7) with $v = v_1$ is

$$f(x,t) = \sum_{m=0}^{\infty} c_m\,\mathrm{e}^{-2m\omega(t-t_0)} \phi_1(x)\, y_m(x/\sigma_0)\,. \tag{4.21}$$

To determine the c_m's we must now impose an initial condition: in particular, the initial condition for the transition pdf's requires

$$c_m = \frac{2}{\sqrt{\sigma_0\sqrt{2\pi}}} \frac{(2m+1)!!}{(2m)!!} \frac{y_m(x_0/\sigma_0)}{y_0(x_0/\sigma_0)}\,. \tag{4.22}$$

As a consequence we will have

$$p(x,t|x_0,t_0) = \frac{\sigma_0^2\phi_1^2(x)}{xx_0}\sum_{m=0}^{\infty}\frac{\mathrm{e}^{-2m\omega(t-t_0)}}{2^{2m}(2m+1)!} \times H_{2m+1}\left(\frac{x}{\sigma_0\sqrt{2}}\right)H_{2m+1}\left(\frac{x_0}{\sigma_0\sqrt{2}}\right). \quad (4.23)$$

A long calculation shows that this series can be summed up to

$$p(x,t|x_0,t_0) = \frac{x}{\alpha(t)}\,\frac{\mathrm{e}^{-[x-\alpha(t)]^2/2\sigma^2(t)} - \mathrm{e}^{-[x+\alpha(t)]^2/2\sigma^2(t)}}{\sigma(t)\sqrt{2\pi}} \quad (4.24)$$

where $\alpha(t)$ and $\sigma^2(t)$ are defined in (4.8). It must be remarked once more that the form (4.24) of the transition pdf must be considered as restricted to $x \geqslant 0$ when $x_0 > 0$ and to $x \leqslant 0$ when $x_0 < 0$, and that only on these intervals it is suitably normalized. In order to take into account at once both these possibilities we can also introduce the Heavyside function $\Theta(x)$ so that for every $x_0 \neq 0$ we will have

$$p(x,t|x_0,t_0) = \Theta(xx_0)\,\frac{x}{\alpha(t)} \times \frac{\mathrm{e}^{-[x-\alpha(t)]^2/2\sigma^2(t)} - \mathrm{e}^{-[x+\alpha(t)]^2/2\sigma^2(t)}}{\sigma(t)\sqrt{2\pi}}. \quad (4.25)$$

This completely solves the problem for $n = 1$ since from (3.14) we can now deduce also the evolution of every other initial pdf. In particular it can be shown that

$$\lim_{t\to+\infty} p(x,t|x_0,t_0) = 2\Theta(xx_0)\,\frac{x^2\,\mathrm{e}^{-x^2/2\sigma_0^2}}{\sigma_0^3\sqrt{2\pi}} = 2\Theta(xx_0)\phi_1^2(x), \quad (4.26)$$

and hence, if $f(x,t_0^+) = f_0(x)$ is the initial pdf, we have for $t > t_0$

$$\lim_{t\to+\infty} f(x,t) = \lim_{t\to+\infty}\int_{-\infty}^{+\infty} p(x,t|y,t_0)\,f_0(y)\,\mathrm{d}y = 2\phi_1^2(x)\int_{-\infty}^{+\infty}\Theta(xy)\,f_0(y)\,\mathrm{d}y = \Gamma(q;x)\phi_1^2(x), \quad (4.27)$$

where we have defined the function

$$\Gamma(q;x) = q\Theta(x) + (2-q)\Theta(-x), \qquad q = 2\int_0^{+\infty} f_0(y)\,\mathrm{d}y. \quad (4.28)$$

Remark that when $q = 1$ (namely when the initial probability is equally shared on the two real half-axis) we have $\Gamma(1;x) = 1$ and the asymptotical pdf coincides with the quantum stationary pdf $\phi_1^2(x)$; if on the other hand $q \neq 1$ the asymptotical pdf has the same shape of $\phi_1^2(x)$ but with different weights on the two half-axis.

If finally $n = 2$ we have $x_0 = -\infty$, $x_1 = -1$, $x_2 = 1$ and $x_3 = +\infty$, and Eq. (4.14) must be solved in the three intervals $(-\infty,-1]$, $[-1,1]$ and $[1,+\infty)$. The two linearly independent solutions are now

$$y^{(1)} = \mathrm{e}^{-x^2/4}M\left(-\frac{\mu+2}{2},\frac{1}{2};\frac{x^2}{2}\right), \qquad y^{(2)} = x\,\mathrm{e}^{-x^2/4}M\left(-\frac{\mu+1}{2},\frac{3}{2};\frac{x^2}{2}\right), \quad (4.29)$$

and it is easy to verify that $\mu = 0$ is an eigenvalue for all the three intervals with eigenfunction

$$y_0(x) = \mathrm{e}^{-x^2/4}M\left(-1,\frac{1}{2};\frac{x^2}{2}\right) = \mathrm{e}^{-x^2/4}H_2\left(\frac{x}{\sqrt{2}}\right) = 2\,\mathrm{e}^{-x^2/4}(x^2-1) \quad (4.30)$$

so that the relation with the quantum eigenfunction is

$$\phi_2(x) = \frac{y_0(x/\sigma_0)}{\sqrt{8\sigma_0\sqrt{2\pi}}}. \quad (4.31)$$

As for the other eigenvalues and eigenfunction they are not too easy to find so that a complete analysis of this case has still to be elaborated. A few indications can be obtained numerically: for example it can be shown that, beyond $\mu_0 = 0$, the first eigenvalues in the interval $[-1,1]$ can be calculated as the first values such that

$$M\left(-\frac{\mu+1}{2},\frac{3}{2};\frac{1}{2}\right) = 0 \quad (4.32)$$

and are $\mu_1 \sim 7.44$, $\mu_2 \sim 37.06$, $\mu_3 \sim 86.41$. Also for the unbounded interval $[1,+\infty)$ (the analysis is similar for $(-\infty,-1]$) the eigenvalues are derivable only numerically.

5. Conclusions

We want to conclude this Letter by remarking that the processes discussed in the previous sections, namely the processes solutions of (3.7) but not associated to quantum mechanical states solutions of (3.1) (in other words, the processes that do not satisfy either the stochastic variational principle [3] or the Nelson dynamical equation [2]), are not without a deep relation with the quantum mechanics. In fact we will show here that to every solution $f(x,t)$ of a Fokker–Planck equation (2.1), with given $v(x)$ and constant diffusion coefficient (3.3), it is possible to associate the wave function of a quantum system with a suitable time-dependent potential. This means in practice that even the *virtual* (non-optimal) processes discussed in this Letter can be associated to proper quantum states, namely can be made optimal albeit with a potential which is different from the $V(x)$ of (3.1).

Let us take a solution $f(x,t)$ of the Fokker–Planck equation (2.1), with given $v(x)$ and constant diffusion coefficient (3.3): if we define the functions $R(x,t)$ and $W(x)$ by

$$R(x,t) = \sqrt{f(x,t)}\,, \quad v(x) = W'(x)\,, \tag{5.1}$$

and we remember that [2] the following relation holds,

$$mv = \partial_x\left(S + \frac{\hbar}{2}\ln R^2\right) = \partial_x\left(S + \frac{\hbar}{2}\ln f\right) \tag{5.2}$$

if $S(x,t)$ is supposed to be the phase of a wave function as in (3.6), we immediately get

$$S(x,t) = mW(x) - \frac{\hbar}{2}\ln f(x,t) - \theta(t) \tag{5.3}$$

which allows us to determine S from f and v (namely W) up to an arbitrary function of the time $\theta(t)$. However, in order that the wave function (3.6) with our R and S be a solution of a Schrödinger equation, we must be also sure that the Hamilton–Jacobi–Madelung equation (3.9) be satisfied. Since S and R are now fixed, this Eq. (3.9) must be considered as a relation defining the new potential which, after a short calculation, becomes

$$V(x,t) = \frac{\hbar^2}{2m}\,\partial_x^2 \ln f + \frac{\hbar}{2}\left(\partial_t \ln f + v\partial_x \ln f\right) - \frac{mv^2}{2} + \dot\theta\,. \tag{5.4}$$

Of course if we start with a quantum wave function for a given potential and if we pick up as a solution of (3.7) exactly $f = R^2$ formula (5.4) will correctly give back the initial potential, as can be seen for the ground state and the first excited state of the harmonic oscillator which (by choosing respectively $\theta(t) = \hbar\omega t/2$ and $\theta(t) = 3\hbar\omega t/2$) give as result the usual harmonic potential (3.1).

On the other hand let us consider now the (non-stationary) fundamental solution (4.7) associated to the velocity field $v_0(x)$ of (4.5) for the case $n = 0$ of the harmonic oscillator (we put $t_0 = 0$ to simplify the notation): a short calculation shows that, by choosing

$$\dot\theta(t) = \frac{\hbar\omega}{2}\left(\frac{2\sigma_0^2}{\sigma^2(t)} - 1\right) = \frac{\hbar\omega}{2}\,\frac{1}{\tanh\omega t} \to \frac{\hbar\omega}{2} \quad (t\to+\infty)\,, \tag{5.5}$$

we get the time-dependent potential

$$V(x,t) = \frac{\hbar\omega}{2}\left[\frac{x-\alpha(t)}{\sigma(t)}\right]^2\frac{\sigma_0^2}{\sigma^2(t)} - \frac{m\omega^2x^2}{2} \to \frac{m\omega^2x^2}{2} \quad (t\to+\infty)\,. \tag{5.6}$$

Of course the fact that for $t\to+\infty$ we recover the harmonic potential is associated to the fact, already noted, that the usual quantum pdf $\phi_0^2(x)$ is also the limit distribution for every initial condition. As for the case $n = 1$, with $v_1(x)$ from (4.5) and the transition probability (4.25) as given non-stationary solution, the calculations are lengthier. However, if we define

$$F(x,t) = \frac{\mathrm{e}^{-[x-\alpha(t)]^2/2\sigma^2(t)}}{\sigma(t)\sqrt{2\pi}}\,,$$

$$G(x,t) = \frac{\mathrm{e}^{-[x+\alpha(t)]^2/2\sigma^2(t)}}{\sigma(t)\sqrt{2\pi}}\,, \tag{5.7}$$

$$T\left[\frac{x\alpha(t)}{\sigma^2(t)}\right] = \frac{x\alpha(t)}{\sigma^2(t)}\,\frac{F(x,t)+G(x,t)}{F(x,t)-G(x,t)}\,,$$

$$T(x) = \frac{x}{\tanh x}\,, \tag{5.8}$$

and if we choose

$$\dot{\theta}(t) = \frac{\hbar\omega}{2}\left(\frac{4\sigma_0^2}{\sigma^2(t)} - \frac{2\sigma_0^2\alpha^2(t)}{\sigma^4(t)} - 1\right) \to \frac{3}{2}\hbar\omega \quad (t \to +\infty) \tag{5.9}$$

we have as time dependent potential for every $x \neq 0$

$$V(x,t) = \frac{m\omega^2x^2}{2}\left(\frac{2\sigma_0^4}{\sigma^4} - 1\right) + \hbar\omega\left[1 - \frac{\sigma_0^2}{\sigma^2}T\left(\frac{x\alpha}{\sigma^2}\right)\right] - \frac{\hbar^2}{4mx^2}\left[1 - T\left(\frac{x\alpha}{\sigma^2}\right)\right] \to \frac{m\omega^2x^2}{2} \quad (t \to +\infty)\,.$$

In this case the asymptotic potential is the usual harmonic potential, but we must consider it separately on the positive and negative x semi axis since in the point $x = 0$ a singular behaviour would show up. This means that, also if asymptotically we recover the right potential, this will be associated with new boundary conditions in $x = 0$ since we will be obliged to keep the system bounded on the positive (for example) semiaxis.

These simple examples show that we can always design a suitable (time dependent) potential which realizes arbitrary diffusive evolutions of classical systems. This has more than a purely formal interest since it can allow one, for instance, to produce a quantum evolution between two stationary states of a system by means of a deformation of the original potential. Since also the measurement processes (which can not be described along a time evolution in the usual formalism of quantum mechanics) can be considered as a transition between two states, these techniques seem to indicate a way to simulate a continuous reduction of a wave packet controlled by a time dependent potential. A subsequent paper will be devoted to a discussion of this point which, however, will require a generalization of the results on the solutions of the Fokker–Planck equation to the case of time dependent velocity fields since in this case we will have to do with non-stationary quantum wave functions.

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