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Entangled states in stochastic mechanics

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Abstract. An axiomatization of the core part of stochastic mechanics (SM) is proposed and this scheme is discussed as a hidden variables theory. We work out in detail an example with entanglement and rigorously prove that SM and quantum mechanics agree in predicting all the observed correlations at different times.

1. Introduction

This paper is devoted to a discussion of Nelson's stochastic mechanics (SM) from the standpoint of the foundations of quantum mechanics (QM). For the time being we will limit ourselves to the part of the theory, here called *representative*, reduced to the Nelson map, which, as it is known, associates a diffusion process to every solution of the Schrödinger equation. This structure will be explicitly redefined as a hidden variables theory in QM for the case of a finite number of spinless particles. In fact the representative part is only the germ of the theory introduced by Nelson in 1966 [1, 2] and subsequently developed in the 1970s and 1980s, not without discussions and polemics. We refer the reader to [3–5] for extensive reviews and large bibliography and to [6] for a survey on applications and new perspectives. A first general analysis of SM from the point of view of the foundations of QM can be found in [7]. The dynamical part of the theory will be revisited in forthcoming papers. We also stress that this work is not related to the problem of giving a description, within SM, of the mechanism which produces the wave collapse. We refer the reader to [8, 9] for some attempts in this direction as far as the microscopic system under observation is considered and to [10] for a general approach, within a class of hidden variables theory which also contains SM, which takes into account both the microscpic system and the apparatus.

Our aim is essentially to provide a sound logical and mathematical basis for discussing entanglement and the various kinds of nonlocality, a subject which is almost ignored in the literature on SM. It is also important to remark that the analysis of examples with entangled states greatly contributed to convince the founder of the theory [11] to reject SM as a physical theory.

The paper is organized as follows: in section 2 we introduce the Nelson map and formulate the basic axiom SM1 of SM. This part somehow exploits mathematical results on the existence of diffusions with singular drift [12, 13] which were not available at the first formulation of the theory by E Nelson in 1966 and covers the case when the wavefunctions presents nodal surfaces [14]. In section 3 we introduce a definition of Nelson's hidden variables which is suitable for the discussion on determinism in section 5 and we carefully discuss conditioning. Section 4 is devoted to the rigorous solution of a problem proposed by Nelson in [11], concerning the calculus of the correlations at different times for position measurements on two separated and non-interacting subsystems which are initially prepared in an entangled state. We prove that a plain application of SM1 and quantum mechanical axioms does not lead to any contradiction and in particular that the oscillating quantum mechanical correlations are exactly reproduced in SM by the correct stochastic description of the evolution in time for the mixture which is produced by the measurement at time zero. We conclude the paper by briefly discussing locality properties and the concept of determinism.

2. The Nelson map

For the sake of clearness and in order to establish the notations we will first of all recall, without pretending any completeness, the axioms of QM which will be assumed and possibly discussed in this paper. We will denote by italic capital letters $\mathcal{A}, \mathcal{X}, \ldots$ the physical observables of a given system, by capital letters $A, X, \ldots$ the results of their measurements and by $\hat{A}, \hat{X}, \ldots$ the corresponding self-adjoint operators in the suitable Hilbert space.

QM1 With every physical system $\mathcal{S}$ is associated a Hilbert space $\mathcal{H}$; the states of $\mathcal{S}$ are represented by the normalized vectors ψ of $\mathcal{H}$, and such a description is as complete as possible.

QM2 The observable quantities are represented by self-adjoint operators on $\mathcal{H}$.

QM3 The time evolution of a state ψ_0 is ruled by the Schrödinger equation

$$\psi(t) = \mathrm{e}^{-\mathrm{i}\hat{H}t/\hbar}\psi_0 \tag{1}$$

where $\hat{H}$ is the Hamiltonian operator.

Next we separately recall the fundamental axioms concerning measurements.

M1 Given any statistical ensemble $\mathcal{E}$ of copies of the system $\mathcal{S}$, the measurement of a discrete observable $\mathcal{A}$ for every element of $\mathcal{E}$ divides $\mathcal{E}$ in subensembles $\mathcal{E}_i$, one for every possible value a_i of $\mathcal{A}$. An immediately repeated measurement of $\mathcal{A}$ on $\mathcal{E}_i$ gives a_i with certainty.

M2 If all the elements of $\mathcal{E}$ are in the state ψ, then the probability of obtaining a_i as the result of a measurement of $\mathcal{A}$ on a randomly chosen element of $\mathcal{E}$ (for short 'on $\mathcal{E}$') is given by

$$\boldsymbol{P}_\psi(A = a_i) = \|\hat{\pi}_i\psi\|^2 \tag{2}$$

where $\hat{\pi}_i$ represents the projection on the invariant subspace M_i corresponding to a_i.

M3 If for a single element of $\mathcal{E}$ in ψ the result of the measurement is a_i, then its state after the measurement is

$$\varphi_i = \frac{\hat{\pi}_i\psi}{\|\hat{\pi}_i\psi\|}. \tag{3}$$

This transformation is instantaneous†.

M4 If $\hat{X}$ is a self-adjoint operator on $\mathcal{H}$ with continuous spectrum, denoting by $\hat{F}_X$ its resolution of identity so that

$$\hat{X} = \int_{-\infty}^{+\infty} x\,\mathrm{d}\hat{F}_X(x) \tag{4}$$

and putting $\Delta\hat{F}_X(x) = \hat{F}_X(x + \Delta x) - \hat{F}_X(x)$ for a given an interval $(x, x + \Delta x]$ on the real line, then if all the elements of $\mathcal{E}$ are in the state ψ, the probability that on $\mathcal{E}$ the result

† This last statement is understood by the authors as an oversimplification.

of the Yes/No experiment designed to verify whether X lies in $(x, x + \Delta x]$ or not is Yes is given by

$$\boldsymbol{P}_\psi(X \in (x, x + \Delta x]) = \|\Delta \hat{F}_X(x)\psi\|. \tag{5}$$

For the systems such that the answer is Yes the initial state ψ is suddenly changed into the new normalized state

$$\varphi_{\Delta x} = \frac{\Delta \hat{F}_X(x)\psi}{\|\Delta \hat{F}_X(x)\psi\|}. \tag{6}$$

Since (when no measurement is performed) the state $\psi(t)$ is uniquely dermined by ψ_0 through (1), we will often adopt in the following the shorthand notations

$$\boldsymbol{P}_{\psi_0}(A(t) = a_i) \qquad \boldsymbol{P}_{\psi_0}(X(t) \in (x, x + \Delta x]) \tag{7}$$

to indicate the probabilities (2) and (5) when $\mathcal{E}$ is in the pure state $\psi(t)$ produced by the equation (1). We will also make use in this paper of the symbols

$$\begin{aligned} &\boldsymbol{P}_{\psi_0}(A(t) = a_i | B(s) = b_j) \\ &\boldsymbol{P}_{\psi_0}(X_1(t) \in (x, x + \Delta x] | X_2(s) \in (x', x' + \Delta x']) \qquad (0 \leqslant s \leqslant t) \end{aligned} \tag{8}$$

with the following meaning: if for a given ensemble initially preparated in the state ψ_0 a selective measurement of B (X_2) with result b_j (result in $(x, x + \Delta x]$) is performed and the corresponding subensemble evolves only by the effect of dynamics up to time t when a non-selective measurement of A (X_1) is performed, the quantities denoted by (8) represent the probability that for a randomly chosen element in this *subensemble* the result of the measurement is a_i (in $(x', x' + \Delta x']$).

Axioms M2 and M3 can also be synthesized by saying that the statistical operator associated with $\mathcal{E}$, namely, $\hat{U} = \hat{\pi}(\psi)$, is suddenly changed by a measurement of $\mathcal{A}$ into the new one $\hat{W} = \sum_i \mathrm{P}_\psi(A = a_i)\hat{\pi}_i$, and that for every element of $\mathcal{E}_i$ the state ψ is suddenly changed into φ_i. Moreover an immediate consequence of the quantum mechanical axioms is that the expected value of any observable $\mathcal{X}$ (both discrete and continuous) on an ensemble $\mathcal{E}$ prepared in the state ψ, denoted by $\langle X\rangle_\psi$, satisfies the equality

$$\langle X\rangle_\psi = (\psi, \hat{X}\psi). \tag{9}$$

We will consider, in this paper, only systems composed of N spinless particles in a flat space and subjected to scalar potentials. Actually SM covers all general situations which have classical analogues and the spin can be incorporated into the theory by the beautiful works by Dankel [15] and Dohrn and Guerra [16]. However, the simplest case mentioned above is sufficient to discuss SM as a hidden variables theory. If now m_i (with $i = 1, \ldots, d$ with $d = 3N$) are the masses of the considered particles†, the dynamics for every ψ_0 of the Hilbert space $\mathcal{H} = \mathrm{L}^2(\boldsymbol{R}^d)$ will be given by the Schrödinger equation

$$\mathrm{i}\hbar\partial_t\psi = -\sum_{i=1}^{d} \frac{\hbar^2}{2m_i}\partial_i^2\psi + V\psi \qquad \psi(0) = \psi_0 \tag{10}$$

with V denoting a scalar potential. The representative part of SM can now be introduced as follows.

Definition (Nelson's map). *With every normalized solution $\psi : [0, +\infty) \times \boldsymbol{R}^d \to \boldsymbol{C}$ of (10) the Nelson map associates a Markov diffusion $\boldsymbol{\xi}(t)$ taking values in the configuration space $\boldsymbol{R}^d$ and such that*

† Of course the masses associated with different coordinates of the same particle will be coincident: we do not specify that in the notation just to keep it as simple as possible.

(1) the random variable $\boldsymbol{\xi}_0 = \boldsymbol{\xi}(0)$ *is distributed according to the probability density* $\rho_0 = |\psi_0|^2$*;*

(2) the transition probability density of the diffusion $p(\boldsymbol{x}, t|\boldsymbol{x}_0, 0)$ *is the solution of the Kolmogorov equation*

$$\partial_t p = \sum_{i=1}^{d} \left[-\partial_i (b_i^{\psi} p) + \frac{\hbar}{2m_i} \partial_i^2 p \right] \qquad p(\boldsymbol{x}, 0|\boldsymbol{x}_0, 0) = \delta(\boldsymbol{x} - \boldsymbol{x}_0) \tag{11}$$

where the components of the velocity field are calculated from ψ *as*

$$b_i^{\psi} = \frac{\hbar}{m_i} \left[\mathrm{Re} \left(\frac{\partial_i \psi}{\psi} \right) + \mathrm{Im} \left(\frac{\partial_i \psi}{\psi} \right) \right]. \tag{12}$$

Notice that, denoting by $\rho(t)$ *the probability density of* $\boldsymbol{\xi}(t)$*, and averaging (11) over the initial conditions with respect to* ρ_0*, we also obtain the familiar Fokker–Planck equation for the probability density, that is*

$$\partial_t \rho = \sum_{i=1}^{d} \left[-\partial_i (b_i^{\psi} \rho) + \frac{\hbar}{2m_i} \partial_i^2 \rho \right] \qquad \rho(0) = |\psi_0|^2. \tag{13}$$

Proposition. *For every* $t \geqslant 0$ *we have* $\rho(\boldsymbol{x}, t) = |\psi(\boldsymbol{x}, t)|^2$ *with* $\boldsymbol{x} \in \boldsymbol{R}^d$*.*

Proof. [1, 2] Representing the wavefunction as $\psi = |\psi| \exp(\mathrm{i}S/\hbar)$ and taking the real part of (10) we have

$$\partial_t |\psi|^2 = -\sum_{i=1}^{d} \partial_i (v_i^{\psi} |\psi|^2) \qquad v_i^{\psi} = \frac{\hbar}{m_i} \mathrm{Im} \left(\frac{\partial_i \psi}{\psi} \right). \tag{14}$$

Now, by using (12), a straightforward calculation shows that $|\psi|^2$ satisfies (13). □

Under weak regularity assumptions which cover all the physically interesting cases and in particular that with unbounded drifts due to possible nodes of the wavefunction (see [12] for the basic references and [13] for further improvements), $\boldsymbol{\xi}(t)$ is proved to satisfy a stochastic differential equation in a weak sense; namely we can claim that there exists a d-dimensional Brownian motion $\boldsymbol{W}(t)$ (independent of $\boldsymbol{\xi}(0)$) with covariance matrix $\sigma_{ij} = \hbar\delta_{ij}/m_i$ such that

$$\mathrm{d}\boldsymbol{\xi}(t) = \boldsymbol{b}^{\psi}(\boldsymbol{\xi}(t), t)\, \mathrm{d}t + \mathrm{d}\boldsymbol{W}(t) \tag{15}$$

or, in the integral notation,

$$\boldsymbol{\xi}(t) = \boldsymbol{\xi}_0 + \int_0^t \boldsymbol{b}^{\psi}(\boldsymbol{\xi}(s), s)\, \mathrm{d}s + \boldsymbol{W}(t). \tag{16}$$

Let us now consider the space $\mathcal{C}^0$ of all the continuous functions from $[0, +\infty)$ to $\boldsymbol{R}^d$ endowed with its Borel σ-algebra of subsets $\mathcal{B}(\mathcal{C}^0)$. Then, by the Markov property of $\boldsymbol{\xi}(t)$, the initial density $|\psi_0|^2$ and the transition density p uniquely determine a probability measure on $(\mathcal{C}^0, \mathcal{B}(\mathcal{C}^0))$. We stress that such a measure is uniquely determined by the solution $\psi(\boldsymbol{x}, t)$ of the Schrödinger equation for the given initial condition $\psi(0) = \psi_0$. As a consequence, once the interaction is given by assigning V, this measure depends uniquely on ψ_0: we shall denote it by $\mathbb{P}_{\psi_0}$ and we will have for every Borel subset B of $\boldsymbol{R}^d$

$$\mathbb{P}_{\psi_0}(\boldsymbol{\xi}(t) \in B) = \int_B |\psi(\boldsymbol{x}, t)|^2\, \mathrm{d}\boldsymbol{x} \tag{17}$$

where of course $(\boldsymbol{\xi}(t) \in B)$ denotes the subset of the realization of the diffusion such that the configurational trajectory visits B at time t.

In the simplest case where $d = 1$, denoting by $\mathcal{X}$ the position observable, from the previous proposition and from M4 we have that on $\mathcal{E}$ initially prepared in ψ_0

$$\boldsymbol{P}_{\psi_0}(X(t) \in (x, x + \Delta x]) = \|\Delta \hat{F}_X(x)\psi(t)\|^2 = \mathbb{P}_{\psi_0}(\boldsymbol{\xi}(t) \in (x, x + \Delta x]) \quad (18)$$

for every $x, \Delta x \in \boldsymbol{R}$, where we exploit the notations specified in (7). Notice that we also have, denoting by $\mathbb{E}_{\psi_0}$ the mathematical expectation with respect to the Nelson measure $\mathbb{P}_{\psi_0}$,

$$\langle X\rangle_{\psi(t)} = \mathbb{E}_{\psi_0}[\xi(t)]. \quad (19)$$

We are now in a position to give a precise formulation of the basic axiom of the representative part of the SM for a system of N spinless particles:

SM1 The time evolution (produced by a dynamics without measurements) of the configuration of an N-spinless-particle system is represented by a curve in $\mathcal{C}^0$ (called a configurational trajectory or sample path). If a statistical ensemble $\mathcal{E}$ is initially prepared in the pure state $\psi_0 \in \mathrm{L}^2(\boldsymbol{R}^d)$ the trajectories of each element of $\mathcal{E}$ are distributed according to the probability measure $\mathbb{P}_{\psi_0}$, namely that representing a diffusion process with values in $\boldsymbol{R}^d$, having initial probability density $|\psi_0|^2$ and transition density p satisfying the Kolmogorov equation (11).

Whether the configurational trajectories, and the trajectories of the individual particles in the three-dimensional space, can be considered as having some physical meaning or not is a problem which will be dealt with in section 5. We can notice that SM1 is stated in a form which avoids the difficulty arising when the wavefunction has nodal surfaces. As it is known in this case one can prove [14] that a particle cannot cross such surfaces, so the 'probability' that a single particle visits certain regions in the physical space is equal to zero also if the square modulus of the wavefunction is positive there. The difficulty arises by not specifying the meaning of 'probability': SM1 explicitly states that such a probability refers to a statistical experiment where a particle is randomly chosen in the homogeneous ensemble $\mathcal{E}$ initially prepared in the state ψ_0, so that (17) remains true.

3. Hidden variables and conditioning

SM in its representative part shares with Bohm mechanics several aspects: in particular the relevance of position observables and the introduction of trajectories. As it is known [17] in Bohm mechanics the hidden variable is given by the initial position of all the particles constituting the system:

$$\lambda^{\mathrm{B}} = \boldsymbol{x}_0 = \{\boldsymbol{r}_1(0), \boldsymbol{r}_2(0), \ldots, \boldsymbol{r}_N(0)\} \in \boldsymbol{R}^{3N} \quad (20)$$

with $\boldsymbol{r}_i(t)$ denoting the position of the ith particle in the physical three-dimensional space at time t. The configuration at time t is then uniquely determined by the solution of the ordinary differential equation in $\boldsymbol{R}^{3N}$:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{v}^{\psi}(\boldsymbol{x}(t), t) \quad (21)$$

where the components of $\boldsymbol{v}^{\psi}$ are defined in (14), or equivalently

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \int_0^t \boldsymbol{v}^{\psi}(\boldsymbol{x}(s), s)\, \mathrm{d}s. \quad (22)$$

The analogous equations in SM are (15) and (16). As a consequence in SM the trajectory depends also on the realization of the $3N$-dimensional Brownian motion† $\boldsymbol{W}(t)$, which appears

† Note that if $\psi(t)$ is not factorized in the product of functions of individual $\boldsymbol{r}_i$ at every time t, then every component of both $\boldsymbol{v}^{\psi}$ and $\boldsymbol{b}^{\psi}$ will depend in general on the position of all the particles. As a consequence the motion of the individual constituents of the system will not in general be Markovian.

in (15). If now $\boldsymbol{w} \in \mathcal{C}^0$ is a function from $[0, +\infty)$ into $\boldsymbol{R}^{3N}$ denoting a generic realization of the process $\boldsymbol{W}(t)$, then the new hidden variable will be

$$\lambda^{\mathrm{N}} = \{\boldsymbol{x}_0, \boldsymbol{w}\} \qquad \boldsymbol{x}_0 \in \boldsymbol{R}^{3N} \qquad \boldsymbol{w} \in \mathcal{C}^0. \tag{23}$$

This way of defining Nelson's hidden variables slightly differs from that adopted by other authors, who prefer to consider as hidden variable the trajectory. We introduce this definition in order to stress that Nelson's hidden variables can be interpreted as both deterministic and intrinsically stochastic, as we will discuss in 5.

We have seen that by the axiom SM1 the probability that a position measurement on $\mathcal{E}$ initially prepared in ψ_0 gives a result in $(x, x + \Delta x]$ is by construction in complete agreement with that predicted by the QM and that this remains true also in the case where nodal surfaces are present, thanks to the well defined role in our axiomatization of the homogeneous ensembles. However, what about the correlations? This is not a trivial point since in [11] the founder of the theory presented an interesting example of a compound system which seemed to lead to a contradiction between SM and QM as far as correlations at different times in composed systems are concerned. To the authors knowledge this example has never been rediscussed in the literature.

Clearly SM1 only gives a representation (by adding hidden variables) of the evolution in time of the initial state ψ_0 by the Schrödinger equation (10). As a consequence the measurements must be ruled by QM measurement axioms M1–M4. Since in the past this point has often been misunderstood, we discuss it in detail within our axiomatization (a qualitative discussion can be found in [7] and a conceptually correct treatment of repeated measurements on the same system is given in [18]). Consider a statistical ensemble $\mathcal{E}$ prepared in the state ψ_0, and let $\psi(t)$ be its evolution at the time $t > 0$ when a measurement of a discrete non-degenerate observable $\mathcal{A}$ is performed. If a_i and φ_i denote respectively the eigenvalues and eigenvectors of $\hat{A}$, a non-selective measurement will divide $\mathcal{E}$ into homogeneous subensembles $\mathcal{E}_i$ such that every system belonging to $\mathcal{E}_i$ is instantaneously put in the state φ_i, which then evolves according to the Schrödinger equation. Thus if we consider a single system in $\mathcal{E}$ its trajectory must be constructed by the stochastic differential equation (15) with the drift $\boldsymbol{b}^{\psi}$ in $[0, t)$, and by a new stochastic differential equation (15) with the new drift $\boldsymbol{b}^{\varphi_i}$ in $[t, +\infty)$.

A similar evolution takes place if the observable $\mathcal{A}$ is replaced by a position observable $\mathcal{X}$: in particular the quantity $\mathbb{E}_{\psi_0}(\xi(0)\xi(t))$ could possibly reproduce on $\mathcal{E}$ initially prepared in ψ_0 the corresponding average of the product of measured values of $\mathcal{X}$ at times 0 and t *only if the measurement at time 0 were non-demolitive*. In fact, if the measurement at time 0 changes the state of our system (according to M3 and M4), it would be wrong to expect that $\mathbb{E}_{\psi_0}(\xi(0)\xi(t))$ reproduces the observed correlations since, by its definition, it refers just to a situation where ψ_0 evolves undisturbed through (1). Remark that, at variance with QM, in SM the calculation of mathematical correlations is possible either by assuming a wavepacket collapse at time 0, or by letting the initial state evolve undisturbed through (1)). A strong 'realistic' philosophical attitude could emphasize the meaning of $\mathbb{E}_{\psi_0}(\xi(0)\xi(t))$ by giving to $\xi(t)$ the meaning of the 'true' value of the configuration at time t and then to $\mathbb{E}_{\psi_0}(\xi(0)\xi(t))$ that of a 'virtual true correlation', which avoids the effects of quantum measurements. Some discussion on this point is shifted to section 5. We stress that at this level we are just checking the consistency of axioms, where only results of measurements performed in actual experiments are mentioned.

In a similar way we can discuss the problems concerning the conditioning. Let us consider the probability that $X \in (x', x' + \Delta x']$ at time t if $X \in (x, x + \Delta x]$ at time s on an ensemble $\mathcal{E}$ prepared in ψ_0 at time 0 (with $0 < s < t$). If $\varphi_{\Delta x}$ is the state produced by the measurement at time s as in (6), then we have, recalling (8),

$$\boldsymbol{P}_{\psi_0}(X(t) \in (x', x' + \Delta x'] | X(s) \in (x, x + \Delta x])$$

$$
\begin{aligned}
&= \boldsymbol{P}_{\varphi_{\Delta x}}(X(t-s) \in (x', x' + \Delta x'])\\
&= \mathbb{P}_{\varphi_{\Delta x}}(\xi(t-s) \in (x', x' + \Delta x'])\\
&\neq \mathbb{P}_{\psi_0}(\xi(t) \in (x', x' + \Delta x'] | \xi(s) \in (x, x + \Delta x]).
\end{aligned} \tag{24}
$$

Remark that, while the first member in (24) must be understood as stated for (8) in section 2, the last member is calculated by letting ψ_0 evolve following an undisturbed Schrödinger evolution (1). In the language of the SM community (see for example [18]) it is sometimes said that *mathematical conditioning is not physical conditioning*: in the author's opinion this is a good way of synthesizing this point (for further discussion on this subject see for example [8, 19]).

The origin of this non-classical phenomenon is of course due to the peculiarity of the quantum measurement and ultimately rests on the uncertainty principle. However one could conjecture that if we consider a system composed by two non-interacting subsystems and we perform position measurements both on the first at time s and on the second at time $t > s$, then the mathematical correlations at times t and s of the two components of the undisturbed Nelson process should in fact reproduce the quantum ones. This is not true in general due to the puzzling effects of entanglement. We will study in the subsequent sections an interesting example of entanglement proposed by Nelson [2] and we will show that also in this case SM1 does not contradict the basic quantum mechanical axioms.

4. Correlations at different times in compound systems

We will consider a quantum system made up of two subsystems 1 and 2 so that the total Hilbert space $\mathcal{H}$ will be the product of two component subspaces $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. We will also suppose that the systems 1 and 2 do not interact so that the total Hamiltonian is $\hat{H} = \hat{H}_1 \otimes \hat{I}_2 + \hat{I}_1 \otimes \hat{H}_2 = \hat{H}_1 + \hat{H}_2$ with $[\hat{H}_1, \hat{H}_2] = [\hat{H}_1 \otimes \hat{I}_2, \hat{I}_1 \otimes \hat{H}_2] = 0$ (we will often write $\hat{H}_1$ in place of $\hat{H}_1 \otimes \hat{I}_2$ and $\hat{H}_2$ in place of $\hat{I}_1 \otimes \hat{H}_2$). Moreover let $\hat{A}$ and $\hat{B}$ be two operators representing the observables A and B respectively for the systems 1 and 2 and such that $[\hat{A} \otimes \hat{I}_2, \hat{I}_1 \otimes \hat{B}] = [\hat{A}, \hat{B}] = 0$. We will assume by now that all our operators have purely discrete, non-degenerate spectra. We put

$$
\hat{H}_1\varphi_j = W_j\varphi_j \qquad \hat{H}_2\psi_k = E_k\psi_k \qquad \hat{A}\alpha_l = a_l\alpha_l \qquad \hat{B}\beta_m = b_m\beta_m \tag{25}
$$

where $\varphi_j, \alpha_l \in \mathcal{H}_1$ and $\psi_k, \beta_m \in \mathcal{H}_2$. We will also adopt the notations $\varphi_j\psi_k = \varphi_j \otimes \psi_k \in \mathcal{H}$ and $\alpha_l\beta_m = \alpha_l \otimes \beta_m \in \mathcal{H}$.

The time evolution of the system can be accounted for in the Schrödinger picture (SP): for a given $\Phi \in \mathcal{H}$, with $\Phi(0) = \Phi$ we have

$$
\Phi(t) = \mathrm{e}^{-\mathrm{i}\hat{H}t/\hbar}\Phi \tag{26}
$$

and in particular we obtain

$$
\mathrm{e}^{-\mathrm{i}\hat{H}_2 t/\hbar}\Phi = \sum_{j,k} (\varphi_j\psi_k, \Phi)\mathrm{e}^{-\mathrm{i}E_k t/\hbar}\varphi_j\psi_k. \tag{27}
$$

Our aim is to calculate the correlation of two quantum observables at different times for a given initial state $\Phi \in \mathcal{H}$: to do that we must go through three steps: first of all since

$$
\Phi(0) = \Phi = \sum_{l,m} (\alpha_l\beta_m, \Phi)\alpha_l\beta_m \tag{28}
$$

a measurement of A at $t = 0$ gives the result a_l with a probability

$$
\boldsymbol{P}_\Phi(A(0) = a_l) = \sum_m |(\alpha_l\beta_m, \Phi)|^2 \tag{29}
$$

and as a consequence Φ collapses into a mixture of the factorized states

$$\Phi_{a_l} = \frac{\sum_m (\alpha_l\beta_m, \Phi)\alpha_l\beta_m}{\sqrt{\sum_m |(\alpha_l\beta_m, \Phi)|^2}} \qquad l = 1, 2, \ldots \tag{30}$$

each with a probability (29). Next we let every Φ_{a_l} evolve in time so that from (26) and (30) we have

$$\begin{aligned}\Phi_{a_l}(t) = \mathrm{e}^{-\mathrm{i}\hat{H}t/\hbar}\Phi_{a_l} &= \frac{\sum_m (\alpha_l\beta_m, \Phi)\sum_{r,s}(\varphi_r, \alpha_l)(\psi_s, \beta_m)\varphi_r\psi_s \mathrm{e}^{-\mathrm{i}(W_r+E_s)t/\hbar}}{\sqrt{\sum_m |(\alpha_l\beta_m, \Phi)|^2}} \\ &= \left(\sum_m (\alpha_l\beta_m, \Phi)\sum_{r,s}(\varphi_r, \alpha_l)(\psi_s, \beta_m)\mathrm{e}^{-\mathrm{i}(W_r+E_s)t/\hbar}\sum_{l',m'}(\alpha_{l'}, \varphi_r)(\beta_{m'}, \psi_s)\alpha_{l'}\beta_{m'}\right) \\ &\quad\times\left\{\sqrt{\sum_m |(\alpha_l\beta_m, \Phi)|^2}\right\}^{-1}.\end{aligned} \tag{31}$$

Finally we measure B at time t and we obtain the outcome $b_{m'}$ with a conditional probability in the sense of (8)

$$\begin{aligned}&\boldsymbol{P}_\Phi(B(t) = b_{m'}|A(0) = a_l) \\ &\quad= \frac{\sum_{l'} |\sum_m (\alpha_l\beta_m, \Phi)\sum_{r,s}(\varphi_r, \alpha_l)(\psi_s, \beta_m)\mathrm{e}^{-\mathrm{i}(W_r+E_s)t/\hbar}(\alpha_{l'}, \varphi_r)(\beta_{m'}, \psi_s)|^2}{\sum_m |(\alpha_l\beta_m, \Phi)|^2}.\end{aligned} \tag{32}$$

Now, since $[\hat{A}(0), \hat{B}(t)] = 0$, we can adopt the usual formulae of classical probability to obtain the joint probabilities, so that the required correlation will be

$$\begin{aligned}\boldsymbol{R}^\Phi_{AB}(t) &= \sum_{l,m'} a_l b_{m'}\boldsymbol{P}_\Phi(B(t) = b_{m'}|A(0) = a_l)\boldsymbol{P}_\Phi(A(0) = a_l) \\ &= \sum_{l,m'} a_l b_{m'}\sum_{l'}\left|\sum_m (\alpha_l\beta_m, \Phi)\sum_{r,s}\right. \\ &\quad\left.\times(\varphi_r, \alpha_l)(\psi_s, \beta_m)\mathrm{e}^{-\mathrm{i}(W_r+E_s)t/\hbar}(\alpha_{l'}, \varphi_r)(\beta_{m'}, \psi_s)\right|^2.\end{aligned} \tag{33}$$

By developing the square module and by taking into account the usual closure relations

$$\sum_m (\psi_s, \beta_m)(\beta_m, \psi_q) = (\psi_s, \psi_q) = \delta_{s,q} \qquad \sum_n (\varphi_k, \alpha_n)(\alpha_n, \varphi_p) = (\varphi_k, \varphi_p) = \delta_{k,p} \tag{34}$$

and so on, we obtain

$$\begin{aligned}\boldsymbol{R}^\Phi_{AB}(t) &= \sum_{j,k}\sum_{p,q}(\Phi, \varphi_j\psi_k)(\varphi_p\psi_q, \Phi)\sum_l (\varphi_j, \alpha_l)a_l(\alpha_l, \varphi_p) \\ &\quad\times\sum_{m'}(\psi_k, \beta_{m'})b_{m'}(\beta_{m'}, \psi_q)\mathrm{e}^{-\mathrm{i}(E_q-E_k)t/\hbar} \\ &= \sum_{j,k}\sum_{p,q}(\Phi, \varphi_j\psi_k)(\varphi_p\psi_q, \Phi)\mathrm{e}^{-\mathrm{i}(E_q-E_k)t/\hbar}(\varphi_j, \hat{A}\varphi_p)(\psi_k, \hat{B}\psi_q).\end{aligned} \tag{35}$$

It is apparent that such an object cannot be immediately calculated as a quantum expected value by a rule similar to (9) since two different states are involved. However, one immediately sees that (35) can be read in the Heisenberg picture as

$$\boldsymbol{R}^\Phi_{AB}(t) = (\Phi, \hat{A}(0)\hat{B}(t)\Phi). \tag{36}$$

Let now $\mathcal{A}$ and $\mathcal{B}$ be substituted by the (one-dimensional) position observables $\mathcal{X}_1$ and $\mathcal{X}_2$ respectively of 1 and 2, and let us consider the (two-component) Nelson process $\boldsymbol{\eta}(t) =$

$(\xi_1(t), \xi_2(t))$ associated with the undisturbed (namely free of demolitive measurements) evolution of Φ: can we expect the corresponding $\mathbb{E}_\Phi(\xi_1(0)\xi_2(t))$ to coincide with the correlation $\boldsymbol{R}_{12}^\Phi(t)$ calculated in a way similar to (35)? A 'classical' way of reasoning would apparently lead to an affirmative answer since the systems 1 and 2 are uncoupled and hence no influence would be expected on the outcomes of a measurement on 2 at time t by any measurement on 1 at the time 0, but thanks to the possible presence of entangled states this is only partially true in QM. In fact we know [20] that the marginal probabilities for separated measurements on 2 are not affected by measurements on 1. However, we also know from Bell's work [21] that the joint probabilities for the outcomes of separated measurements on 1 and 2 are more sensitive. Thus, since $\mathbb{E}_\Phi(\xi_1(0)\xi_2(t))$ would correspond to a situation which is free of demolitive measurements it should not necessarily be supposed to be equal to the QM correlations $\boldsymbol{R}_{12}^\Phi(t)$ calculated as in (35).

To investigate this point let us consider, following [11], a couple of quantum dynamically uncoupled harmonic oscillators with equal masses and elastic constants with total Hamiltonian $\hat{H} = \hat{H}_1 + \hat{H}_2$ where

$$\hat{H}_1 = \frac{\hat{P}_1^2}{2m} + \frac{m\omega^2}{2}\hat{X}_1^2 \qquad \hat{H}_2 = \frac{\hat{P}_2^2}{2m} + \frac{m\omega^2}{2}\hat{X}_2^2. \tag{37}$$

Eigenvalues and eigenfuncions of $\hat{H}_1$ and $\hat{H}_2$ are very well known [22]:

$$\begin{aligned} &E_k = W_k = (k + \tfrac{1}{2})\hbar\omega \qquad \varphi_k(x) = \psi_k(x) = N_k Q_k(\alpha x)\mathrm{e}^{-\alpha^2 x^2/2} \\ &k = 0, 1, 2, \ldots \end{aligned} \tag{38}$$

where

$$\alpha = \sqrt{\frac{m\omega}{\hbar}} \qquad N_k = \sqrt{\frac{\alpha}{\sqrt{\pi}2^k k!}} \qquad Q_k(y) = (-1)^k \mathrm{e}^{y^2}\frac{\mathrm{d}^k}{\mathrm{d}y^k}\mathrm{e}^{-y^2}. \tag{39}$$

The correlation will now be evaluated for an initial state Φ whose wavefunction is chosen as

$$\Phi(\boldsymbol{x}) = \sqrt{C}\mathrm{e}^{-\boldsymbol{x}\cdot\mathbb{A}\boldsymbol{x}/4} \tag{40}$$

where $\boldsymbol{x} = (x_1, x_2)$ and

$$C = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} \qquad \mathbb{A} = \mathbb{R}^{-1} = \frac{1}{\sigma_1^2\sigma_2^2(1-r^2)}\begin{pmatrix} \sigma_2^2 & -r\sigma_1\sigma_2 \\ -r\sigma_1\sigma_2 & \sigma_1^2 \end{pmatrix} \tag{41}$$

while of course we have

$$\langle X_1\rangle_\Phi = \langle X_2\rangle_\Phi = 0 \qquad \mathbb{R} = \begin{pmatrix} \langle X_1^2\rangle_\Phi & \langle X_1X_2\rangle_\Phi \\ \langle X_1X_2\rangle_\Phi & \langle X_2^2\rangle_\Phi \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & r\sigma_1\sigma_2 \\ r\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}. \tag{42}$$

This means that the two oscillators are normally distributed and statistically correlated (while dynamically independent) at time $t = 0$ and that their initial correlation is $\boldsymbol{R}_{12}^\Phi(0) = r\sigma_1\sigma_2$, a feature characteristic of the so-called entangled states.

We expect of course that, in analogy with the discrete case, also for this compound system the same result is obtained by working in SP, after applying M4, or by flatly rewriting (36) for the two continuous observables $\mathcal{X}_1$ and $\mathcal{X}_2$. We stress that, while the coincidence is of course still expected, the compact expression of the type (36) does not come too directly from M4 since some limiting procedure is needed. We will treat this point in detail since both QM axioms and SM1 are formulated in the SP.

Let us first give the result using an expression analogous to (36). Since

$$(\varphi_j, \hat{X}_1\varphi_p) = (\psi_j, \hat{X}_2\psi_p) = \frac{1}{\alpha}\left(\delta_{j+1,p}\sqrt{\frac{j+1}{2}} + \delta_{j-1,p}\sqrt{\frac{j}{2}}\right) \tag{43}$$

a standard calculation yields, recalling (35),

$$(\Phi, \hat{X}_1(0)\hat{X}_2(t)\Phi) = \sum_{j,k}\sum_{p,q}(\Phi, \varphi_j\psi_k)(\varphi_p\psi_q, \Phi)\mathrm{e}^{-\mathrm{i}(E_q-E_k)t/\hbar} \times(\varphi_j, \hat{X}_1\varphi_p)(\psi_k, \hat{X}_2\psi_q) = \lambda\cos\omega t \tag{44}$$

where the constant λ can be easily calculated from $\lambda = \boldsymbol{R}_{12}^{\Phi}(0) = \langle X_1X_2\rangle_\Phi = r\sigma_1\sigma_2$, so we finally have

$$(\Phi, \hat{X}_1(0)\hat{X}_2(t)\Phi) = r\sigma_1\sigma_2\cos\omega t. \tag{45}$$

To restart in the SP, let us now consider two partitions of the real line $(a_i, a_i + \Delta a_i]$, $i \in \boldsymbol{N}$ and $(b_j, b_j + \Delta b_j]$, $j \in \boldsymbol{N}$. A non-selective measurement of $\mathcal{X}_1$ at time 0 designed to check whether Yes or No $X_1(0) \in (a_i, a_i + \Delta a_i]$, $i \in \boldsymbol{N}$, will produce a mixture of states of the type

$$\Phi_{\Delta a_i} = \frac{(\Delta\hat{F}_1(a_i)\otimes\hat{I}_2)\Phi}{\|(\Delta\hat{F}_1(a_i)\otimes\hat{I}_2)\Phi\|} = \frac{\Delta\hat{F}_1(a_i)\Phi}{\|\Delta\hat{F}_1(a_i)\Phi\|}. \tag{46}$$

We will suppress the indices i and j in most of the following. After a Hamiltonian time evolution

$$\Phi_{\Delta a}(t) = \mathrm{e}^{-\mathrm{i}\hat{H}t/\hbar}\Phi_{\Delta a} = \frac{\mathrm{e}^{-\mathrm{i}\hat{H}t/\hbar}\Delta\hat{F}_1(a)\Phi}{\|\Delta\hat{F}_1(a)\Phi\|} \tag{47}$$

we perform a measurement of X_2 at time t and we find $X_2 \in (b, b + \Delta b]$ with a conditional probability (in the sense of (8))

$$\begin{aligned}\boldsymbol{P}_\Phi(X_2(t) \in (b, b+\Delta b] \mid X_1(0) \in (a, a+\Delta a]) &= \boldsymbol{P}_{\Phi_{\Delta a}}(X_2(t) \in (b, b+\Delta b]) \\ &= \|\Delta\hat{F}_2(b)\Phi_{\Delta a}(t)\|^2 \\ &= \frac{\|\Delta\hat{F}_2(b)\mathrm{e}^{-\mathrm{i}\hat{H}t/\hbar}\Delta\hat{F}_1(a)\Phi\|^2}{\|\Delta\hat{F}_1(a)\Phi\|^2}.\end{aligned} \tag{48}$$

Since $\hat{X}_1$ and $\hat{X}_2$ commute we can also calculate the joint probability as

$$\boldsymbol{P}_\Phi(X_2(t) \in (b, b+\Delta b] \mid X_1(0) \in (a, a+\Delta a])\boldsymbol{P}_\Phi(X_1(0) \in (a, a+\Delta a]) = \|\Delta\hat{F}_2(b)\mathrm{e}^{-\mathrm{i}\hat{H}t/\hbar}\Delta\hat{F}_1(a)\Phi\|^2. \tag{49}$$

From the definition of our operators we have now

$$\begin{aligned}\|\Delta\hat{F}_2(b)\mathrm{e}^{-\mathrm{i}\hat{H}t/\hbar}\Delta\hat{F}_1(a)\Phi\|^2 &= (\Delta\hat{F}_2(b)\mathrm{e}^{-\mathrm{i}\hat{H}t/\hbar}\Delta\hat{F}_1(a)\Phi, \Delta\hat{F}_2(b)\mathrm{e}^{-\mathrm{i}\hat{H}t/\hbar}\Delta\hat{F}_1(a)\Phi) \\ &= (\mathrm{e}^{-\mathrm{i}\hat{H}_2t/\hbar}\Phi, \Delta\hat{F}_1(a)\Delta\hat{F}_2(b)\mathrm{e}^{-\mathrm{i}\hat{H}_2t/\hbar}\Phi) \\ &= \left(\sum_{jk}\mathrm{e}^{-\mathrm{i}E_kt/\hbar}(\varphi_j\psi_k, \Phi)\varphi_j\psi_k, \Delta\hat{F}_1(a)\Delta\hat{F}_2(b)\sum_{pq}\mathrm{e}^{-\mathrm{i}E_qt/\hbar}(\varphi_p\psi_q, \Phi)\varphi_p\psi_q\right) \\ &= \sum_{jk}\sum_{pq}\mathrm{e}^{-\mathrm{i}(E_q-E_k)t/\hbar}(\Phi, \varphi_j\psi_k)(\varphi_p\psi_q, \Phi)(\varphi_j, \Delta\hat{F}_1(a)\varphi_p)(\psi_k, \Delta\hat{F}_2(b)\psi_q)\end{aligned} \tag{50}$$

and since for $\Delta a \to 0$ and $\Delta b \to 0$ we can write

$$(\varphi_j, \Delta\hat{F}_1(a)\varphi_p) = \int_a^{a+\Delta a}\varphi_j^*(x)\varphi_p(x)\,\mathrm{d}x = \varphi_j^*(a)\varphi_p(a)\Delta a + \mathrm{o}(\Delta a)$$

$$(\psi_k, \Delta\hat{F}_2(b)\psi_q) = \int_b^{b+\Delta b}\psi_k^*(x)\psi_q(x)\,\mathrm{d}x = \psi_k^*(b)\psi_q(b)\Delta b + \mathrm{o}(\Delta b)$$

we finally obtain the joint probability density function $p_\Phi(x_2, t; x_1, 0)$ for the joint probability defined in (49)

$$p_\Phi(x_2, t; x_1, 0) = \sum_{jk}\sum_{pq} \mathrm{e}^{-\mathrm{i}(E_q-E_k)t/\hbar}(\Phi, \varphi_j\psi_k)(\varphi_p\psi_q, \Phi)\varphi_j^*(x_1)\varphi_p(x_1)\psi_k^*(x_2)\psi_q(x_2). \quad (51)$$

A classical calculation shows now that the total quantum correlation for the mixture, at the continuous limit for the two partitions, is in fact

$$\begin{aligned}\boldsymbol{R}_{12}^{\Phi}(t) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1x_2p_\Phi(x_2, t; x_1, 0)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\ &= \sum_{jk}\sum_{pq}(\Phi, \varphi_j\psi_k)(\varphi_p\psi_q, \Phi)\mathrm{e}^{-\mathrm{i}(E_q-E_k)t/\hbar}(\varphi_j, \hat{X}_1\varphi_p)(\psi_k, \hat{X}_2\psi_q) \\ &= (\Phi\hat{X}_1(0)\hat{X}_2(t)\Phi). \end{aligned} \quad (52)$$

To now obtain the right SM description we must apply SM1 to the evolution of each pure state $\Phi_{\Delta a_i}(t)$ constituting the mixture: this will give rise, by the Nelson map, to a family of two-component Nelson diffusions $\boldsymbol{\xi}^{\Phi_{\Delta a_i}}(t) = (\xi_1^{\Phi_{\Delta a_i}}, \xi_2^{\Phi_{\Delta a_i}})$ each with its own particular two-component drift field

$$\begin{pmatrix} b_1^{\Phi_{\Delta a_i}} \\ b_2^{\Phi_{\Delta a_i}} \end{pmatrix} = \frac{\hbar}{m}\begin{pmatrix} \mathrm{Re}\left(\frac{\partial_1\Phi_{\Delta a_i}}{\Phi_{\Delta a_i}}\right) + \mathrm{Im}\left(\frac{\partial_1\Phi_{\Delta a_i}}{\Phi_{\Delta a_i}}\right) \\ \mathrm{Re}\left(\frac{\partial_2\Phi_{\Delta a_i}}{\Phi_{\Delta a_i}}\right) + \mathrm{Im}\left(\frac{\partial_2\Phi_{\Delta a_i}}{\Phi_{\Delta a_i}}\right) \end{pmatrix} \qquad i \in \boldsymbol{N}. \quad (53)$$

For every branch of the mixture we then have, recalling (8), the basic equality

$$\begin{aligned}&\boldsymbol{P}_\Phi(X_2(t) \in (b, b + \Delta b]|X_1(0) \in (a_i, a_i + \Delta a_i]) \\ &\qquad = \mathbb{P}_{\Phi_{\Delta a_i}}(\xi_2^{\Phi_{\Delta a_i}}(t) \in (b, b + \Delta b]|\xi_1^{\Phi_{\Delta a_i}}(0) \in (a_i, a_i + \Delta a_i]).\end{aligned} \quad (54)$$

We are then in a position to exploit again (49) and the subsequent purely mathematical manipulations. Then, recombining in a purely classical way the contributions to the total correlation coming from the different branches of the mixture and going to the continuous limit, as done from (49) to (52), we obtain the oscillating correlation (45).

It may be worthwhile to stress that, even if the averaged object $\boldsymbol{R}_{12}^{\Phi}(t)$ exists in the continuous limit, the Nelson map cannot be directly applied in the same limit since the limit of the square root of the initial density, for every branch of the mixture, would not belong to $\mathrm{L}^2(\boldsymbol{R}^{3N})$. This is in nice agreement with the fact that M4 is actually stated for a finite interval. We conclude this section by observing that we have just described a particular effect of the entanglement on the observed correlations, at different times, for measurements on two separate subsystems. We stress again that, due to the collapse of the initial state Φ, the mathematical correlations of the two-component Nelson process associated with the purely dynamical evolution of Φ does not represent any statistical object related to actually performed experiments. To fully appreciate the discrepancy between quantum correlations and the just mentioned 'virtual correlations' we report in the appendix the explicit expressions for the unperturbed Nelson process and its correlations which, as expected, go exponentially fast to zero for the time going to infinity.

5. Discussion and conclusions

We have worked out an example of entanglement within SM starting on a well defined logical basis and fully rigorously from a mathematical point of view. This leaves no doubt of the fact that entanglement is well described in this stochastic framework (see [23] for pioneering work). To briefly discuss locality we first recall here some well known characteristics of

Bohm mechanics which are conserved even in SM as a hidden variable theory, since they do not depend on the type of configurational trajectories which are introduced but just on their 'existence'.

(1) The positions of all the particles constituting a system can be considered as having a precise value independent of the occurrence of a measurement, and that without violating the basic axioms of QM (with the exception of the statement on the completeness of the theory). This in particular, being true also in the case of superpositions of pure states, gives an answer to all the paradoxes of the *Schrödinger's cat* type.
(2) The same form of *microreality* seems not to be true for all the observables (see, for example, the argument about spin 1 variables in [24]).
(3) The hidden variables are in general not accessible since the observation of a trajectory associated with a solution of the Schrödinger equation would need infinitely precise and non-demolitive measurements at every time.

Thus the configurational trajectories, and consequently the trajectories in the three dimensional space described by every particle constituting the system, have physical meaning, but only in the sense of point 1.

Of course both the ordinary differential equations (21) of the Bohm theory and the stochastic differential equations (15) of the Nelson theory are written in terms of a velocity field ($\boldsymbol{v}^{\psi}$ and $\boldsymbol{b}^{\psi}$ respectively), whose components depend in general on the positions of all the particles constituting the system unless the state is factorized. Thus, as it is well known, SM shares with the Bohm theory a particular sort of non-locality at the level of trajectories. On the other hand, since these are not accessible, such a fact does not necessarily require a drastic change in the usual philosophical attitude. The situation is different as far as the celebrated Bell locality is concerned. Such a form of locality constraint requires a factorization of joint probabilities for results of experiments performed on space-like separated systems. The requirement yields Bell's inequality, which is violated by QM in various examples, as well as in experiment [25]. Since the violation of Bell's inequality comes from the basic axioms of QM which are not contradicted by SM, we obtain that SM also violates Bell's inequality in the case of entangled systems: such a violation is at the level of correlations. Since correlations are in fact observed in statistical experiments, such a violation seems to pose one of the most challenging problems to any hidden variables theory. This point would be particularly interesting in rediscussing the dynamical part of SM since, at variance with Bohm theory, the aim of SM is firstly to provide new principles which independently 'produce quantization'. We stress that the violation of Bell's locality looks like an intrinsically statistical phenomenon (it cannot produce superluminal signals!) and thus a stochastic framework should be the most appropriate for facing such a challenging problem.

As a final point we would like to discuss the concept of *deterministic* and *stochastic* hidden variables. It is apparent that there are no conceptual differences between λ^{B} and λ^{N} of (20) and (23); hence whether the theory is deterministic or stochastic depends on the interpretation. Bohm's theory is called deterministic since all the probabilistic aspects are reduced to the ignorance of the initial positions, which in any case have a precise value, although they are not accessible (in other words one could say that the probabilities are purely *epistemic*). On the other hand the Nelson approach is called stochastic since the trajectory depends also on the random realization of the Brownian motion $\boldsymbol{W}(t)$. Now, if such a realization is interpreted as describing some (unknown) physical effect (see, for example, the quantum fluctuations quoted in [5]), the situation is the same as for the Bohm theory and all the probabilistic aspects are originated by some sort of ignorance. In this sense SM would also be a deterministic hidden variable theory, but if the interpretation is that the introduction of the Brownian motion

$\boldsymbol{W}(t)$ is not more deeply analysable, but is rather a purely mathematical device and does not represent any physical (unknown) phenomenon, then the theory is not deterministic in the above explained sense. One can of course notice that the first interpretation of SM would fit with the celebrated Einstein sentence 'God does not play dice'. It is worth mentioning that there is some current research, starting from different, non-trivial conjectures, on the origin of the quantum fluctuations: see, for example, [26–28]. Clearly the just quoted works are in principle the most ambitious: indeed not only do the authors accept the idea that QM is not complete and introduce hidden variables, but, in addition, they study the origin of quantization as a physical (mysterious) phenomenon leading in particular to the violation of Bell's inequality (see in particular [28]).

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Appendix A. The correlations of the unperturbed Nelson process

The aim of this appendix is to show in an explicit way that, when no actual measurement is performed at time 0 on one of the two entangled oscillators of section 4, the SM forecasts an exponential time decay of the two time correlations. First of all let us recall a few classical results which will be exploited in the following. Let us consider a one-dimensional quantum harmonic oscillator whose Schrödinger equation is

$$\mathrm{i}\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\partial_x^2\psi + \frac{m\omega^2x^2}{2}\psi; \tag{A.1}$$

it is well known (see for example [29]) that the (non-normalizable) solution of (A.1) corresponding to the initial condition

$$\psi(x, 0^+) = \delta(x - y) \tag{A.2}$$

has the form

$$\psi(x, t|y, 0) = \frac{\mathrm{e}^{\mathrm{i}[(x^2+y^2)\cos^2\omega t - 2xy]/4\sigma_0^2\sin\omega t}}{\sqrt{4\pi\mathrm{i}\sigma_0^2\sin\omega t}} \qquad \sigma_0^2 = \frac{\hbar}{2m\omega} \tag{A.3}$$

and the corresponding velocity field is

$$b(x, t) = A(t) + B(t)x \qquad A(t) = -\frac{\omega y}{\sin\omega t} \qquad B(t) = \frac{\omega}{\tan\omega t}. \tag{A.4}$$

The knowledge of (A.3) allows us now to calculate the solutions of (A.1) with a Gaussian initial condition (with σ in general different from σ_0):

$$\psi(x, 0^+) = \frac{\mathrm{e}^{-(x-y)^2/4\sigma^2}}{\sqrt{\sigma\sqrt{2\pi}}}. \tag{A.5}$$

These solutions have the form

$$\psi_{y\sigma}(x, t) = R(x, t)\mathrm{e}^{-\mathrm{i}S(x,t)/\hbar} \tag{A.6}$$

where

$$R(x, t) = \frac{\mathrm{e}^{-(x-\mu(t))^2/4\sigma^2(t)}}{\sqrt{\sigma(t)\sqrt{2\pi}}} \qquad \mu(t) = y\cos\omega t$$
$$\sigma^2(t) = \frac{\sigma_0^2}{\alpha^2}(\alpha^4\cos^2\omega t + \sin^2\omega t) \qquad \alpha = \frac{\sigma}{\sigma_0} \tag{A.7}$$

$$S(x,t) = \frac{\hbar}{2}\arctan\left(\frac{\tan\omega t}{\alpha^2}\right) + \frac{\hbar\sin\omega t}{4\sigma_0^2}\frac{x^2(\alpha^4-1)\cos\omega t + 2xy - y^2\cos\omega t}{\alpha^4\cos^2\omega t + \sin^2\omega t}. \tag{A.8}$$

The corresponding velocity field is

$$\begin{aligned} &b(x,t) = A(t) + B(t)x \qquad A(t) = \frac{\alpha^2\cos\omega t - \sin\omega t}{\alpha^4\cos^2\omega t + \sin^2\omega t}\omega y \\ &B(t) = -\frac{(\alpha^4-1)\cos\omega t\sin\omega t + \alpha^2}{\alpha^4\cos^2\omega t + \sin^2\omega t}\omega. \end{aligned} \tag{A.9}$$

Let us consider the Hamiltonian

$$\hat{H} = \frac{1}{2m}(\hat{P}_1^2 + \hat{P}_2^2) + \frac{m\omega^2}{2}(\hat{X}_1^2 + \hat{X}_2^2). \tag{A.10}$$

We want to analyse the time evolution of the initial state (40) as ruled by the Hamiltonian (A.10) and with no measurement at time 0. We put $\sigma_1 = \sigma_2 = \sigma$, so that our initial wavefunction will be

$$\begin{aligned} &\Phi(x_1, x_2; 0) = \frac{\mathrm{e}^{-\boldsymbol{x}\cdot\mathbb{A}\boldsymbol{x}/4}}{\sqrt{2\pi\sigma^2\sqrt{1-r^2}}} \\ &\mathbb{A} = \mathbb{R}^{-1} = \frac{1}{\sigma^2(1-r^2)}\begin{pmatrix} 1 & -r \\ -r & 1 \end{pmatrix} \\ &\mathbb{R} = \sigma^2\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} \end{aligned} \tag{A.11}$$

and the wave equation becomes

$$\mathrm{i}\hbar\partial_t\Phi = -\frac{\hbar^2}{2m}(\partial_1^2 + \partial_2^2)\Phi + \frac{m\omega^2}{2}(x_1^2 + x_2^2)\Phi. \tag{A.12}$$

Thanks to the circular symmetry ($\omega_1 = \omega_2 = \omega$) we can perform a rotation of x_1, x_2 axes, which, with an invariant form of (A.12), allows us to rewrite the initial state (A.11) as a factorized wavefunction. Since the two harmonic oscillators are dynamically uncoupled, this factorization makes the evolution completely independent in the two variables x_1 and x_2. It is easy to see that the rotation

$$\boldsymbol{y} = \mathbb{O}\boldsymbol{x} \qquad \mathbb{O} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \tag{A.13}$$

transforms the wavefunction into

$$\begin{aligned} &\Psi(y_1, y_2; 0) = \frac{\mathrm{e}^{-\boldsymbol{y}\cdot\mathbb{B}\boldsymbol{y}/4}}{\sqrt{2\pi\sigma^2\sqrt{1-r^2}}} = \varphi_1(y_1, 0)\varphi_2(y_2, 0) \\ &\varphi_1(y_1, 0) = \frac{\mathrm{e}^{-y_1^2/4\sigma_1^2}}{\sqrt{\sigma_1\sqrt{2\pi}}} \qquad \varphi_2(y_2, 0) = \frac{\mathrm{e}^{-y_2^2/4\sigma_2^2}}{\sqrt{\sigma_2\sqrt{2\pi}}} \\ &\sigma_1 = \sigma\sqrt{1-r^2} \qquad \sigma_2 = \sigma\sqrt{1+r^2} \\ &\mathbb{B} = \mathbb{O}\mathbb{A}\mathbb{O}^T = \frac{1}{\sigma^2}\begin{pmatrix} \frac{1}{1-r} & 0 \\ 0 & \frac{1}{1+r} \end{pmatrix} \\ &\mathbb{B}^{-1} = \mathbb{O}\mathbb{R}\mathbb{O}^T = \sigma^2\begin{pmatrix} 1-r & 0 \\ 0 & 1+r \end{pmatrix} \end{aligned} \tag{A.14}$$

while the wave equation (A.12) still has the same form in the new coordinates. As a consequence φ_1 and φ_2 evolve independently in $\varphi_1(x_1, t)$ and $\varphi_2(x_2, t)$, each following (A.6) with its own parameters $\sigma_1, \sigma_2, \alpha_1$ and α_2. From the evolution $\Psi(y_1, y_2; t) = \varphi_1(x_1, t)\varphi_2(x_2, t)$

we recover the evolution $\Phi(x_1, x_2; t)$ in the original coordinates by means of the inverse rotation $\boldsymbol{x} = \mathbb{O}^T \boldsymbol{y}$.

By SM1 we can associate Nelson vector processes with both Φ and Ψ: let us denote them respectively by $\boldsymbol{\xi} = (\xi_1, \xi_2)$ and $\boldsymbol{\eta} = (\eta_1, \eta_2)$. Their components will be connected by the rotations

$$\boldsymbol{\eta} = \mathbb{O}\boldsymbol{\xi} \qquad \boldsymbol{\xi} = \mathbb{O}^T \boldsymbol{\eta}. \tag{A.15}$$

Clearly $\eta_1(0)$ and $\eta_2(0)$ are independent, Gaussian and respectively distributed as $\mathcal{N}(0, \sigma_1^2)$ and $\mathcal{N}(0, \sigma_2^2)$. Moreover their evolution is completely independent (the stochastic differential equations are not coupled), so $\eta_1(t)$ and $\eta_2(t')$ remain independent, Gaussian and respectively distributed as $\mathcal{N}(0, \sigma_1^2(t))$ and $\mathcal{N}(0, \sigma_2^2(t'))$ ($\sigma_1^2(t)$ and $\sigma_2^2(t')$ are of the form (A.7) with $\alpha_1 = \sigma_1/\sigma_0$ and $\alpha_2 = \sigma_2/\sigma_0$). Now, since $\mathbb{E}\eta_1(0)\eta_2(t) = \mathbb{E}\eta_1(0)\mathbb{E}\eta_2(t) = 0$ and $\mathbb{E}\eta_2(0)\eta_1(t) = 0$, keeping account of (A.15) we have

$$\mathbb{E}\xi_1(0)\xi_2(t) = \tfrac{1}{2}\mathbb{E}[(\eta_1(0) + \eta_2(0))(-\eta_1(t) + \eta_2(t))] = \tfrac{1}{2}[\mathbb{E}\eta_2(0)\eta_2(t) - \mathbb{E}\eta_1(0)\eta_1(t)]. \tag{A.16}$$

As a consequence we can study $\mathbb{E}\xi_1(0)\xi_2(t)$ by analysing $\mathbb{E}\eta_1(0)\eta_1(t)$ and $\mathbb{E}\eta_2(0)\eta_2(t)$.

Since the two processes η_1 and η_2 have a completely identical behaviour (except for the value of one parameter, their initial variance) let us drop the indices 1 and 2 and let us calculate $\mathbb{E}\eta(0)\eta(t)$ for a process $\eta(t)$ such that $\eta(0) = \mathcal{N}(0, \sigma^2)$, and such that the drift velocity field is

$$\begin{aligned} &b(y, t) = yB(t) \\ &B(t) = -\omega\frac{(\alpha^4 - 1)\sin\omega t\cos\omega t + \alpha^2}{\alpha^4\cos^2\omega t + \sin^2\omega t} = -\omega\frac{2\alpha^2 + (\alpha^4 - 1)\sin 2\omega t}{(\alpha^4 + 1) + (\alpha^4 - 1)\cos 2\omega t} \\ &\alpha = \frac{\sigma}{\sigma_0}. \end{aligned} \tag{A.17}$$

The stochastic differential equation (15) will take the form

$$\mathrm{d}\eta\,(t) = b(\eta(t), t)\,\mathrm{d}t + \mathrm{d}W\,(t) = B(t)\eta(t)\,\mathrm{d}t + \mathrm{d}W\,(t) \tag{A.18}$$

or in integral form

$$\eta(t) = \eta(0) + \int_0^t B(s)\eta(s)\,\mathrm{d}s + W(t). \tag{A.19}$$

Now from

$$\eta(0)\eta(t) = \eta^2(0) + \int_0^t B(s)\eta(0)\eta(s)\,\mathrm{d}s + \eta(0)W(t) \tag{A.20}$$

we immediately obtain

$$\mathbb{E}\eta(0)\eta(t) = \mathbb{E}\eta^2(0) + \int_0^t B(s)\mathbb{E}\eta(0)\eta(s)\,\mathrm{d}s. \tag{A.21}$$

Hence, by differentiating, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}\eta(0)\eta(t) = B(s)\mathbb{E}\eta(0)\eta(s) \tag{A.22}$$

with the initial condition $\mathbb{E}\eta^2(0) = \sigma^2$, and finally

$$\mathbb{E}\eta(0)\eta(t) = \sigma^2 \mathrm{e}^{\int_0^t B(s)\,\mathrm{d}s}. \tag{A.23}$$

The fact that the right-hand side of (A.23) will go exponentially fast to zero for $t \to +\infty$ would be apparent if $B(t) < 0$ for every $t > 0$; but this happens only if $\sqrt{\sqrt{2}-1} < \alpha < \sqrt{\sqrt{2}+1}$. However, it is easy to show from (A.17) that in general we have

$$\mathbb{E}\eta(0)\eta(t) = \sigma^2\sqrt{\frac{(\alpha^4+1)+(\alpha^4-1)\cos 2\omega t}{2\alpha^4}}\,\mathrm{e}^{-\arctan(\alpha^{-2}\tan\omega t)} \tag{A.24}$$

and a simple geometrical argument proves that (A.24) for $t \to +\infty$ is infinitesimal, of the same order as $\mathrm{e}^{-\omega t}$.

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