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THE HALO FORMATION IN CHARGED PARTICLE BEAMS: A STOCHASTIC–HYDRODYNAMIC APPROACH

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The formation of the beam halo in charged particle accelerators is studied in the framework of a stochastic-hydrodynamic model for the collective motion of the particle beam. We take into account space-charge effects, which lead to a set of self–consistent coupled nonlinear hydrodynamic equations. Solutions of the dynamical equations describe quasi-stationary beam configurations with enhanced transverse dispersion and transverse emittance growth. Finally, potentials and drifts leading to ring–shaped halos are studied.

1. Introduction

In high intensity beams of charged particles, proposed in recent years for a wide variety of accelerator–related applications, it is very important to keep at low level the beam loss to the wall of the beam pipe, since even small fractional losses in a high–current machine can cause exceedingly high levels of radioactivation. One of the possible relevant mechanisms for these losses is the formation of a low intensity beam halo more or less far from the core. These halos have been observed [1] or studied in experiments [2], and have also been subjected to an extensive simulation analysis [3,4,5,6,7,8,9,10,11]. It is however widely believed that for the next generation of high intensity machines it is still necessary to obtain a more quantitative understanding not only of the physics of the halo, but also of the beam transverse distribution in general [12,13,14]. In fact "because there is not a consensus about its definition, halo remains an imprecise term" [15] so that several proposals have been put forward for its description.

The charged particle beams are usually described in terms of classical dynamical systems. The standard model is that of a collisionless plasma where the corresponding dynamics is embodied in a suitable phase space (see for example [16]). We propose and develop a different approach: a model for the halo formation in particle beams based on the idea that the trajectories are samples of a stochastic process, rather than usual deterministic (differentiable) trajectories. Indeed the authors believe that a plasma (with collisions) described in terms of controlled stochastic processes seems a good candidate to explain the rare escape of particles from a quasi-stable beam core by statistically taking into account the random inter–particle interactions that can not be described in detail. Of course the idea of a stochastic approach is hardly new [16,17,18], but there are several different ways to implement it. In fact, the system we want to describe is endowed with some measure of invariance under time reversal [19], and this looks reasonable in the intermediate regime of stability. In a few previous papers [20,21] we explicitly introduced a stochastic description which involves both a kinematical, diffusion equation and a dynamical equation with external potentials. We showed that this method allows also to implement techniques of active control for the dynamics of the beam. These techniques have been proposed to improve the beam focusing and to independently change the frequency of the betatron oscillations. As a first step to approach the halo problem, we implement the method to quantitatively investigate the nature, the size and the dynamical characteristics of a possible stationary beam halo.

Time–reversal invariant diffusion processes are obtained by promoting deterministic kinematics to stochastic kinematics, and by adding a further dynamical prescription [22,23]. The simpler, and most elegant, way to obtain the equations of such processes is to impose stochastic variational principles which generalise the usual ones of the deterministic mechanics to the case of diffusive kinematics [22,24]. This method can be applied to classical, conservative many–particle systems, whose complex dynamics can be effectively described by a representative particle performing stochastic trajectories. If ρ is the (normalized) density of the particles and $\mathbf{v}$ the current velocity, the stochastic variational principle leads to the following gradient form for the velocity field $\mathbf{v}$

$$m\mathbf{v}(\mathbf{r},t) = \nabla S(\mathbf{r},t)\,, \tag{1}$$

and to the couple of nonlinear hydrodynamic equations

$$\partial_t \rho = -\nabla \cdot (\rho \mathbf{v})\,, \tag{2}$$

$$\partial_t S + \frac{m}{2}\mathbf{v}^2 - 2mD^2 \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} + V(\mathbf{r},t) = 0\,. \tag{3}$$

Here, D is the diffusion coefficient, and $V(\mathbf{r},t)$ is the external potential energy applied to the system. Due to the non differentiability of the stochastic trajectories, it is not possible to define the standard velocity. One then introduces the forward velocity $\mathbf{v}_+$ (connected to the mean time-derivative from the right) and the backward velocity $\mathbf{v}_-$ (connected to the mean time-derivative from the left). In the conservative diffusions, the two velocities are exchanged under time-reversal. Furthermore, the current velocity is the balanced mean of $\mathbf{v}_\pm$; it describes the velocity of the center of the density profile, and obviuosly reduces to the standard deterministic velocity if the noise is removed putting $D = 0$. In this last case, the two hydrodynamic equations reduces to the equations for an ideal fluid. The equation (2) can be also explicitely written in the form of Fokker–Planck equation

$$\partial_t \rho = -\nabla \cdot [\mathbf{v}_{(+)}\rho] + D\,\nabla^2 \rho \tag{4}$$

formally associated to the Itô equation. It is finally important to remark that, introducing the representation [25]

$$\psi(\mathbf{r},t) = \sqrt{\rho(\mathbf{r},t)}\, e^{iS(\mathbf{r},t)/\alpha}\,, \tag{5}$$

(with $\alpha = 2mD$) the coupled equations (2) and (3) are made equivalent to a single linear equation of the form of the Schrödinger equation in the function ψ, with the Planck action constant replaced by α:

$$i\alpha\partial_t \psi = -\frac{\alpha^2}{2m}\nabla^2\psi + V\psi\,. \tag{6}$$

We will refer to it as a Schrödinger–like (S-l) equation. In this formulation the phenomenological "wave function" ψ carries the information on the dynamics of both: the bunch density, and the velocity field of the bunch, since the velocity field is determined through equation (1) by the phase function $S(\mathbf{r},t)$. This shows that our procedure, starting from a different point of view, leads to a description formally analogous to that of the so called Quantum-like approaches to beam dynamics [26]. In this last framework, an interesting analysis of the halo formation in particle beams has been performed in Ref. 27.

2. Self consistent equations

One of the possible mechanisms for the formation of the halo in particle beams is that due to the unavoidable presence of space charge effects. In this Section we will investigate this possibility in the framework of our hydrodynamic-stochastic model of beam dynamics. To this end, we take into account the space charge effects by coupling the hydrodynamic equations of stochastic mechanics with the Maxwell equations which describe the mutual electromagnetic interactions between the particles of the beam. We thus obtain a self-consistent, stochastic magnetohydrodynamic system of coupled nonlinear differential equations that can be numerically solved to show the effect of the space charge.

In the following, the reference physical system will be an ensemble of N identical copies of a single charged particle embedded in a particle beam and subject to both an external and a space-charge potential. In a reference frame comoving with the beam, our system is then described by the Schrödinger equation (6), where α is the unit of action (emittance) and $\widehat{H}$ the Hamiltonian operator which will be explicitly determined in the following. Since in general ψ is not normalized, we introduce the following notation for its constant norm

$$\|\psi\|^2 = \int_{\mathbf{R}^3} |\psi(\mathbf{r},t)|^2 \, d^3\mathbf{r} \,, \tag{7}$$

so that, if N is the number of particles with individual charge q_0, the space charge density of the beam will be

$$\rho_{sc}(\mathbf{r},t) = N q_0 \frac{|\psi(\mathbf{r},t)|^2}{\|\psi\|^2} \,. \tag{8}$$

In this paper we study the case of zero current density, so that the wave function is stationary:

$$\psi(\mathbf{r},t) = u(\mathbf{r}) \, e^{-iEt/\alpha} \,. \tag{9}$$

We couple Eq. (6) with the Poisson equation for the scalar potential Φ_{sc} generated by the space charge, and we consider that in a reference frame comoving with the beam we can always assume that the wave function is of the stationary form (9). Moreover, if the beam with space-charge interactions stays cylindrically symmetric the function u will depend only on the modulus of the cylindrical radius $\mathbf{r}$. We define the potential energies $V_{ext} = q_0 \Phi_{ext}$, and $V_{sc} = q_0 \Phi_{sc}$. We then choose as external potential energy a cylindrically symmetric, harmonic potential with a proper frequency ω in absence of space charge: $V_{ext} = (m\omega^2 r^2)/2$, with $r = \sqrt{x^2 + y^2}$, and introduce the dimensionless quantities

$$s = \frac{r}{\sigma\sqrt{2}},$$

$$w(s) = w\left(\frac{r}{\sigma\sqrt{2}}\right) \equiv \sigma^{3/2} u(r), \tag{10}$$

$$v(s) = v\left(\frac{r}{\sigma\sqrt{2}}\right) \equiv \frac{4m\sigma^2}{\alpha^2} V_{sc}(r),$$

where $\sigma^2 = \alpha/2m\omega$ is the variance of the ground state of the cylindrical harmonic oscillator without space charge.

Then, the coupled (Schrödinger and Poisson) equations become, in terms of the adimensional unities,

$$s w''(s) + w'(s) + \left[\beta - s^2 - v(s)\right] s w(s) = 0, \tag{11}$$

$$s v''(s) + v'(s) + b\, s w^2(s) = 0, \tag{12}$$

where

$$\beta = \frac{4m\sigma^2}{\alpha^2} E = \frac{2E}{\alpha\omega},$$

$$b = \frac{4m\sigma^2}{\alpha^2} \frac{N q_0^2}{2\pi\epsilon_0 L} \frac{1}{B} = \frac{2}{\alpha\omega} \frac{N q_0^2}{2\pi\epsilon_0 L} \frac{1}{B}, \tag{13}$$

$$B = \frac{A\sigma}{2} = \int_0^\infty s w^2(s)\, ds.$$

The effect of the space charge will be accounted for by comparing the normalized solution $w(s)$ with the unperturbed ground state of the cylindrical harmonic oscillator:

$$\psi_{000}(r) = \frac{e^{-r^2/4\sigma^2}}{\sigma\sqrt{2\pi L}}. \tag{14}$$

We have solved numerically the system (11), (12) by tentatively fixing one of the two free parameters b and β, and then searching by an iterative trial and error method a value of the other such that the solution shows, in a given interval of values of s, the correct infinitesimal asymptotic behavior for large values of s. We have then normalized the solutions $w(s)$ by calculating numerically the value of B. It is clear from the definition of the dimensionless parameters B, b and β (13) that the value of β is a sort of reduced energy eigenvalue of the system, while the product

$$\gamma = Bb\,, \tag{15}$$

which is by definition a non negative number, will play the role of the interaction strength, since it depends on the space-charge density along the linear extension of the beam. Reverting to dimensional quantities, since $\alpha^2/2m\sigma^2$ has the dimensions of an energy, the two relevant parameters are

$$E = \beta\,\frac{\alpha^2}{4m\sigma^2}\,, \qquad \frac{Nq_0^2}{2\pi\epsilon_0 L} = \gamma\,\frac{\alpha^2}{4m\sigma^2}\,, \tag{16}$$

which are respectively the energy of the individual particle embedded in the beam, and the strength of the space-charge interaction.

In the following we will limit ourselves to discuss solutions of Eqns. (11), and (12) without nodes (a sort of ground state for the system). It is possible to see that no solution without nodes can be found for values of the space-charge strength γ beyond about 22.5 and that for values of γ ranging from 0 to 22.5, the energy β decreases monotonically from 2.0000 to -0.0894. If the unit of action α is fixed at a given value, it is apparent that the value of γ is directly proportional to the charge per unit length Nq_0/L of the beam: a small value of γ means a rarefied beam; a large value of γ indicates that the beam is intense. A halo is supposed to be present in intense beams, while in rarefied beams the behavior of every single particle tends to be affected only by the external harmonic potential.

It is useful to provide a numerical estimate (in MKS units) of the relevant parameters. We are assuming a beam made of protons, so that m and q_0 are the proton mass and charge, while ϵ_0 is the vacuum permittivity. From empirical data, a reasonable estimate for σ yields $\sigma \approx 10^{-3}$ m, while for N/L usually a value of 10^{11} particles per meter is considered realistic. On the ground of accepted experimental values of the beam emittance (usually measured in units of length) we can assume α/mc of the order of 10^{-7}m, and in particular from now on we will fix α at its approximate central value $\frac{\alpha}{mc} \approx 4.0 \times 10^{-7}$ m. Finally, we note that by changing the beam intensity (namely N/L and γ) one correspondingly changes the energy (namely E and β) of the individual particle embedded in the beam. The quantities β and γ are dimensionless: the true physical quantities (energies) are obtained by multiplicating them by the unit of energy $\frac{\alpha^2}{4m\sigma^2} \approx 37.5$ eV.

The overall effect of the space charge in this model is a conspicuous spreading of the transverse distribution of the particles in the beam with respect to the unperturbed ground state distribution (14) k[32]. When $\gamma = 0$ the potential due to space charge vanishes, and the solution exactly coincides with the ground state of the cylindrical harmonic oscillator with variance σ^2. When $\gamma > 0$ the transverse distribution begins to spread [32]. To give a more quantitative measure of the flattening and broadening of the transverse distribution we compare numerically the probabilities of finding a particle at a relatively large distance from the beam longitudinal axis with and without space charge. The quantity

$$\mathbf{P}_\gamma(c) = \int_{c/\sqrt{2}}^{\infty} s\, w^2(s)\, ds \tag{17}$$

is the probability of finding a particle at a distance greater than $c\sigma$ for systems with a given strength γ of the space charge coupling. For instance, considering the two different situations $\gamma = 0$ (no space charge) and $\gamma = 22.5$ (strong space charge) we have:

$$\mathbf{P}_0(10) \approx 1.9 \times 10^{-22}\,, \quad \mathbf{P}_{22.5}(10) \approx 1.7 \times 10^{-6}\,. \tag{18}$$

We see that the probability of finding particles at a distance larger than 10σ from the core of the beam is enhanced by space charge by many orders of magnitude. This means, for example, that if in the beam there are 10^{11} particles per meter, while practically no one is found beyond 10σ in absence of space charge, for very strong space-charge intensity we can find up to 10^5 particles per meter at that distance from the core.

The above analysis shows that in the hydrodynamic-stochastic theory of charged beams, the space-charge potential induces a strong broadening of the unperturbed transverse density distribution of the beam, thus yielding a small, but finite probability of having particles at a distance well away from the core of the beam.

The hydrodynamic-stochastic model allows also an estimate on the growth of the emittance due to the presence of the space charge. The emittance can be calculated by exploiting a structure of uncertainty products that is inherent to the SM. In particular, the transverse emittance can be calculated as $\Delta x \cdot \Delta p_x$, where we have:

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}\,, \quad \Delta p_x = \sqrt{\langle p_x^2 \rangle - \langle p_x \rangle^2}\,. \tag{19}$$

In our case, it is easy to show that the root mean square deviation for the position is:

$$\Delta x = \sqrt{\langle x^2 \rangle} = \sigma \sqrt{\int_0^\infty s^3 w^2(s)\, ds} \,. \tag{20}$$

As for the momentum $\mathbf{p}$, it can be recovered from the velocity field which is well defined in SM. Since we are considering a stationary state, only the osmotic part of the velocity field is non zero, and the momentum field reads

$$\mathbf{p}(\mathbf{r}) = \alpha \, \frac{\nabla \rho(\mathbf{r})}{\rho(\mathbf{r})} = 2\alpha \, \frac{\nabla u(r)}{u(r)} \,. \tag{21}$$

As a consequence, the root mean square deviation for the x–component of the momentum is:

$$\Delta p_x = \sqrt{\langle p_x^2 \rangle} = \frac{\alpha}{\sigma} \sqrt{\int_0^\infty w'^2(s) s \, ds} \,. \tag{22}$$

We can then define the emittance $\mathcal{E}$ as the position-momentum uncertainty product in phase space in the following way:

$$\mathcal{E} = \Delta x \cdot \Delta p_x = \alpha \sqrt{\int_0^\infty s^3 w^2(s) ds \cdot \int_0^\infty w'^2(s) s ds}. \tag{23}$$

The emittance (23) can be thus estimated from knowledge of the numerical solutions $w(s)$ and their first derivatives $w'(s)$. It can be seen [32] that the emittance is exactly α for $\gamma = 0$ (namely in absence of space charge), and grows with γ to a value $\approx 1.2 \times \alpha$ for $\gamma \approx 22.5$. This result is consistent with the expected growth of emittance produced by space-charge effects, and it provides evidence that in the model of SM of charged beams the constant α plays the role of a lower bound for the phase-space emittance.

We have seen that in our scheme the space–charge density leads to an enlargement of the particle density, that can be interpreted as a possible halo effect. However, the space–charge is only one possible source of halo. It has been also suggested that the halo distribution could be described, in first approximation, as a Gaussian core distribution plus a small ring of particles surrounding it and constituting the halo. Thus, we now will follow a different route. In fact, since it is not clearly established that a halo can be due only to space-charge effects, starting with a realistic ring distribution and trying to understand the dynamics that can produce it could be very useful in this respect. Then, we will simply assume a specific form of a beam with a halo without deriving it as an effect of space charge interactions, and resorting to the techniques introduced in Refs. 28, 29 and 20, we reverse the point of view: we insert into the equations (3) and (4) the supposed distribution, and obtain the drift and the potential which realize it. For a three-dimensional, cylindrically symmetric beam we introduce the normalized radial density distribution

$$\rho(r) = A\,\frac{e^{-r^2/2\sigma^2}}{\sigma^2} + (1-A)\frac{e^{-r^2/2p^2\sigma^2}}{p^2\sigma^2\Gamma(q+1)}\left(\frac{r^2}{2p^2\sigma^2}\right)^q \tag{24}$$

which is composed of a Gaussian core with variance σ^2 (the simple harmonic oscillator ground state), plus a ring-like distribution whose size is fixed through the two parameters $p > 0$ and $q \geq 0$. The parameter $0 \leq A \leq 1$ is the relative weight of the two parts. This density distribution has the required form for suitable values of the parameters, but it has no nodes. This is convenient for two main reasons: first because it is a rather general requirement for a ground state to have no nodes (this fact is a rigorous theorem for one-dimensional systems). Moreover, it has been shown in Refs. 28, 30 and 31 that stationary distributions without nodes are also attractors for every other possible (non extremal) initial distribution: a property that will be useful in a future discussion of the possible relaxation of the system toward a stable beam halo.

Using, as usual, adimensional quantities, and using the methods of Refs. 28, 29 and 20, we can obtain the radial drift and the radial potential associated to the distribution (24). It results that the drift undergoes a sudden jump if compared to the umperturbed drift associated to the pure Gaussian, halo–free distribution (14). Correspondingly, the potential is deformed with respect to the harmonic shape, and displays a narrow peak: this last generates the escape of a fraction of the particles (for details, see Ref. 32).

In order to illustrate more explicitely the main effects involved, let us give here the simpler formulae relative to the 1d case. We consider only 1d processes denoting by x one of the transverse space coordinates. We assume that the longitudinal and the transverse beam dynamics can be deemed independent, with the further simplification of considering decoupled evolutions along the transverse directions x and y. Under these conditions the density distribution with a halo ring now reads

$$\rho(x) = A\,\frac{e^{-x^2/2\sigma^2}}{\sigma\sqrt{2\pi}} + (1-A)\frac{e^{-x^2/2p^2\sigma^2}}{p\sigma\sqrt{2}\Gamma\left(q+\frac{1}{2}\right)}\left(\frac{x^2}{2p^2\sigma^2}\right)^q . \tag{25}$$

It is then straightforward to compute the corresponding velocities and potential, obtaining [32]

$$\begin{aligned} u(x) &= \sqrt{\rho(x)}\,, \\ v_{(+)}(x) &= \frac{\alpha}{m}\frac{u'(x)}{u(x)}\,, \\ V(x) &= \frac{\alpha^2}{4m\sigma^2} + \frac{\alpha^2}{2m}\frac{u''(x)}{u(x)}\,. \end{aligned} \tag{26}$$

Using these expressions, it can be verified a behavior similar to that of the three–dimensional case: sudden jumps of the drift, with corresponding narrow peaks in the potential.

In conclusion, we have presented a dynamical, stochastic approach to the description of the beam transverse distribution in the particle accelerators. We have shown that in this framework it is possible to have tranverse distribution which show a broadening and an emittance growth, typical of the halo formation. In forthcoming papers we plan to extend these techniques to the problem of engineering suitable time–dependent potentials for the control and the elimination of the beam halo.

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