Controlled quantum evolutions and transitions

Nicola Cufaro Petroni†, Salvatore De Martino‡, Silvio De Siena‡ and
Fabrizio Illuminati‡

† INFN Sezione di Bari, INFM Unità di Bari and Dipartimento Interateneo di Fisica
dell'Università e del Politecnico di Bari, via Amendola 173, 70126 Bari, Italy
‡ INFM Unità di Salerno, INFN Sezione di Napoli—Gruppo collegato di Salerno and
Dipartimento di Fisica dell'Università di Salerno, via S Allende, 84081 Baronissi, Salerno, Italy

E-mail: cufaro@ba.infn.it, demartino@physics.unisa.it,
desiena@physics.unisa.it and illuminati@physics.unisa.it

Abstract. We study the nonstationary solutions of Fokker–Planck equations associated to either stationary or non stationary quantum states. In particular, we discuss the stationary states of quantum systems with singular velocity fields. We introduce a technique that allows arbitrary evolutions ruled by these equations to account for controlled quantum transitions. As a first signficant application we present a detailed treatment of the transition probabilities and of the controlling time-dependent potentials associated to the transitions between the stationary, the coherent, and the squeezed states of the harmonic oscillator.

1. Introduction

In a few recent papers [1] the analogy between diffusive classical systems and quantum systems has been reconsidered from the standpoint of the stochastic simulation of quantum mechanics [2–4] and particular attention has been devoted there to the evolution of the classical systems associated to a quantum wavefunction when the conditions imposed by the stochastic variational principle are not satisfied (nonextremal processes). The problem studied in those papers was the convergence of an arbitrary evolving probability distribution, solution of the Fokker–Planck equation, toward a suitable quantum distribution. In [1] it was pointed out that, while the correct convergence is achieved for a few quantum examples, these results cannot be considered general, as was shown in some counterexamples: in fact, not only for particular nonstationary wavefunctions (as for a minimal uncertainty packet), but also for stationary states with nodes one does not recover in a straightforward way the correct quantum asymptotic behaviour. For stationary states with nodes the problem is that the corresponding velocity field to consider in the Fokker–Planck equation shows singularities at the locations of the nodes of the wavefunction. These singularities effectively separate the available interval of the configurational variables into noncommunicating sectors which trap any amount of probability initially attributed and make the system nonergodic.

In a more recent paper [5], it has been shown that for transitive systems with stationary velocity fields (as, for example, a stationary state without nodes) we always have an exponential convergence to the correct quantum probability distribution associated to the extremal process, even if we initially start from an arbitrary nonextremal process. These results can also be

extended to an arbitrary stationary state if we consider separately the process as confined in every region of the configuration space between two subsequent nodes.

In the same paper [5] it has been further remarked that while the nonextremal processes should be considered virtual, as the nonextremal trajectories of classical Lagrangian mechanics, they can however, become physical, real solutions if we suitably modify the potential in the Schrödinger equation. The interest of this remark lies not so much in the fact that nonextremal processes are exactly what is lacking in quantum mechanics in order to interpret it as a totally classical theory of stochastic processes (for example, in order to have a classical picture of a double-slit experiment [6]), but rather in the much more interesting possibility of engineering and controlling physically realizable evolutions of quantum states. This observation would be of great relevance, for instance, to the study and the description of: (a) transitions between quantum states, (b) possible models for quantum measurements [3] and (c) control of the dynamics of quantum-like systems (for instance, charged beams in particle accelerators) [7].

In particular, case (c) is being studied in the framework of Nelson stochastic mechanics which is an independent and self-consistent reformulation of quantum mechanics [2, 3] and can be applied in other areas of physical phenomenology. For instance, it can usefully account for systems not completely described by the quantum formalism, but whose evolution is, however, strongly influenced by quantum fluctuations, i.e. the so-called mesoscopic or quantum-like systems. This behaviour characterizes, for example, the beam dynamics in particle accelerators and there is evidence that it can be described by the stochastic formalism of Nelson diffusions [1, 7] since in these quantum-like systems, trajectories and transition probabilities acquire a clear physical meaning, at variance with the case of quantum mechanics.

On the other hand, quantum behaviours can be simulated by means of classical stochastic processes in a by now well defined and established framework [2]. A stochastic variational principle provides a foundation for that, in close analogy with classical mechanics and field theory [3]. In this scheme the deterministic trajectories of classical mechanics are replaced by the random trajectories of diffusion processes in configuration space. The programming equations derived from the stochastic variational principle are formally identical to the equations of the Madelung fluid [8], the hydrodynamical equivalent of the Schrödinger equation [9]. On this basis, it is possible to develop a model whose phenomenological predictions coincide with those of quantum mechanics for all the experimentally measurable quantities. Within this interpretative code stochastic mechanics is nothing but a quantization procedure, different from the canonical one only formally, but completely equivalent from the point of view of the physical consequences: a probabilistic simulation of quantum mechanics, providing a bridge between this fundamental physical theory and stochastic differential calculus. However, it is well known that the central objects in the theory of classical stochastic processes, namely the transition probability densities, seldom play any observable role in stochastic mechanics and must be considered as a type of gauge variable. Several generalizations of Nelson stochastic quantization have been recently proposed to allow for the observability of the transition probabilities: for instance, stochastic mechanics could be modified by means of nonconstant diffusion coefficients [1]; alternatively, it has been suggested that the stochastic evolution might be modified during the measurement process [10].

The aim of the present paper is instead to show how the transition probabilities associated to Nelson diffusion processes can play a very useful role in standard quantum mechanics, in particular with regard to describing and engineering the dynamics of suitably controlled quantum evolutions and transitions. More precisely, we consider the following problem in the theory of quantum control: *given an initial probability distribution ρ_i associated to an arbitrarily assigned quantum state ψ_i, we study its time evolution with the drift associated to another arbitrarily assigned quantum state ψ_f, to determine the controlling time-dependent*

potential $V_c(x, t)$ such that (1) at any instant of time the evolving probability distribution is that associated to the wavefunction solution of the Schrödinger equation in the potential $V_c(x, t)$, and that (2) asymptotically in time the evolving distribution converges to the distribution ρ_f associated to ψ_f.

After introducing the formalism of Nelson stochastic mechanics to describe quantum evolutions in section 2, in sections 3 and 4 we provide a self-contained review of the Sturm–Liouville problem for the Fokker–Planck equation and the techniques of solution for the Nelson diffusions associated to stationary quantum states. In section 5 we discuss, in detail, the example of the harmonic oscillator, explicitly solving for the transition probability densities of the ground and of the low-lying excited states. Sections 6–8 are devoted to the study and the solution of the problem outlined above, discussing the potentials associated to the definition of controlled quantum evolution, and modelling transitions. Two explicit examples are studied in detail: the controlled transition between the invariant probability densities associated to the ground and the first excited state of the harmonic oscillator, and the controlled evolution between pairs of coherent or squeezed wavepackets. In these cases the problem can be solved completely, yielding the explicit analytic form of the evolving transition probabilities and of the evolving controlling potentials at all times. Finally, in section 9 we present our conclusions and discuss possible future extensions and applications of the technique introduced in the present paper, with regard to the discussion of anharmonic quantum and quantum-like systems, the role of instabilities in the initial conditions, and the implementation of optimization procedures.

2. Fokker–Planck equations and quantum systems

Here we will recall a few notions of stochastic mechanics in order to fix the notation. The configuration of a classical particle is promoted to a vector Markov process $\xi(t)$ taking values in $\mathbb{R}^3$. This process is characterized by a probability density $\rho(r, t)$ and a transition probability density $p(r, t | r', t')$ and its components satisfy an Itô stochastic differential equation of the form

$$\mathrm{d}\xi_j(t) = v_{(+)j}(\xi(t), t)\,\mathrm{d}t + \mathrm{d}\eta_j(t) \tag{2.1}$$

where $v_{(+)j}$ are the components of the forward velocity field. However, here the fields $v_{(+)j}$ are not given *a priori*, but play the role of dynamical variables and are consequently determined by imposing a specific dynamics. The noise $\eta(t)$ is a standard Wiener process independent of $\xi(t)$ and such that

$$\boldsymbol{E}_t(\mathrm{d}\eta_j(t)) = 0 \qquad \boldsymbol{E}_t(\mathrm{d}\eta_j(t)\,\mathrm{d}\eta_k(t)) = 2D\delta_{jk}\,\mathrm{d}t \tag{2.2}$$

where $\mathrm{d}\eta_j(t) = \eta_j(t + \mathrm{d}t) - \eta_j(t)$ (for $\mathrm{d}t > 0$), D is the diffusion coefficient, and $\boldsymbol{E}_t$ are the conditional expectations with respect to $\xi(t)$. In what follows, for the sake of notational simplicity, we will limit ourselves to the case of one-dimensional trajectories, but the results that will be obtained can be immediately generalized to any number of dimensions. We will suppose, for the time being, that the forces will be defined by means of purely configurational potentials, possibly time-dependent $V(x, t)$. A suitable definition of the Lagrangian and of the stochastic action functional for the system described by the dynamical variables ρ and $v_{(+)}$ allows one to select the processes which reproduce the correct quantum dynamics [2, 3]. In fact, while the probability density $\rho(x, t)$ satisfies, as usual, the forward Fokker–Planck equation associated to the stochastic differential equation (2.1)

$$\partial_t \rho = D\partial_x^2 \rho - \partial_x(v_{(+)}\rho) = \partial_x(D\partial_x\rho - v_{(+)}\rho) \tag{2.3}$$

the choice for the Lagrangian field of

$$L(x, t) = \frac{m}{2}v_{(+)}^2(x, t) + mD\partial_x v_{(+)}(x, t) - V(x, t) \tag{2.4}$$

enables one to define a stochastic action functional

$$\mathcal{A} = \int_{t_0}^{t_1} \boldsymbol{E}[L(\xi(t), t)] \, \mathrm{d}t \tag{2.5}$$

which leads, through the stationarity condition $\delta\mathcal{A} = 0$, to the equation

$$\partial_t S + \frac{(\partial_x S)^2}{2m} + V - 2mD^2 \frac{\partial_x^2 \sqrt{\rho}}{\sqrt{\rho}} = 0. \tag{2.6}$$

The field $S(x, t)$ is defined as

$$S(x, t) = -\int_t^{t_1} \boldsymbol{E}[L(\xi(s), s)|\xi(t) = x] \, \mathrm{d}s + \boldsymbol{E}[S_1(\xi(t_1))|\xi(t) = x] \tag{2.7}$$

where $S_1(\cdot) = S(\cdot, t_1)$ is an arbitrary final condition. By introducing the function $R(x, t) \equiv \sqrt{\rho(x, t)}$ and the de Broglie ansatz

$$\psi(x, t) = R(x, t)\mathrm{e}^{\mathrm{i}S(x,t)/2mD} \tag{2.8}$$

equation (2.6) takes the form

$$\partial_t S + \frac{(\partial_x S)^2}{2m} + V - 2mD^2 \frac{\partial_x^2 R}{R} = 0 \tag{2.9}$$

and the complex function ψ satisfies the Schrödinger-like equation

$$\mathrm{i}(2mD)\partial_t \psi = \hat{H}\psi = -2mD^2 \partial_x^2 \psi + V\psi. \tag{2.10}$$

If the diffusion coefficient is chosen to be

$$D = \frac{\hbar}{2m} \tag{2.11}$$

we exactly recover the Schrödinger equation of quantum mechanics. Different choices of D also allow one to describe the effective quantum-like dynamics of more general systems.

On the other hand, if we start from the (one-dimensional) Schrödinger equation (2.10) with the de Broglie ansatz (2.8) and the diffusion coefficient (2.11), separating the real and the imaginary parts as usual in the hydrodynamical formulation [8], we recover equations (2.3) and (2.6) with $\rho = R^2 = |\psi|^2$ and the forward velocity field

$$v_{(+)}(x, t) = \frac{1}{m}\partial_x S + \frac{\hbar}{2m}\partial_x(\ln R^2). \tag{2.12}$$

3. The Sturm–Liouville problem and the solutions of the Fokker–Planck equation

Let us recall (see, for example, [11]) a few generalities about the techniques of solution of the Fokker–Planck equation with D and $v_{(+)}$ two time-independent continuous and differentiable functions defined for $x \in [a, b]$ and $t \geqslant t_0$, such that $D(x) > 0$, and $v_{(+)}(x)$ has no singularities in (a, b). The Fokker–Planck equation then reads

$$\partial_t \rho = \partial_x^2(D\rho) - \partial_x(v_{(+)}\rho) = \partial_x[\partial_x(D\rho) - v_{(+)}\rho]. \tag{3.1}$$

The conditions imposed on the probabilistic solutions are of course

$$\begin{aligned} \rho(x, t) &\geqslant 0 \qquad a < x < b \qquad t_0 \leqslant t \\ \int_a^b \rho(x, t) \, \mathrm{d}x &= 1 \qquad t_0 \leqslant t \end{aligned} \tag{3.2}$$

and from the form of (3.1) the second condition takes the form

$$[\partial_x(D\rho) - v_{(+)}\rho]_{a,b} = 0 \qquad t_0 \leqslant t. \tag{3.3}$$

Suitable initial conditions will be added to produce the required evolution: for example, the transition probability density $p(x, t|x_0, t_0)$ will be selected by the initial condition

$$\lim_{t \to t_0^+} \rho(x, t) = \rho_{in}(x) = \delta(x - x_0). \tag{3.4}$$

It is also possible to show by direct calculation that

$$h(x) = N^{-1} \mathrm{e}^{-\int [D'(x) - v_{(+)}(x)]/D(x)\,\mathrm{d}x} \qquad N = \int_a^b \mathrm{e}^{-\int [D'(x) - v_{(+)}(x)]/D(x)\,\mathrm{d}x}\,\mathrm{d}x \tag{3.5}$$

is always an invariant (time independent) solution of (3.1) satisfying the conditions (3.2) (here the prime symbol denotes differentiation). One should observe, however, that relation (3.1) is not in the standard self-adjoint form [12]; this fact notwithstanding, if we define the new function $g(x, t)$ by means of

$$\rho(x, t) = \sqrt{h(x)}\, g(x, t) \tag{3.6}$$

it is easy to show that $g(x, t)$ obeys an equation of the form

$$\partial_t g = \mathcal{L}g \tag{3.7}$$

where the operator $\mathcal{L}$ acting on positive normalizable functions $\varphi(x)$ and defined by

$$\mathcal{L}\varphi = \frac{\mathrm{d}}{\mathrm{d}x}\left[r(x) \frac{\mathrm{d}\varphi(x)}{\mathrm{d}x} \right] - q(x)\varphi(x) \tag{3.8}$$

with

$$\begin{aligned} r(x) &= D(x) > 0 \\ q(x) &= \frac{[D'(x) - v_{(+)}(x)]^2}{4D(x)} - \frac{[D'(x) - v_{(+)}(x)]'}{2} \end{aligned} \tag{3.9}$$

is now self-adjoint. By separating the variables by means of $g(x, t) = \gamma(t)G(x)$ we have $\gamma(t) = \mathrm{e}^{-\lambda t}$ while G must be a solution of a typical Sturm–Liouville problem associated to the equation

$$\mathcal{L}G(x) + \lambda G(x) = 0 \tag{3.10}$$

with the boundary conditions

$$\begin{aligned} [D'(a) - v_{(+)}(a)]G(a) + 2D(a)G'(a) &= 0 \\ [D'(b) - v_{(+)}(b)]G(b) + 2D(b)G'(b) &= 0. \end{aligned} \tag{3.11}$$

It is easy to see that $\lambda = 0$ is always an eigenvalue for the problem (3.10) with (3.11), and that the corresponding eigenfunction is $\sqrt{h(x)}$ as defined from (3.5).

For the differential problem (3.10) with (3.11) we have that [12] the simple eigenvalues λ_n will constitute an infinite, monotonically increasing sequence and the corresponding eigenfunction $G_n(x)$ will have n simple zeros in (a, b). This means that $\lambda_0 = 0$, corresponding to the eigenfunction $G_0(x) = \sqrt{h(x)}$ which never vanishes in (a, b), is the lowest eigenvalue, and that all the other eigenvalues are strictly positive. Moreover, the eigenfunctions will form a complete orthonormal set of functions in $L^2([a, b])$ [13]. As a consequence, the general solution of equation (3.1) satisfying the conditions (3.2) will have the form

$$\rho(x, t) = \sum_{n=0}^{\infty} c_n \mathrm{e}^{-\lambda_n t} \sqrt{h(x)} G_n(x) \tag{3.12}$$

with $c_0 = 1$ for normalization (remember that $\lambda_0 = 0$). The coefficients c_n for a particular solution are selected by an initial condition

$$\rho(x, t_0^+) = \rho_{in}(x) \tag{3.13}$$

and are then calculated from the orthonormality relations as

$$c_n = \int_a^b \rho_{in}(x) \frac{G_n(x)}{\sqrt{h(x)}} \, dx. \tag{3.14}$$

In particular, for the transition probability density we have from (3.4) that

$$c_n = \frac{G_n(x_0)}{\sqrt{h(x_0)}}. \tag{3.15}$$

Since $\lambda_0 = 0$ and $\lambda_n > 0$ for $n > 1$, the general solution (3.12) of (3.1) has a precise time evolution: all the exponential factors in (3.12) vanish as $t \to +\infty$ with the only exception being the term $n = 0$ which is constant, so that exponentially fast we will always have

$$\lim_{t \to +\infty} \rho(x, t) = c_0 \sqrt{h(x)} G_0(x) = h(x). \tag{3.16}$$

Therefore, the general solution will always relax in time toward the invariant solution $h(x)$. As a consequence the eigenvalues λ_n which solve the Sturm–Liouville problem define the physical time scales of the decay. By the structure of equations (3.7)–(3.11) we see that tuning the choice of the physical parameters that enter in the diffusion coefficient and in the forward velocity field allows for different sets of eigenvalues which define different sets of timescales. Hence, the rate of convergence can be fixed as to yield fast decay, slow decay, or even, on proper observational scales, quasi-metastable behaviours, according to what kind of physical evolution between quantum states one wants to realize. This point will be further discussed and elucidated in section 6.

4. Processes associated to stationary quantum states

Let us consider now a Schrödinger equation (2.10) with a time-independent potential $V(x)$ which gives rise to a purely discrete spectrum and bound, normalizable states. Let us introduce the following notations for stationary states, eigenvalues and eigenfunctions:

$$\psi_n(x, t) = \phi_n(x) e^{-iE_n t/\hbar}$$
$$\hat{H}\phi_n = -\frac{\hbar^2}{2m}\phi_n'' + V\phi_n = E_n\phi_n. \tag{4.1}$$

Taking into account relation (2.11), the previous eigenvalue equation can also be recast in the following form:

$$D\phi_n'' = \frac{V - E_n}{\hbar}\phi_n. \tag{4.2}$$

For these stationary states the probability densities are the time-independent, real functions

$$\rho_n(x) = |\psi_n(x, t)|^2 = \phi_n^2(x) \tag{4.3}$$

while the phase and the amplitude of ψ_n from (2.8) are

$$S_n(x, t) = -E_n t \qquad R_n(x, t) = \phi_n(x) \tag{4.4}$$

so that the associated velocity fields from (2.12) are

$$v_{(+)n}(x) = 2D\frac{\phi_n'(x)}{\phi_n(x)}. \tag{4.5}$$

Each $v_{(+)n}$ is time independent and presents singularities in the nodes of the associated eigenfunction. Since the nth eigenfunction of a quantum system with bound states has exactly n simple nodes $x_1, \ldots, x_n$, the coefficients of the Fokker–Planck equation (2.3) are not defined in these n points and it is necessary to solve it in separate intervals by imposing the correct

boundary conditions connecting the different sectors. In fact, these singularities effectively separate the real axis in $n + 1$ sub-intervals with walls impenetrable to the probability current. Hence the process will not have a unique invariant measure and will never cross the boundaries fixed by the singularities of $v_{(+)}(x)$: if the process starts in one of the sub-intervals, it will always remain there [14].

As a consequence, the normalization integral (3.2) (with $a = -\infty$ and $b = +\infty$) is the sum of $n + 1$ integrals over the sub-intervals $[x_k, x_{k+1}]$ with $k = 0, 1, \ldots, n$ (where we understand, to unify the notation, that $x_0 = -\infty$ and $x_{n+1} = +\infty$). Hence, for $n \geqslant 1$ equation (2.3) must be restricted in each interval $[x_k, x_{k+1}]$ with the integrals

$$\int_{x_k}^{x_{k+1}} \rho(x, t) \, dx \tag{4.6}$$

constrained to a constant value for $t \geqslant t_0$. This constant is not, in general, equal to one (only the sum of these $n + 1$ integrals amounts to one) and, since the separate intervals cannot communicate, it will be fixed by the choice of the initial conditions. Therefore, due to the singularities appearing in the forward velocity fields $v_{(+)n}$ for $n \geqslant 1$, we deal with a Fokker–Planck problem with barriers. The boundary conditions associated to (2.3) then require the conservation of probability in each sub-interval $[x_k, x_{k+1}]$, i.e. the vanishing of the probability current at the end points of the interval:

$$[D\partial_x \rho - v_{(+)}\rho]_{x_k, x_{k+1}} = 0 \qquad t \geqslant t_0. \tag{4.7}$$

To obtain a particular solution one must specify the initial conditions. In particular, we are interested in the transition probability density $p(x, t|x_0, t_0)$, which is singled out by the initial condition (3.4), because the asymptotic convergence in L^1 of the solutions of equation (2.3) is ruled by the asymptotic behaviour of $p(x, t|x_0, t_0)$ through the Chapman–Kolmogorov equation [1]

$$\rho(x, t) = \int_{-\infty}^{+\infty} p(x, t|y, t_0)\rho(y, t_0^+) \, dy. \tag{4.8}$$

It is clear, at this point, that in every interval $[x_k, x_{k+1}]$ (both finite or infinite) we can write the solution of equation (2.3) along the guidelines sketched in section 3. We must only keep in mind that in $[x_k, x_{k+1}]$ we already know the invariant, time-independent solution $\phi_n^2(x)$ which is never zero inside the interval itself, with the exception of the end points x_k and x_{k+1}. Hence, as we have seen in the general case, with the position

$$\rho(x, t) = \phi_n(x)g(x, t) \tag{4.9}$$

we can reduce (2.3) to the form

$$\partial_t g = \mathcal{L}_n g \tag{4.10}$$

where $\mathcal{L}_n$ is now the self-adjoint operator defined on $[x_k, x_{k+1}]$ by

$$\mathcal{L}_n \varphi(x) = \frac{d}{dx}\left[r(x)\frac{d\varphi(x)}{dx}\right] - q_n(x)\varphi(x) \tag{4.11}$$

with

$$r(x) = D > 0 \qquad q_n(x) = \frac{v_{(+)n}^2(x)}{4D} + \frac{v_{(+)n}'(x)}{2}. \tag{4.12}$$

Equation (4.10) is solved by separating the variables, so that we immediately have $\gamma(t) = e^{-\lambda t}$ while the spatial part $G(x)$ of g must be the solution of

$$\mathcal{L}_n G(x) + \lambda G(x) = 0 \tag{4.13}$$

with the boundary conditions

$$[2DG'(x) - v_{(+)n}(x)G(x)]_{x_k,x_{k+1}} = 0. \tag{4.14}$$

The general behaviour of the solutions of this Sturm–Liouville problem obtained as expansions in the system of the eigenfunctions of (4.13) has already been discussed in section 3. In particular, we deduce from (3.12) that for the stationary quantum states (more precisely, in every sub-interval defined by two subsequent nodes) all the solutions of (2.3) always converge in time toward the correct quantum solution $|\phi_n|^2$. As a further consequence, any quantum solution ϕ_n^2 defined on the entire interval $(-\infty, +\infty)$ will be stable under deviations from its initial condition.

5. An explicit example: the harmonic oscillator

To provide an explicit evolution of the probability and the transition probability densities of stochastic mechanics, we consider, in detail, the example of a harmonic oscillator associated to the potential

$$V(x) = \frac{m}{2}\omega^2 x^2 \tag{5.1}$$

with energy eigenvalues

$$E_n = \hbar\omega(n + \tfrac{1}{2}) \qquad n = 0, 1, 2\ldots. \tag{5.2}$$

Introducing the notation

$$\sigma_0^2 = \frac{\hbar}{2m\omega} \tag{5.3}$$

the time-independent part of the eigenfunctions (4.1) reads

$$\phi_n(x) = \frac{1}{\sqrt{\sigma_0\sqrt{2\pi}\,2^n n!}}e^{-x^2/4\sigma_0^2}H_n\left(\frac{x}{\sigma_0\sqrt{2}}\right) \tag{5.4}$$

where H_n are the Hermite polynomials. The corresponding forward velocity fields for the lowest-lying levels are

$$\begin{aligned} v_{(+)0}(x) &= -\omega x \\ v_{(+)1}(x) &= 2\frac{\omega\sigma_0^2}{x} - \omega x \\ v_{(+)2}(x) &= 4\omega\sigma_0^2\frac{x}{x^2 - \sigma_0^2} - \omega x \end{aligned} \tag{5.5}$$

with singularities in the zeros of the Hermite polynomials. When $n = 0$ equation (2.3) takes the form

$$\partial_t \rho = \omega\sigma_0^2\partial_x^2\rho + \omega x\,\partial_x\rho + \omega\rho \tag{5.6}$$

and the fundamental solution turns out to be the Ornstein–Uhlenbeck transition probability density

$$p_0(x, t|x_0, t_0) = \frac{1}{\sigma(t)\sqrt{2\pi}}e^{-[x-\alpha(t)]^2/2\sigma^2(t)} \qquad (t \geqslant t_0) \tag{5.7}$$

where we have introduced the notation

$$\alpha(t) = x_0 e^{-\omega(t-t_0)} \qquad \sigma^2(t) = \sigma_0^2\left[1 - e^{-2\omega(t-t_0)}\right] \qquad (t \geqslant t_0). \tag{5.8}$$

The stationary Markov process associated to the transition probability density (5.7) is selected by the initial, invariant probability density

$$\rho_0(x) = \frac{1}{\sigma_0\sqrt{2\pi}} e^{-x^2/2\sigma_0^2} \tag{5.9}$$

which is also the asymptotic probability density for every other initial condition when the evolution is ruled by equation (5.6) (see [1]) so that the invariant distribution also plays the role of the limit distribution. Since this invariant probability density also coincides with the quantum one $\phi_0^2 = |\psi_0|^2$, the process associated by stochastic mechanics to the ground state of the harmonic oscillator is nothing but the stationary Ornstein–Uhlenbeck process.

For $n \geqslant 1$ the solutions of (2.3) are determined in the following way. As discussed in the previous section, one has to solve the eigenvalue problem (4.13) which can now be written as

$$-\frac{\hbar^2}{2m} G''(x) + \left(\frac{m}{2}\omega^2 x^2 - \hbar\omega\frac{2n+1}{2} \right) G(x) = \hbar\lambda G(x) \tag{5.10}$$

in every interval $[x_k, x_{k+1}]$ between two subsequent singularities of the forward velocity fields $v_{(+)n}$. The boundary conditions at the end points of these intervals, deduced from (4.14) through (4.5), are

$$[\phi_n G' - \phi_n' G]_{x_k, x_{k+1}} = 0. \tag{5.11}$$

Remembering that ϕ_n (but not ϕ_n') vanishes in x_k, x_{k+1}, the conditions to impose are

$$G(x_k) = G(x_{k+1}) = 0 \tag{5.12}$$

where it is understood that for x_0 and x_{n+1} we mean, respectively

$$\lim_{x\to-\infty} G(x) = 0 \qquad \lim_{x\to+\infty} G(x) = 0. \tag{5.13}$$

At this point, it is also useful to state the eigenvalue problem in adimensional form by using the reduced eigenvalue $\mu = \lambda/\omega$, and the adimensional variable x/σ_0 which, by a slight abuse of notation, will still be denoted by x. In this way equation (5.10) with conditions (5.12) becomes

$$y''(x) - \left(\frac{x^2}{4} - \frac{2n+1}{2} - \mu \right) y(x) = 0$$
$$y(x_k) = y(x_{k+1}) = 0. \tag{5.14}$$

If μ_m and $y_m(x)$ are the eigenvalues and eigenfunctions of (5.14), the general solution of the corresponding Fokker–Planck equation (2.3) will be (reverting to dimensional variables)

$$\rho(x, t) = \sum_{m=0}^{\infty} c_m e^{-\mu_m \omega t} \phi_n(x) y_m \left(\frac{x}{\sigma_0} \right). \tag{5.15}$$

The values of the coefficients c_m will be fixed by the initial conditions and by the obvious requirements that $\rho(x, t)$ must be non-negative and normalized during the whole time evolution. Two linearly independent solutions of (5.14) are

$$y^{(1)} = e^{-x^2/4} M\left(-\frac{\mu+n}{2}, \frac{1}{2}; \frac{x^2}{2} \right) \qquad y^{(2)} = x e^{-x^2/4} M\left(-\frac{\mu+n-1}{2}, \frac{3}{2}; \frac{x^2}{2} \right) \tag{5.16}$$

where $M(a, b; z)$ are the confluent hypergeometric functions. The complete specification of the solutions obviusly requires the knowledge of all the eigenvalues μ_m.

We consider first the instance $n = 1$ ($x_0 = -\infty$, $x_1 = 0$ and $x_2 = +\infty$), which can be completely solved [5]. In this case equation (5.14) must be solved separately for $x \leqslant 0$ and for $x \geqslant 0$ with the boundary conditions $y(0) = 0$ and

$$\lim_{x\to-\infty} y(x) = \lim_{x\to+\infty} y(x) = 0. \tag{5.17}$$

A long calculation [5] shows that the transition probability density is now

$$p(x, t|x_0, t_0) = \frac{x}{\alpha(t)} \frac{e^{-[x-\alpha(t)]^2/2\sigma^2(t)} - e^{-[x+\alpha(t)]^2/2\sigma^2(t)}}{\sigma(t)\sqrt{2\pi}} \tag{5.18}$$

where $\alpha(t)$ and $\sigma^2(t)$ are defined in (5.8). However, it must be noted that (5.18) must be considered as restricted to $x \geqslant 0$ when $x_0 > 0$ and to $x \leqslant 0$ when $x_0 < 0$, and that only on these intervals is it suitably normalized. In order to take into account both these possibilities we can introduce the Heaviside function $\Theta(x)$ so that for every $x_0 \neq 0$ we will have

$$p_1(x, t|x_0, t_0) = \Theta(xx_0) \frac{x}{\alpha(t)} \frac{e^{-[x-\alpha(t)]^2/2\sigma^2(t)} - e^{-[x+\alpha(t)]^2/2\sigma^2(t)}}{\sigma(t)\sqrt{2\pi}}. \tag{5.19}$$

From equation (4.8) we can deduce the evolution of every other initial probability density. In particular, it can be shown that, with $\rho_1(x) = \phi_1^2(x)$

$$\lim_{t\to+\infty} p_1(x, t|x_0, t_0) = 2\Theta(xx_0)\frac{x^2 e^{-x^2/2\sigma_0^2}}{\sigma_0^3\sqrt{2\pi}} = 2\Theta(xx_0)\rho_1(x). \tag{5.20}$$

Hence, if $\rho(x, t_0^+) = \rho_{in}(x)$ is the initial probability density, we have for $t > t_0$

$$\lim_{t\to+\infty} \rho(x, t) = \lim_{t\to+\infty} \int_{-\infty}^{+\infty} p(x, t|y, t_0)\rho_{in}(y)\,\mathrm{d}y$$

$$= 2\phi_1^2(x) \int_{-\infty}^{+\infty} \Theta(xy)\rho_{in}(y)\,\mathrm{d}y = \Gamma(\epsilon; x)\rho_1(x) \tag{5.21}$$

where we have defined the function

$$\Gamma(\epsilon; x) = \epsilon\Theta(x) + (2-\epsilon)\Theta(-x) \qquad \epsilon = 2\int_0^{+\infty} \rho_{in}(y)\,\mathrm{d}y. \tag{5.22}$$

When $\epsilon = 1$ (with symmetric initial probability, equally shared on the two real semi-axes) we have $\Gamma(1; x) = 1$ and the asymptotical probability density coincides with the quantum stationary density $\rho_1(x) = \phi_1^2(x)$. If, on the other hand, $\epsilon \neq 1$ the asymptotic probability density has the same shape of $\phi_1^2(x)$ but with different weights on the two semi-axes.

If we consider the higher excited states, the Sturm–Liouville problem (5.14) must be solved numerically in each sub-interval. For instance, in the case $n = 2$ we have $x_0 = -\infty$, $x_1 = -1$, $x_2 = 1$ and $x_3 = +\infty$. Considering, in particular, the sub-interval $[-1, 1]$, it can be shown that beyond $\mu_0 = 0$ the first few eigenvalues are determined as the first possible values such that

$$M\left(-\frac{\mu+1}{2}, \frac{3}{2}; \frac{1}{2}\right) = 0. \tag{5.23}$$

This gives $\mu_1 \sim 7.44$, $\mu_2 \sim 37.06$, $\mu_3 \sim 86.41$.

6. Controlled evolutions

In this section we move on to implement the programme outlined in the introduction, that is to exploit the transition probabilities of Nelson stochastic mechanics to model controlled quantum evolutions to arbitrarily assigned final quantum states. We start by observing that to every solution $\rho(x, t)$ of the Fokker–Planck equation (3.1), with a given $v_{(+)}(x, t)$ and constant diffusion coefficient (2.11), we can always associate the wavefunction of a quantum system. To this end, it is sufficient to introduce a suitable time-dependent potential.

Let us take a solution $\rho(x,t)$ of the Fokker–Planck equation (3.1), with a given $v_{(+)}(x,t)$ and a constant diffusion coefficient D: introduce the functions $R(x,t)$ and $W(x,t)$ from

$$\rho(x,t) = R^2(x,t) \qquad v_{(+)}(x,t) = \partial_x W(x,t) \tag{6.1}$$

and rememeber from (2.12) that the relation

$$mv_{(+)} = \partial_x S + \hbar \frac{\partial_x R}{R} = \partial_x S + \frac{\hbar}{2}\frac{\partial_x \rho}{\rho} = \partial_x \left(S + \frac{\hbar}{2}\ln \tilde{\rho} \right) \tag{6.2}$$

must hold, where $\tilde{\rho}$ is an adimensional function (argument of a logarithm) obtained from the probability density ρ by means of a suitable and arbitrary dimensional multiplicative constant. If we now impose that the function $S(x,t)$ must be the phase of a wavefunction as in (2.8), we immediately obtain from (6.1) and (6.2) the equation

$$S(x,t) = mW(x,t) - \frac{\hbar}{2}\ln \tilde{\rho}(x,t) - \theta(t) \tag{6.3}$$

which allows one to determine S from ρ and $v_{(+)}$ (namely W) up to an additive arbitrary function of time $\theta(t)$. However, in order that the wavefunction (2.8) with R and S given above be a solution of a Schrödinger problem in quantum mechanics, we must also make sure that the Hamilton–Jacobi–Madelung equation (2.9) is satisfied. Since S and R are now fixed, equation (2.9) must be considered as a relation (constraint) defining the controlling potential V_c, which, after straightforward calculations, yields

$$V_c(x,t) = \frac{\hbar^2}{4m}\partial_x^2 \ln \tilde{\rho} + \frac{\hbar}{2}(\partial_t \ln \tilde{\rho} + v_{(+)}\partial_x \ln \tilde{\rho}) - \frac{mv_{(+)}^2}{2} - m\partial_t W + \dot{\theta}. \tag{6.4}$$

Of course, if we start with a quantum wavefunction $\psi(x,t)$ associated to a given time-independent potential $V(x)$ and we select as a solution of (2.3) exactly $\rho = |\psi|^2$, then formula (6.4) always yields back the given potential, as it should. This can be explicitly seen (to become familiar with this kind of approach) in the examples of the ground state and the first excited state of the harmonic oscillator potential (5.1), by choosing, respectively, in equation (6.4) $\theta(t) = \hbar\omega t/2$ and $\theta(t) = 3\hbar\omega t/2$, which amounts to suitably fixing the zero of the potential energy.

On the other hand, the nonstationary fundamental solution (5.7) associated to the velocity field $v_{(+)0}(x)$ of (5.5) for the case $n = 0$ (we put $t_0 = 0$ to simplify the notation) does not correspond to a quantum wavefunction of the harmonic oscillator whatsoever. However, it is easy to show that, by choosing

$$\dot{\theta}(t) = \frac{\hbar\omega}{2}\left(\frac{2\sigma_0^2}{\sigma^2(t)} - 1 \right) = \frac{\hbar\omega}{2}\frac{1}{\tanh \omega t} \to \frac{\hbar\omega}{2} \qquad (t \to +\infty) \tag{6.5}$$

and the time-dependent controlling potential

$$V_c(x,t) = \frac{\hbar\omega}{2}\left[\frac{x - \alpha(t)}{\sigma(t)} \right]^2 \frac{\sigma_0^2}{\sigma^2(t)} - \frac{m\omega^2 x^2}{2} \to \frac{m\omega^2 x^2}{2} \qquad (t \to +\infty) \tag{6.6}$$

with $\alpha(t)$, $\sigma(t)$ and σ_0 defined in equations (5.8) and (5.3), we can define a quantum state, i.e. a wavefunction $\psi_c(x,t)$ solution of the Schrödinger equation in the potential (6.6). At the same time ψ_c is associated to the transition probability density of the form (5.7) which is its modulus squared. Of course, the fact that for $t \to +\infty$ we recover the harmonic potential is connected to the already stated fact that the usual quantum probability density is also the limit distribution for every initial condition and, in particular, also for the evolution (5.7). In the case $n = 1$, with the $v_{(+)1}(x)$ as given by equation (5.5) and the transition probability density (5.19), we define

$$T(x) = \frac{x}{\tanh x} \tag{6.7}$$

and then we choose

$$\dot{\theta}(t) = \frac{\hbar\omega}{2}\left(\frac{4\sigma_0^2}{\sigma^2(t)} - \frac{2\sigma_0^2\alpha^2(t)}{\sigma^4(t)} - 1\right) \to \frac{3}{2}\hbar\omega \qquad (t \to +\infty) \tag{6.8}$$

so that we have the following time-dependent controlling potential (for every $x \neq 0$):

$$V_c(x,t) = \frac{m\omega^2 x^2}{2}\left(\frac{2\sigma_0^4}{\sigma^4} - 1\right) + \hbar\omega\left[1 - \frac{\sigma_0^2}{\sigma^2}T\left(\frac{x\alpha}{\sigma^2}\right)\right] - \frac{\hbar^2}{4mx^2}\left[1 - T\left(\frac{x\alpha}{\sigma^2}\right)\right]$$

$$\to \frac{m\omega^2 x^2}{2} \qquad (t \to +\infty). \tag{6.9}$$

The limit $t \to +\infty$ must obviously be intended in a physical sense, i.e. for times much longer than λ_1^{-1}, the largest characteristic time decay in expansion (3.12). In this particular case $\lambda_1 = \omega$. In fact, here the asymptotic potential is also the usual one of the harmonic oscillator, but it must be considered separately on the positive and negative x semi-axes, since at the point $x = 0$ a singular behaviour would show up when $t \to 0$. This means that, if we also asymptotically recover the right potential, it will be associated with a new boundary condition in $x = 0$ since the system must be confined on one of the two semi-axes.

7. Modelling transitions

Given any couple $(\rho, v_{(+)})$ associated to a Fokker–Planck equation, the possibility of promoting it to a solution of a Schrödinger problem by a suitable controlling potential $V_c(x,t)$ enables one to model quantum evolutions driving, for instance, the probability density of a given quantum stationary state to another (decays and excitations). Moreover, an immediate generalization of this scheme might open the way to modelling evolutions from a given, arbitrary quantum state to an eigenfunction of a given observable. As a first example let us consider the transition between the invariant probability densities associated to the ground and the first excited state of the harmonic oscillator potential (5.1):

$$\rho_0(x) = \phi_0^2(x) = \frac{1}{\sigma_0\sqrt{2\pi}}e^{-x^2/2\sigma_0^2}$$
$$\rho_1(x) = \phi_1^2(x) = \frac{x^2}{\sigma_0^3\sqrt{2\pi}}e^{-x^2/2\sigma_0^2}. \tag{7.1}$$

If we choose to describe the decay $\phi_1 \to \phi_0$ we may exploit the Chapman–Kolmogorov equation (4.8) with the transition probability density (5.7), and the initial probability density $\rho_1(x)$. In this case, an elementary integration shows that the resulting evolution takes the form $(t_0 = 0)$

$$\rho_{1\to 0}(x,t) = \beta^2(t)\rho_0(x) + \gamma^2(t)\rho_1(x) \tag{7.2}$$

where

$$\beta^2(t) = 1 - e^{-2\omega t} \qquad \gamma(t) = e^{-\omega t}. \tag{7.3}$$

Recalling $v_{(+)0}(x)$ as given in (5.5) and the evolving probability density (7.2), and inserting them in equation (6.4) we obtain the following form of the controlling potential:

$$V_c(x,t) = \frac{m\omega^2 x^2}{2} - 2\hbar\omega U(x/\sigma_0; \beta/\gamma) \tag{7.4}$$

where

$$U(x;b) = \frac{x^4 + b^2 x^2 - b^2}{(b^2 + x^2)^2}. \tag{7.5}$$

The parameter

$$b^2(t) = \frac{\beta^2(t)}{\gamma^2(t)} = e^{2\omega t} - 1 \tag{7.6}$$

is such that $b^2(0^+) = 0$ and $b^2(+\infty) = +\infty$. Thus U goes to zero as $t \to +\infty$ for any x, and as $t \to 0^+$ is one for every x, except for a negative singularity in $x = 0$. As a consequence, while for $t \to +\infty$ the controlling potential (7.4) simply tends to the potential (5.1), for $t \to 0^+$ it presents an unessential shift of $-2\hbar\omega$ in the zeroth level, and a deep negative singularity in $x = 0$.

The singular behaviour of the controlling potential at the initial time of the evolution is a problem connected to the proper definition of the phase function S. In fact, from (6.3) we have

$$S(x, t) = -\frac{\hbar}{2} \ln\left[\beta^2(t) + \frac{x^2}{\sigma_0^2}\gamma^2(t)\right] - \frac{\hbar\omega}{2}t \tag{7.7}$$

so that in particular we have

$$S(x, 0^+) = -\frac{\hbar}{2} \ln\frac{x^2}{\sigma_0^2}. \tag{7.8}$$

Instead, we would have expected that initially the phase be independent of x as for every stationary wavefunction. This means that in the constructed evolution $S(x, t)$ presents a discontinuous behaviour for $t \to 0^+$. The problem arises here from the fact that we initially have a stationary state characterized by a probability density $\rho_1(x)$ and a velocity field $v_{(+)1}(x)$, and then suddenly, in order to activate the decay, we impose that the same ρ_1 be embedded in a different velocity field $v_{(+)0}(x)$ which drags it toward a new stationary $\rho_0(x)$. This discontinuous change from $v_{(+)1}$ to $v_{(+)0}$ is, of course, responsible for the remarked discontinuous change in the phase of the wavefunction. We have therefore modelled a transition which starts with a sudden, discontinuous kick. At present, this only seems to have a mathematical meaning since it would be difficult to either implement it physically or to simulate it numerically. However, in many instances discontinuous models can be relevant as simplification of more complicated processes (as, for example, in rigid, instantaneous classical collisions disregarding interaction details): here, in particular, an impulsive external field turned on very quickly could well approximate our instantaneous interaction. Moreover, we claim that it would also be possible to construct transitions that evolve smoothly for $t \to 0^+$ by taking into account a continuous and smooth modification of the initial velocity field into the final one. This requirement would compel us to consider a new class of Fokker–Planck equations with time-dependent forward velocity fields $v_{(+)}(x, t)$. In particular, to achieve the proposed smooth controlled decay between two stationary states of the harmonic oscillator, we should solve an evolution equation with a continuous velocity field $v_{(+)}(x, t)$ which evolves smoothly from $v_{(+)1}(x)$ to $v_{(+)0}(x)$. Clearly, the smoothing procedure can be realized in several different ways and the selection must be dictated by the actual physical requirements and outputs one is interested in. A suitable smoothing for our transitions which leads to manageable equations still has to be found; however, in the following section we will study a problem in which the smoothness of the evolution is *a priori* granted.

8. Smooth transitions: coherent and squeezed wavepackets

As anticipated at the end of the previous section we will now consider an instance of controlled evolution that does not require an extra smoothing procedure for the driving velocity field, i.e. the transition between pairs of coherent wavepackets. In particular, we will consider both the

transition from a coherent oscillating packet (coherent state) to the ground state of the same harmonic oscillator, and a dynamical procedure of squeezing a coherent wavepacket.

To this end we will recall a simple result [1] which indicates how to find the solutions of a particular class of evolution equations (2.3) which includes the situation of our proposed examples. If the velocity field of the evolution equation (2.3) has the linear form

$$v_{(+)}(x, t) = A(t) + B(t)x \tag{8.1}$$

with $A(t)$ and $B(t)$ continuous functions of time, then there are always solutions of the Fokker–Planck equation in the normal form $\mathcal{N}(\mu(t), v(t))$, where $\mu(t)$ and $v(t)$ are solutions of the differential equations

$$\dot{\mu}(t) - B(t)\mu(t) = A(t) \qquad \dot{v}(t) - 2B(t)v(t) = 2D \tag{8.2}$$

with suitable initial conditions. The first case that we consider is the coherent wavepacket with a certain initial displacement a:

$$\psi(x, t) = \left(\frac{1}{2\pi \sigma_0^2} \right)^{1/4} \exp\left[-\frac{(x - a\cos\omega t)^2}{4\sigma_0^2} - i\left(\frac{4ax\sin\omega t - a^2 \sin 2\omega t}{8\sigma_0^2} + \frac{\omega t}{2} \right) \right] \tag{8.3}$$

whose forward velocity field reads

$$v_{(+)}(x, t) = a\omega(\cos\omega t - \sin\omega t) - \omega x. \tag{8.4}$$

The field (8.4) is of the required form (8.1), with $A(t) = a\omega(\cos\omega t - \sin\omega t)$ and $B(t) = -\omega$, while the configurational probability density is

$$\rho(x, t) = |\psi(x, t)|^2 = \rho_0(x - a\cos\omega t) \tag{8.5}$$

with ρ_0 that one of the ground state of the harmonic oscillator given by (7.1). It is easy to show that when $B(t) = -\omega$, as in the case of the wavepackets we are considering, there are coherent solutions of (2.3) with $v(t) = \sigma_0^2$ of the form $\mathcal{N}(\mu(t), \sigma_0^2)$, i.e. of the form

$$\rho(x, t) = \rho_0(x - \mu(t)). \tag{8.6}$$

Now, the time evolution of such coherent solutions can be determined in one step, without implementing the two-step procedure of first calculating the transition probability density and then, through the Chapman–Kolmogorov equation, the evolution of an arbitrary initial probability density. On the other hand, if we compare (5.5) and (8.4) we see that the difference between $v_{(+)0}$ and $v_{(+)}$ consists in the first, time-dependent term of the latter; hence it is natural to consider the problem of solving evolution equation (2.3) with a velocity field of the type

$$v_{(+)}(x, t) = A(t) - \omega x$$
$$A(t) = a\omega(\cos\omega t - \sin\omega t)F(t) \tag{8.7}$$

where $F(t)$ is an arbitrary function varying smoothly between 1 and 0, or vice versa. In this case evolution equation (2.3) still has coherent solutions of the form (8.6) with a $\mu(t)$ dependent on our choice of $F(t)$ through equation (8.2).

A completely smooth transition from the coherent, oscillating wavefunction (8.3) to the ground state ϕ_0 (5.4) of the harmonic oscillator can now be achieved, for example, by means of the following choice of the function $F(t)$:

$$F(t) = 1 - (1 - e^{-\Omega t})^N = \sum_{k=1}^{N} (-1)^{k+1} \binom{N}{k} e^{-\omega_k t} \tag{8.8}$$

where

$$\Omega = \frac{\ln N}{\tau} \qquad \omega_k = k\Omega \qquad \tau > 0 \qquad N \geqslant 2. \tag{8.9}$$

In fact, a function $F(t)$ of this form goes monotonically from $F(0) = 1$ to $F(+\infty) = 0$ with a flex point in τ (which can be considered as the arbitrary instant of the transition) where its derivative $F'(\tau)$ is negative and grows, in absolute value, logarithmically with N. The condition that the exponent $N \geqslant 2$ also guarantees that $F'(0) = 0$, and hence that the controlling potential $V_c(x, t)$ given in equation (6.4) will continuously start at $t = 0$ from the harmonic oscillator potential (5.1), and asymptotically return to it for $t \to +\infty$. Finally, the phase function $S(x, t)$ will also change continuously from that of ψ given in (8.3) to that of the harmonic oscillator ground state ψ_0. A long calculation yields the explicit form of the controlling potential:

$$V_c(x, t) = m\omega^2 \frac{x^2}{2} - m\omega ax \sum_{k=1}^{N} (-1)^{k+1} \binom{N}{k} [U_k(t)\omega_k e^{-\omega_k t} - W_k \omega e^{-\omega t}] \tag{8.10}$$

where

$$U_k(t) = \sin \omega t + \frac{2\omega^2 \sin \omega t - \omega_k^2 \cos \omega t}{(\omega_k - \omega)^2 + \omega^2}$$
$$W_k = 1 + \frac{2\omega^2 - \omega_k^2}{(\omega_k - \omega)^2 + \omega^2} = \sqrt{2} U_k \left(\frac{\pi}{4\omega} \right). \tag{8.11}$$

The parameters τ and N, apart the constraints (8.9), are free and can be fixed by the particular form of the transition that we want to implement, according to what specific physical situations we are interested in. Finally we remark that, in a harmonic oscillator, the transition between a coherent, oscillating wavepacket and the ground state is a transition between a (Poisson) superposition of all the energy eigenstates to just one energy eigenstate: an outcome which, at first sight, looks similar to that of an energy measurement. However, here the result (the energy eigenstate) is deterministically controlled by a time-dependent potential. In fact, the controlled transition that we have constructed does not produce mixtures, but pure states (eigenstates). Moreover, all these different final states are not realized by just one apparatus, as in the usual quantum measurement theory: indeed, here the Hamiltonian depends on the desired outcome, and thus, the 'measurement apparatus' would have to be chosen differently for different outcomes.

Until now we have considered transitions between Gaussian wavepackets with constant width. However, it is also of great interest to discuss the case of controlling potentials able to produce a wavepacket evolution with variable width: a kind of controlled squeezing of the wavepacket. This could be very useful in instances such as the shaping of the Gaussian output in the manufacturing of molecular reactions, or in the design of focusing devices for beams in particle accelerators, in which the width of the bunch must be properly squeezed. We will now discuss a simple case which also shows that, in the particular conditions chosen, it is also possible to avoid the integration of the differential equations (8.2).

Let us remember that when the forward velocity field has the form (8.1) the Fokker–Planck equation (2.3) always possesses Gaussian solutions of the form

$$\rho(x, t) = \frac{e^{-[x-\mu(t)]^2/2\nu(t)}}{\sqrt{2\pi \nu(t)}} \tag{8.12}$$

if $\mu(t)$ and $\nu(t)$ are solutions of (8.2). We now plan to describe evolutions of the quantum state (2.8) such that (1) both $V_c(x, t)$ and $S(x, t)$ are continuous and regular at every instant, and (2) the variance $\nu(t)$ satisfy the relations

$$\nu(-\infty) = \sigma_0^2 \qquad \nu(+\infty) = \sigma_1^2. \tag{8.13}$$

In practice this means that, if for example we require for the sake of simplicity $\mu(t) = 0$ at every time, we will describe a transition from the ground state of a harmonic oscillator

with frequency $\omega_0 = D/\sigma_0^2$ to the ground state of another harmonic oscillator with frequency $\omega_1 = D/\sigma_1^2$. It is convenient to remark here that this very simple transition cannot be achieved by means of an arbitrary time-dependent potential $V_c(x, t)$, given that it goes from $m\omega_0^2 x^2/2$ for $t \to -\infty$ to $m\omega_1^2 x^2/2$ for $t \to +\infty$. The intermediate evolution, indeed, when not suitably designed, would introduce components of every other energy eigenstate of the final harmonic oscillator which will not, in general, asymptotically disappear.

Let us recall here that the relevant quantities are the phase function

$$S(x, t) = mW(x, t) - mD \ln \tilde{\rho}(x, t) - \theta(t) \tag{8.14}$$

(where $\theta(t)$ is arbitrary and, from (6.1) and (8.1), $W(x, t) = A(t)x + B(t)x^2/2$), and the controlling potential

$$V_c(x, t) = mD^2 \, \partial_x^2 \ln \tilde{\rho} + mD(\partial_t \ln \tilde{\rho} + v_{(+)}\partial_x \ln \tilde{\rho}) - \frac{mv_{(+)}^2}{2} - m\partial_t W + \dot{\theta}. \tag{8.15}$$

Both these two functions are determined from the knowledge of the forward velocity field $v_{(+)}(x, t)$ and of the adimensional density $\tilde{\rho}(x, t) = \sigma_0 \rho(x, t)$. However, in this coherent evolution it will not be necessary to integrate the differential equations (8.2) to obtain an explicit form of S and V_c. Indeed, since $A(t)$, $B(t)$ and $\ln \rho(x, t)$ can be expressed through (8.2) in terms of $\mu(t)$, $\nu(t)$, $\dot{\mu}(t)$, $\dot{\nu}(t)$ and D, it is a straightforward matter to show that the phase is of the general form

$$S(x, t) = \frac{m}{2}[\Omega(t)x^2 - 2U(t)x + \Delta(t)] \tag{8.16}$$

with

$$\Omega(t) = \frac{\dot{\nu}(t)}{2\nu(t)}$$

$$U(t) = \frac{\mu(t)\dot{\nu}(t) - 2\nu(t)\dot{\mu}(t)}{2\nu(t)} \tag{8.17}$$

$$\Delta(t) = D\frac{\mu^2(t)}{\nu(t)} + D \ln \frac{2\pi \nu(t)}{\sigma_0^2} - \frac{2\theta(t)}{m}$$

while the controlling potential reads

$$V_c(x, t) = \frac{m}{2}[\omega^2(t)x^2 - 2a(t)x + c(t)] \tag{8.18}$$

where

$$\omega^2(t) = \frac{4D^2 - 2\nu(t)\ddot{\nu}(t) + \dot{\nu}^2(t)}{4\nu^2(t)}$$

$$a(t) = \frac{4D^2\mu(t) + 4\nu^2(t)\ddot{\mu}(t) - 2\mu(t)\nu(t)\ddot{\nu}(t) + \mu(t)\dot{\nu}^2(t)}{4\nu^2(t)} \tag{8.19}$$

$$c(t) = \frac{8D^2\mu^2(t) - 4D\nu(t)\dot{\nu}(t) - 8D^2\nu(t) - (2\nu(t)\dot{\mu}(t) - \mu(t)\dot{\nu}(t) + 2D\mu(t))}{4\nu^2(t)} + \frac{2\dot{\theta}(t)}{m}.$$

We can simplify our notation by imposing that $\mu(t) = 0$ (and hence $\dot{\mu}(t) = \ddot{\mu}(t) = 0$) for every t, obtaining

$$S(x, t) = \frac{m}{2}[\Omega(t)x^2 + \Delta(t)] \tag{8.20}$$

with

$$\Omega(t) = \frac{\dot{\nu}(t)}{2\nu(t)}$$

$$\Delta(t) = D \ln \frac{2\pi \nu(t)}{\sigma_0^2} - \frac{2\theta(t)}{m}. \tag{8.21}$$

The controlling potential also reduces to

$$V_c(x, t) = \frac{m}{2}[\omega^2(t)x^2 + c(t)] \tag{8.22}$$

where

$$\begin{aligned}
\omega^2(t) &= \frac{4D^2 - 2v(t)\ddot{v}(t) + \dot{v}^2(t)}{4v^2(t)} \\
c(t) &= \frac{2\dot{\theta}(t)}{m} - D\frac{v(t)\dot{v}(t) + 2Dv(t)}{v^2(t)}.
\end{aligned} \tag{8.23}$$

Hence the evolution is completely defined, through the four functions $\Omega(t)$, $\Delta(t)$, $\omega^2(t)$ and $c(t)$, by $\theta(t)$ and $v(t)$. In particular, it is expedient to choose

$$\theta(t) = \frac{mD}{2} \ln \frac{2\pi v(t)}{\sigma_0^2} + \frac{mD^2 t}{v(t)}. \tag{8.24}$$

In this way

$$\Delta(t) = -\frac{2D^2 t}{v(t)} \tag{8.25}$$

because from (8.13) we have $\dot{v}(\pm\infty) = 0$ so that (see (8.20)):

$$\begin{aligned}
S(x, t) &\sim -\frac{mD^2 t}{\sigma_0^2} \qquad t \to -\infty \\
S(x, t) &\sim -\frac{mD^2 t}{\sigma_1^2} \qquad t \to +\infty.
\end{aligned} \tag{8.26}$$

This was to be expected from the fact that $mD^2/\sigma_0^2 = \hbar\omega_0/2$ and $mD^2/\sigma_1^2 = \hbar\omega_1/2$ are the energy eigenvalues of the ground states of the two harmonic oscillators. Moreover, from the choice (8.24) it also follows that $c^2(\pm\infty) = 0$ so that the controlling potential (8.22) will show no asymptotical extra terms with respect to the initial and final harmonic potentials.

In order to completely specify the controlled evolution we are now left with the determination of the form of $v(t)$. If $b = \sigma_1/\sigma_0$, then take

$$v(t) = \sigma_0^2 \left(\frac{b + e^{-t/\tau}}{1 + e^{-t/\tau}}\right)^2 \qquad (\tau > 0) \tag{8.27}$$

so that the transition happens around the instant $t = 0$ and τ controls its velocity. We thus obtain the explicit expressions for the four functions (8.21) and (8.23):

$$\begin{aligned}
\Omega(t) &= \frac{b - 1}{\tau} \frac{e^{-t/\tau}}{(b + e^{-t/\tau})(1 + e^{-t/\tau})} \\
\Delta(t) &= -\frac{2D^2 t}{\sigma_0^2} \left(\frac{1 + e^{-t/\tau}}{b + e^{-t/\tau}}\right)^2 \\
\omega^2(t) &= \frac{D^2}{\sigma_0^4} \left(\frac{1 + e^{-t/\tau}}{b + e^{-t/\tau}}\right)^4 + \frac{b - 1}{\tau^2} \frac{e^{-t/\tau}(1 - e^{-t/\tau})}{(1 + e^{-t/\tau})^2(b + e^{-t/\tau})} \\
c(t) &= -\frac{4D^2(b - 1)}{\sigma_0^2} \frac{e^{-t/\tau}(1 + e^{-t/\tau})}{(b + e^{-t/\tau})^3} \frac{t}{\tau}.
\end{aligned} \tag{8.28}$$

Their form is displayed in the figures 1–4, where to fix an example we have chosen the values $\tau = 1, b = 2, \omega_0 = 1$ and $\sigma_0 = 1$ (as a consequence the units will be chosen so that $D = 1$). As can be seen, in this case the behaviour of the potential time-dependent parameters is not trivial even for the very simple squeezing of a static Gaussian wavepacket from a given variance to another. How to precisely follow this time dependence in a stable way will be the argument of a forthcoming paper, as discussed in the next section.
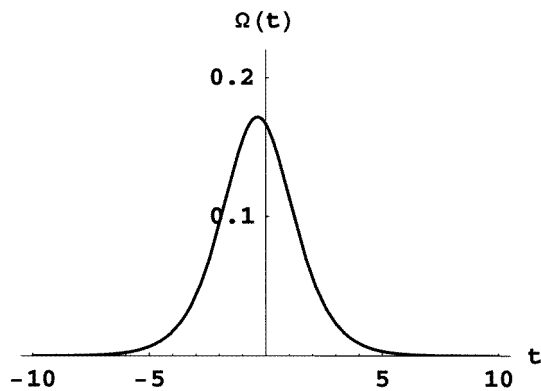
Figure 1. The parameter $\Omega(t)$ defined in (8.28) with $\tau = 1$ and $b = 2$. Dimensionally, it is a frequency and it appears in the phase $S(x, t)$ (8.20) of a wavefunction smoothly evolving between the two Gaussian distributions (8.12) with different variances (8.13). The unit system in these diagrams is chosen so that $D = \hbar/2m = 1$.
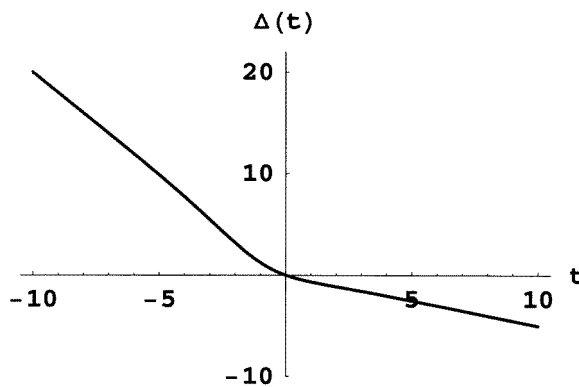

Figure 2. The parameter $\Delta(t)$ defined in (8.28) with $\tau = 1$, $b = 2$ and $\sigma_0 = 1$. Dimensionally, it is a diffusion coefficient and appears in the form (8.20) of the phase function $S(x, t)$.
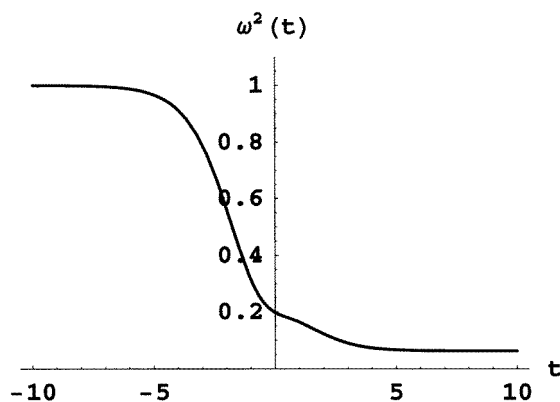
Figure 3. The parameter $\omega(t)$ from (8.28). Dimensionally it is a frequency and it determines the form of the control potential $V_c(x, t)$ in (8.22). Here too $\tau = 1$, $b = 2$ and $\sigma_0 = 1$. The template of its time evolution is a smoothed step allowing for a rather quick transition to a final harmonic potential with a new frequency.

9. Conclusions and outlook

We have shown how to treat the typical inverse problem in quantum control, i.e. that of determining a controlling potential for a given quantum evolution, in the framework of Nelson stochastic mechanics. In this way we have been able to determine the general characteristics of controlled evolutions between assigned initial and final quantum states. The solution techniques and the relation between the transition probabilities, phase functions and controlling potentials have been discussed on general grounds. Detailed, explicit calculations have also been shown in the paradigmatic test arena provided by the harmonic oscillator.
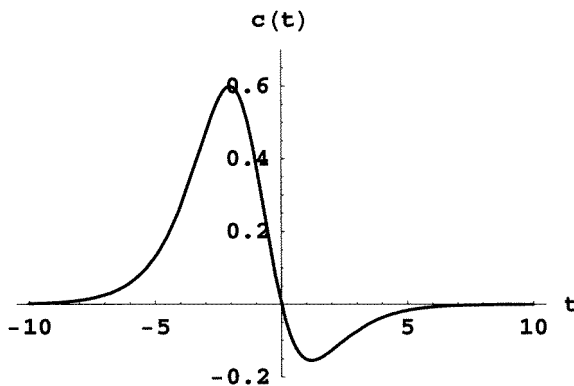
Figure 4. The parameter $c(t)$ from (8.28) dimensionally is the square of a velocity and appears in the potential $V_c(x, t)$ in (8.22). It allows for a time-dependent shift of the potential zero value and is plotted here for $\tau = 1$, $b = 2$ and $\sigma_0 = 1$.

Further extensions of the method outlined in the present paper are currently under study. One immediate application to be faced is the generalization of the analysis performed for the harmonic oscillator to anharmonic systems. The difficulty to be faced on the way toward this aim is that one is, in general, forced to deal with approximate quantum wavefunctions, as in the case of the quartic oscillator. Therefore, the controlled evolution must be supplemented by a suitable feedback mechanism ensuring that the error initially made in choosing a certain initial approximate state does not grow during the controlled time evolution. For example, self-consistent variational methods of Hartree–Fock type could, on the one hand, give the right approximation for the initial state, and on the other a control on the deviations from the required evolution. One extremely interesting application would be the description of a controlled evolution driving initial approximate quantum states of anharmonic systems to stable wavepackets generalizations of the coherent states of the harmonic oscillator [15]. Besides the obvious interest in several areas of quantum phenomenology, the above is also of great potential interest in discussing the control and the reduction of aberrations in quantum-like systems, i.e. deviations from the harmonic evolution that are detected in systems such as charged beams in particle accelerators. Moreover, there is a number of interesting potentials (rigid square wells, Morse, sestic oscillator and so on) such that at least a few eigenstates and eigenvalues are perfectly known: hence it should be possible to calculate—at least in the form of a series—the transition probability densities and hence the required controlled evolutions between initial and final states.

Another very interesting future line of research that has been left virtually unexplored in the present paper is the introduction of optimization procedures. We have barely touched upon this problem when discussing the smoothing of the controlled transitions. Optimization of suitable functionals, chosen according to the kind of physical evolution one needs or desires to manufacture, would provide a powerful criterion of selection among the different possible smoothed evolutions. Instances of functionals to be optimized during the controlled dynamics that come naturally to mind are the uncertainty products of conjugate observables (to be optimized to a relative minimum under the constraint of Schrödinger dynamics [16]), or the relative entropy between the initial and final states. But many more can be imagined and devised, according to the nature of the physical problem considered.

One last, but important, consideration is in order. When we implement a controlled evolution by means of a suitable controlling potential we must also bear in mind that in practice small deviations away from the designed potential and from the desired wavefunction are always possible. In general, such deviations are not subsequently reabsorbed but rather tend to drag the state away from the required evolution. Hence to really control these

quantum evolutions it will be very important to study their stability under small deviations and perturbations: this is of crucial importance from the standpoint of confronting the formal, theoretical scheme with the practical applications. Work is currently in progress in all the above-mentioned extensions of the present research, and we plan to report on it soon.

References

[1] Cufaro Petroni N and Guerra F 1995 *Found. Phys.* **25** 297
  Cufaro Petroni N 1995 Asymptotic behaviour of densities for Nelson processes *Quantum Communications and Measurement* ed V P Belavkin *et al* (New York: Plenum)
  Cufaro Petroni N, De Martino S and De Siena S 1997 Non equilibrium densities of Nelson processes *New Perspectives in the Physics of Mesoscopic Systems* ed S De Martino *et al* (Singapore: World Scientific)
[2] Nelson E 1966 *Phys. Rev.* **150** 1079
  Nelson E 1967 *Dynamical Theories of Brownian Motion* (Princeton: Princeton University Press)
  Nelson E 1985 *Quantum Fluctuations* (Princeton: Princeton University Press)
  Guerra F 1981 *Phys. Rep.* **77** 263
  Blanchard Ph, Combe Ph and Zheng W 1987 *Mathematical and Physical Aspects of Stochastic Mechanics* (Berlin: Springer)
[3] Guerra F and Morato L M 1983 *Phys. Rev.* D **27** 1774
  Guerra F and Marra R 1983 *Phys. Rev.* D **28** 1916
  Guerra F and Marra R 1984 *Phys. Rev.* D **29** 1647
  Morato L M 1985 *Phys. Rev.* D **31** 1982
  Loffredo M I and Morato L M 1989 *J. Math. Phys.* **30** 354
[4] Bohm D and Vigier J P 1954 *Phys. Rev.* **96** 208
[5] Cufaro Petroni N, De Martino S and De Siena S 1998 *Phys. Lett.* A **245** 1
[6] Cufaro Petroni N 1989 *Phys. Lett.* A **141** 370
  Cufaro Petroni N 1991 *Phys. Lett.* A **160** 107
  Cufaro Petroni N and Vigier J P 1992 *Found. Phys.* **22** 1
[7] Cufaro Petroni N, De Martino S, De Siena S and Illuminati F 1998 A stochastic model for the semiclassical collective dynamics of charged beams in particle accelerators *15th ICFA Advanced Beam Dynamics Workshop (Monterey, CA, USA)* Bari Univeristy preprint (webpage http://www.ba.infn.it/~cufaro/scientific.html)
  Cufaro Petroni N, De Martino S, De Siena S, Fedele R, Illuminati F and Tzenov S 1998 Stochastic control of beam dynamics *EPAC'98 Conf. (Stockholm, Sweden)* Bari Univeristy preprint (webpage http://www.ba.infn.it/~cufaro/scientific.html)
[8] Madelung E 1926 *Z. Phys.* **40** 332
  de Broglie L 1926 *C. R. Acad. Sci., Paris* **183** 447
[9] Bohm D 1952 *Phys. Rev.* **85** 166
  Bohm D 1952 *Phys. Rev.* **85** 180
  Cufaro Petroni N, Dewdney C, Holland P, Kyprianidis T and Vigier J P 1985 *Phys. Rev.* D **32** 1375
[10] Guerra F 1997 The problem of the physical interpretation of Nelson stochastic mechanics as a model for quantum mechanics *New Perspectives in the Physics of Mesoscopic Systems* ed S De Martino *et al* (Singapore: World Scientific)
[11] Risken H 1989 *The Fokker–Planck Equation* (Berlin: Springer)
[12] Tricomi F 1948 *Equazioni Differenziali* (Torino: Einaudi)
[13] Tricomi F 1985 *Integral Equations* (New York: Dover)
[14] Albeverio S and Høgh-Krohn R 1974 *J. Math. Phys.* **15** 1745
[15] De Martino S, De Siena S and Illuminati F 1997 *J. Phys. A: Math. Gen.* **30** 4117
[16] Illuminati F and Viola L 1995 *J. Phys. A: Math. Gen.* **28** 2953