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Lévy–Student processes for a stochastic model of beam halos

N. Cufaro Petroni[a,*], S. De Martino[b], S. De Siena[b], F. Illuminati[b]

[a]Department of Mathematics, University of Bari, and INFN Sezione di Bari, via E. Orabona 4, 70125 Bari, Italy
[b]Department of Physics, University of Salerno, and INFN Sezione di Napoli (gruppo di Salerno), Via S. Allende, I–84081 Baronissi (SA), Italy

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Abstract

We describe the transverse beam distribution in particle accelerators within the controlled, stochastic dynamical scheme of the stochastic mechanics which produces time reversal invariant diffusion processes. In this paper we analyze the consequences of introducing the generalized Student laws, namely non-Gaussian, Lévy *infinitely divisible* (but not *stable*) distributions. We will analyze this idea from two different standpoints: (a) first by supposing that the stationary distribution of our (Wiener powered) stochastic model is a Student distribution; (b) by supposing that our model is based on a (non-Gaussian) Lévy process whose increments are Student distributed. In the case (a) the longer tails of the power decay of the Student laws, and in the case (b) the discontinuities of the Lévy–Student process can well account for the rare escape of particles from the beam core, and hence for the formation of a halo in intense beams.

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Keywords: Beam halo; Lévy processes; Stochastic mechanics

1. Introduction

The charged particle beams dynamics and their possible halos are described in this paper in terms of stochastic processes. To have *time reversal invariance* a dynamics must be added; but since the position $\mathbf{Q}(t)$ of our process is Markovian and not derivable, we are obliged to drop the momentum equation and to work in a *configuration space*. Consequently the dynamics is introduced by means of a *stochastic variational principle*. This scheme, the stochastic mechanics (SM), is known for its application to classical stochastic models for quantum mechanics [1,2], but is suitable for a large number of other systems [3,4]. This leads to a linearized theory summarized in a Schrödinger-like (S-ℓ) equation as the basis for a model of beam dynamics [5]. The space charge effects have been introduced in more recent papers [6] by coupling this S-ℓ equation with the Maxwell equations.

A new role in the beam dynamics can be played by non-Gaussian Lévy distributions [7]. Their today's popularity is mainly confined to the *stable* laws [3,8]. We introduce instead a family of non-Gaussian Lévy laws which are *infinitely divisible but not stable* [9]: the generalized Student laws. This has two advantages: first, at variance with stable non-Gaussian laws, the Student laws can have *finite variances*; second, the Student laws can incrementally *approximate the Gaussian laws*. On the other hand an i.d. law is all that is required to build the *Lévy processes* used to represent the evolution of our particle beam. The Student laws will be used here in two ways:

- in the framework of the traditional SM, with randomness supplied by a *Gaussian Wiener noise*, we study the self-consistent potentials which can produce a *Student distribution as stationary transverse distribution* of a particle beam, and we focus our attention on the increase of the probability of finding the particles far away from the beam core.
- we define a *Lévy–Student process*, and we show that these processes can help to explain how a particle can be expelled from the bunch by means of some kind of hard collision. In fact the trajectories of our Lévy–Student

*Corresponding author.
E-mail address: cufaro@ba.infn.it (N.C. Petroni).

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process show the *typical jumps of the non-Gaussian Lévy processes*: a feature that we propose to use as a model for the halo formation.

2. Stochastic beam dynamics

The position $\mathbf{Q}(t)$ of a representative particle in the beam is a process ruled by the Itô stochastic differential equation (SDE)

$$\mathrm{d}\mathbf{Q}(t) = \mathbf{v}_{(+)}(\mathbf{Q}(t), t)\,\mathrm{d}t + \sqrt{D}\,\mathrm{d}\mathbf{W}(t), \tag{1}$$

where $\mathbf{v}_{(+)}(\mathbf{r}, t)$ is the forward velocity, $\mathrm{d}\mathbf{W}(t)$ is the increment process of a standard Wiener noise, the diffusion coefficient D is constant, and the action $\alpha = 2mD$ will be later connected to the emittance of the beam. To add a dynamics we introduce a stochastic least action principle and we get a Nelson process [2]. If $\rho(\mathbf{r}, t)$ is the pdf of $\mathbf{Q}(t)$, and we define the backward, current and osmotic velocities

$$\mathbf{v}_{(-)} = \mathbf{v}_{(+)} - 2D\,\frac{\nabla\rho}{\rho}$$

$$\mathbf{v} = \frac{\mathbf{v}_{(+)} + \mathbf{v}_{(-)}}{2}, \quad \mathbf{u} = \frac{\mathbf{v}_{(+)} - \mathbf{v}_{(-)}}{2}$$

from the stochastic least action principle we get that the current velocity is irrotational

$$m\mathbf{v}(\mathbf{r}, t) = \nabla S(\mathbf{r}, t) \tag{2}$$

and the Lagrange equations of motion for ρ and S are

$$\partial_t \rho = -\frac{1}{m}\,\nabla\cdot(\rho\nabla S), \tag{3}$$

$$\partial_t S = -\frac{1}{2m}\,\nabla S^2 + 2mD^2\,\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} - V, \tag{4}$$

where V is an external potential. This system is time-reversal invariant; in fact the forward velocity $\mathbf{v}_{(+)}(\mathbf{r}, t)$ is not given a priori, but it is dynamically determined by the evolution equation (4). With the representation

$$\Psi(\mathbf{r}, t) = \sqrt{\rho(\mathbf{r}, t)}\,\mathrm{e}^{\mathrm{i}S(\mathbf{r},t)/\alpha}, \quad \alpha = 2mD \tag{5}$$

the coupled equations (3) and (4) become a single linear equation of the form of the Schrödinger equation, with the Planck action constant replaced by α:

$$\mathrm{i}\alpha\partial_t\Psi = -\frac{\alpha^2}{2m}\,\nabla^2\psi + V\Psi. \tag{6}$$

We analyzed in several papers [10] both the stationary and the non-stationary behavior of this S-ℓ equation.

3. Self-consistent equations

In this SM scheme $|\Psi(\mathbf{r}, t)|^2$ is the pdf of a Nelson process; hence when the N-particles are a pure ensemble, $N|\Psi(\mathbf{r}, t)|^2\,\mathrm{d}^3\mathbf{r}$ is the number of particles in a small neighborhood of $\mathbf{r}$. Our N particles, however, are not a pure ensemble due to their mutual e.m. interaction: we should take into account the space charge effects by coupling the S-ℓ equation with the Maxwell equations [6]. The space charge and current densities are

$$\rho_{\mathrm{sc}}(\mathbf{r}, t) = Nq_0|\Psi(\mathbf{r}, t)|^2, \tag{7}$$

$$\mathbf{j}_{\mathrm{sc}}(\mathbf{r}, t) = Nq_0\,\frac{\alpha}{m}\,\Im\{\Psi^*(\mathbf{r}, t)\boldsymbol{\nabla}\Psi(\mathbf{r}, t)\}. \tag{8}$$

so that he e.m. potentials $(\mathbf{A}_{\mathrm{sc}}, \Phi_{\mathrm{sc}})$ and Ψ obey the following system of wave equations and gauge conditions:

$$0 = \nabla\cdot\mathbf{A}_{\mathrm{sc}}(\mathbf{r}, t) + \frac{1}{c^2}\,\partial_t\Phi_{\mathrm{sc}}(\mathbf{r}, t),$$

$$\mu_0\mathbf{j}_{\mathrm{sc}}(\mathbf{r}, t) = \nabla^2\mathbf{A}_{\mathrm{sc}}(\mathbf{r}, t) - \frac{1}{c^2}\,\partial_t^2\mathbf{A}_{\mathrm{sc}}(\mathbf{r}, t),$$

$$\frac{\rho_{\mathrm{sc}}(\mathbf{r}, t)}{\varepsilon_0} = \nabla^2\Phi_{\mathrm{sc}}(\mathbf{r}, t) - \frac{1}{c^2}\,\partial_t^2\Phi_{\mathrm{sc}}(\mathbf{r}, t),$$

$$\mathrm{i}\,\frac{\alpha}{2m}\,\partial_t\Psi = \left[\mathrm{i}\alpha\nabla - \frac{q_0}{c}(\mathbf{A}_{\mathrm{sc}} + \mathbf{A}_{\mathrm{e}})\right]^2\Psi + q_0(\Phi_{\mathrm{sc}} + \Phi_{\mathrm{e}})\Psi.$$

For stationary wave functions

$$\Psi(\mathbf{r}, t) = \psi(\mathbf{r})\mathrm{e}^{-\mathrm{i}Et/\alpha} \tag{9}$$

with potential energies $V_{\mathrm{e}}(\mathbf{r}) = q_0\Phi_{\mathrm{e}}(\mathbf{r})$, $V_{\mathrm{sc}}(\mathbf{r}) = q_0\Phi_{\mathrm{sc}}(\mathbf{r})$, for cylindrical symmetry with constant p_z and beam length L

$$\psi(\mathbf{r}) = \chi(r, \varphi)\,\frac{\mathrm{e}^{\mathrm{i}p_z z/\alpha}}{\sqrt{L}}, \quad p_z = \frac{2k\pi\alpha}{L}, \ k = 0, \pm 1, \ldots$$

for $\mathcal{N} = N/L$, $E_T = E - p_z^2/2m$, $\chi(r, \varphi) = u(r)\Phi(\varphi)$, zero angular momentum, and dimensionless quantities (η, λ are dimensional constants)

$$s = \frac{r}{\lambda}, \quad \beta = \frac{E_T}{\eta}, \quad \xi = \frac{\mathcal{N}q_0^2}{2\pi\varepsilon_0\eta} \quad \text{(perveance)}$$

$$w(s) = \lambda u(\lambda s), \quad v(s) = \frac{V_{\mathrm{sc}}(\lambda s)}{\eta}, \quad v_{\mathrm{e}}(s) = \frac{V_{\mathrm{e}}(\lambda s)}{\eta}$$

we get the radial, stationary, cylindrical, dimensionless equations

$$sw''(s) + w'(s) = [v_{\mathrm{e}}(s) + v(s) - \beta]sw(s), \tag{10}$$

$$sv''(s) + v'(s) = -\xi sw^2(s). \tag{11}$$

We can look at these equations in two different ways:

- v_{e} is a given external potential and we solve the system for w and v: no simple analytical solution—playing the role of the Kapchinskij–Vladimirskij distribution—is available.
- w is a given radial distribution and we solve the system for v_{e} and v: analytical solutions are available.

We adopted the first in previous papers [6] where we numerically solved Eqs. (10) and (11); here we will instead elaborate a few ideas from the second standpoint. In fact the Poisson equation (11) with $v(0^+) = v'(0^+) = 0$ gives the space charge potential

$$v(s) = -\xi \int_0^s \frac{\mathrm{d}y}{y} \int_0^y x w^2(x)\,\mathrm{d}x \tag{12}$$

and the first equation (10) gives the external potential

$$v_e(s) = v_0(s) + \xi \int_0^s \frac{\mathrm{d}y}{y} \int_0^y x w^2(x)\,\mathrm{d}x, \tag{13}$$

$$v_0(s) = \frac{w''(s)}{w(s)} + \frac{1}{s}\frac{w'(s)}{w(s)} + \beta, \tag{14}$$

where $v_0(s)$ is the zero perveance potential ($\xi = 0$) that we would get without space charge.

4. Self–consistent potentials

First take as stationary distribution the radial ground state of a harmonic oscillator: for zero perveance

$$u_0(r) = \frac{e^{-r^2/4\sigma^2}}{\sigma}, \qquad E_T = \alpha\omega, \tag{15}$$

$$V_e(r) = \frac{m\omega^2}{2} r^2 = \frac{\alpha^2}{8m\sigma^4} r^2, \quad \sigma^2 = \frac{\alpha}{2m\omega} \tag{16}$$

and in dimensionless notation ($\eta = \alpha\omega/2,\ \lambda = \sigma\sqrt{2}$):

$$w(s) = \sqrt{2}\,e^{-s^2/2}, \qquad \beta = 2, \quad v_e(s) = s^2. \tag{17}$$

Then, by taking into account the space charge, the potentials that produce (17) as stationary wave function are (see Fig. 1)

$$v(s) = -\frac{\xi}{2}[\log(s^2) + \mathbb{C} - \mathrm{Ei}(-s^2)], \tag{18}$$

$$v_0(s) = s^2, \tag{19}$$

$$v_e(s) = s^2 + \frac{\xi}{2}[\log(s^2) + \mathbb{C} - \mathrm{Ei}(-s^2)], \tag{20}$$

where $\mathbb{C} \approx 0.577$ is the Euler constant and

$$\mathrm{Ei}(x) = \int_{-\infty}^x \frac{e^t}{t}\,\mathrm{d}t, \quad x < 0$$

is the exponential–integral function. On the other hand, if a halo is produced by large deviations from the beam axis, we alternatively suppose that the stationary transverse distributions are non-Gaussian: consider the family of univariate, two-parameters probability laws $\Sigma(v, a^2)$ with pdfs

$$f(x) = \frac{\Gamma(v+1/2)}{\Gamma(1/2)\Gamma(v/2)} \frac{a^v}{(x^2+a^2)^{v+1/2}}, \quad v > 0 \tag{21}$$

with mode and median in $x = 0$; a is a scale parameter, while v rules the power decay of the tails: for large x the tails go as $x^{-(v+1)}$. Comparison with a Gauss law $\mathcal{N}(0, \sigma^2)$ is given in Fig. 2: when v grows the difference between the pdfs becomes smaller. Since $\Sigma(n, n)$ with $n = 1, 2, \ldots$ are the classical t-Student laws, we call $\Sigma(v, a^2)$ generalized Student laws. They have finite variance

$$\sigma^2 = \frac{a^2}{v-2}. \tag{22}$$

only when $v > 2$. The circularly symmetric, bivariate Student laws $\Sigma_2(v, a^2)$ are

$$f(x, y) = \frac{v}{2\pi} \frac{a^v}{(x^2+y^2+a^2)^{v+2/2}}. \tag{23}$$

They have non-correlated, (but not independent), marginals $\Sigma(v, a^2)$. The radial transverse beam distribution with finite variance σ^2 ($v > 2$) is then

$$\rho(r) = r\,\frac{v}{2\pi L} \frac{[(v-2)\sigma^2]^{v/2}}{[r^2 + (v-2)\sigma^2]^{v+2/2}}$$

and in dimensionless form

$$w^2(s) = \frac{2v}{v-2} \frac{1}{(1+z^2)^{v+2/2}}, \quad z = \frac{s\sqrt{2}}{\sqrt{v-2}}, \tag{24}$$

with dimensional constants $\eta = \alpha^2/4m\sigma^2$ and $\lambda = \sigma\sqrt{2}$. Then, with $\beta = 2 + 8/(v-2)$, the potentials are

$$v(s) = -\frac{\xi}{2}\left[\frac{2z^{-v}}{v}\,{}_2F_1\left(\frac{v}{2}, \frac{v}{2}; \frac{v+2}{2}; -\frac{1}{z^2}\right) + \log z^2 + \mathbb{C} + \psi\left(\frac{v}{2}\right)\right], \tag{25}$$

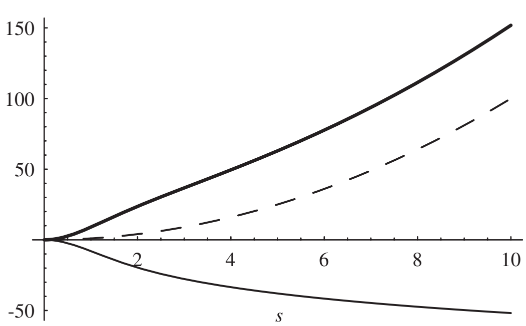

Fig. 1. The dimensionless potentials $v(s)$ (thin line), $v_0(s) = s^2$ (dashed line) and $v_e(s)$ (thick line) for $\xi = 20$. The self-consistent wave function coincides with that of a harmonic oscillator for zero perveance (17).

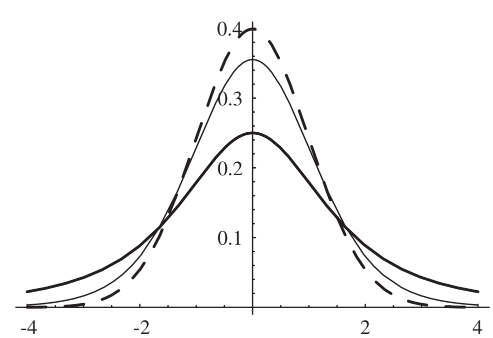

Fig. 2. The Gauss pdf $\mathcal{N}(0, 1)$ (dashed line) compared with the $\Sigma(2, 2)$ (thick line) and the $\Sigma(10, 12)$ (thin line). The flexes of the three curves coincide.

$$v_0(s) = \frac{\nu+2}{\nu-2}\,\frac{z^2(4z^2+\nu+10)}{2(1+z^2)^2}, \tag{26}$$

$$v_e(s) = v_0(s) - v(s), \tag{27}$$

where ${}_2F_1(a,b;c;w)$ is a hypergeometric function and $\psi(w) = \Gamma'(w)/\Gamma(w)$ is the logarithmic derivative of the Euler Gamma function (digamma function). Comparing them with the potentials of a Gaussian distribution we see that the space charge potentials $v(s)$ (Fig. 3) are similar: both behave as $-\xi \log s$ for $s \to +\infty$. On the other hand the zero perveance potentials $v_0(s)$ (Fig. 4) look different for large s, but the difference fades away for large ν: in the Gauss case $v_0(s)$ diverges as s^2, while in the Student case it goes to β as s^{-2}. The total external potentials $v_e(s) = v_0(s) - v(s)$ are plotted in Fig. 5. Hence, even if the potential near the beam axis is harmonic, deviations from this behavior in a region removed form the core can produce a deformation of the distribution from the Gaussian to the Student. We can finally calculate the probability $P(c)$ of being beyond a distance $c\sigma$ from the beam axis: in the Gauss case we get

$$P(c) = e^{-c^2/2}, \quad P(10) \simeq 1.9 \times 10^{-22}$$

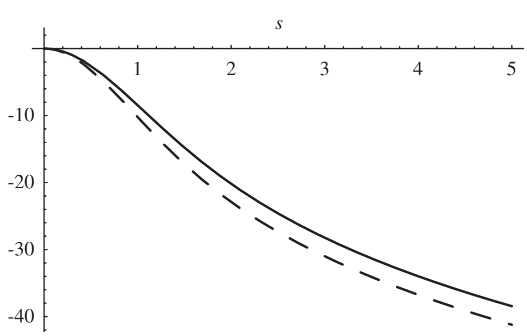

Fig. 3. The space charge potentials $v(s)$ respectively for a Student (solid line) distribution $\Sigma_2(22, 20\sigma^2)$, and for a Gauss (dashed line) distribution. Dimensionless perveance $\xi = 20$.

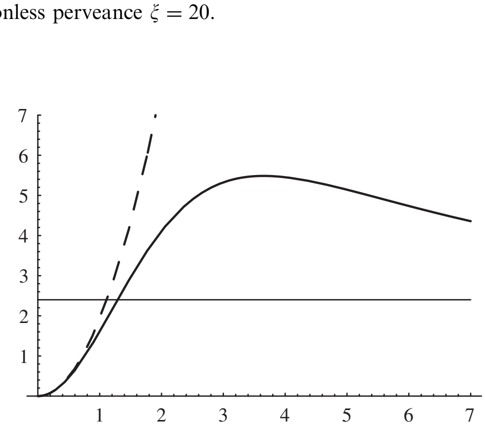

Fig. 4. The zero perveance potential $v_0(s)$ of a Student $\Sigma_2(22, 20\sigma^2)$ (solid line) compared with that of a Gauss law (dashed line) with the same behavior near the beam axis.

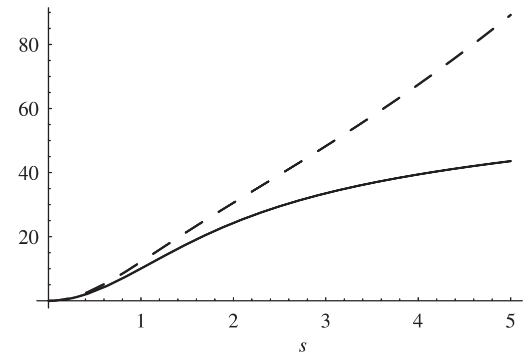

Fig. 5. The total external potential $v_e(s)$ (27) that should be applied to get a stationary Student transverse distribution $\Sigma_2(22, 20\sigma^2)$ (solid line), compared with that (20) needed for a Gauss distribution (dashed line).

while in the Student case

$$P_\nu(c) = \left(1 + \frac{c^2}{\nu-2}\right)^{-\nu/2} \quad \begin{cases} P_{10}(10) \simeq 2.2 \times 10^{-6}, \\ P_{22}(10) \simeq 2.8 \times 10^{-9}. \end{cases}$$

Hence, for $\mathcal{N} = 10^{11}$ particle per meter of beam, we find beyond 10σ practically no particle in the Gaussian case, but between 10^3 and 10^5 in the Student case. We got similar numbers [6] in the numerical solutions for $\xi = 20$.

5. Lévy–Student processes

The Student laws $\Sigma(\nu, a^2)$ are a family of Lévy infinitely divisible (i.d.) laws. Present interest about non-Gaussian Lévy laws (from physics to finance) is mostly confined to the stable laws: a sub-family of the i.d. laws. The i.d. laws constitute both the more general form of possible limit laws for the generalized central limit theorem, and the class of all the laws of the increments for every stationary, stochastically continuous, independent increments process (Lévy process). Non-Gaussian Lévy process have trajectories with moving discontinuities (e.g. compound Poisson process): a possible model for the relatively rare escape of particles from the beam core. In the following we will limit ourselves to 1-DIM systems.

The relevant mathematical concepts used in this paper are better discussed in the framework of the theory of the addition of independent random variables (r.v.): for more details see [11]. A law $\mathscr{L}$ with characteristic function (ch.f.) φ is i.d. when for every n there is a law $\mathscr{L}_n$ with ch.f. φ_n such that $\varphi = \varphi_n^n$, i.e. when the r.v. X can always be decomposed in the sum of n independent and identically distributed r.v.'s. The laws $\mathscr{L}_n$, however, are not in general of the same type as $\mathscr{L}$: two laws are said to be of the same type when they differ by a centering and a rescaling, namely $e^{ia\kappa}\varphi(b\kappa)$ for every a and $b>0$. A law $\mathscr{L}$ is stable when it is i.d. and the component laws are of the same type as $\mathscr{L}$; more precisely a law is stable if for every $b, b'>0$, there exist a and c such that

$$\varphi(c\kappa) = e^{ia\kappa}\varphi(b\kappa)\varphi(b'\kappa).$$

Gauss and Cauchy laws are stable; Poisson laws are only i.d. The Lévy–Khintchin formulas give the ch.f.s of i.d. and stable laws. For stable laws these ch.f.s are explicitly known in terms of elementary functions, while for i.d. laws the ch.f.s are given through an integral containing a Lévy function $L(x)$ associated to every particular law. For most i.d. laws the Lévy functions are not known.

Let us consider the sequence of r.v.s $X_{n,k}$ with $n \in \mathbb{N}$ and $k = 1, \ldots, n$ with $X_{n,1}, \ldots, X_{n,n}$ independent for every n. The modern formulation of the central limit problem asks to find the more general laws which are limits of the laws of the *consecutive sums*

$$S_n = \sum_{k=1}^{n} X_{n,k}. \tag{28}$$

Under very general technical conditions the central limit theorem now states that the family of all the limit laws of the consecutive sums (28) coincides with the family of i.d. laws. The stable laws come into play only when we specialize the form of our consecutive sums: when we have

$$X_{n,k} = \frac{X_k}{a_n} - \frac{b_n}{n},$$

where a_n and b_n are sequences of numbers, and X_k are independent r.v.s, the consecutive sums take the form of the usual *normed sums* (centered and rescaled sums of independent r.v.s)

$$S_n = \frac{S_n^*}{a_n} - b_n, \quad S_n^* = \sum_{k=1}^{n} X_k. \tag{29}$$

Then, if the X_k are also identically distributed, the family of the limit laws of the normed sums (29) coincides with the family of the stable laws. The classical (Gaussian) central limit theorem is an example of convergence toward a stable law; on the other hand the Poisson theorem (convergence of Binomial laws toward Poisson laws) is an example of convergence toward an i.d. law.

The general formulation of the central limit theorem is strictly connected to the definition of the processes with independent increments (*decomposable processes*). It is apparent in fact that if the increments $\Delta X(t) = X(t + \Delta t) - X(t)$ for non superposed intervals are independent, the previous forms of the central limit theorem imply that the laws of the increments must be i.d. laws. Moreover, since the decomposable process are also Markov processes, the laws of the increments are also all that is needed to completely define them. If a decomposable processes $X(t)$ is *stationary* (namely the law of $X(t+s) - X(s)$ does not depend on s) and *stochastically continuous* (namely for every t we have $X(t + \Delta t) - X(t) \to 0$ in probability when $\Delta t \to 0$) we will call it a *Lévy process*. Remark that its trajectories can now have moving—as opposed to fixed—discontinuities: for instance a Poisson process is a Lévy process since, despite its discontinuities, it is stochastically continuous. In fact these discontinuities do not impair the stochastic continuity of the process because they are *moving* (as opposed to *fixed*) discontinuities. On the other hand it is possible to prove that only the Gaussian Lévy processes (for example the Wiener, or the Ornstein–Uhlenbeck processes) are *pathwise continuous*, namely: almost every sample path is everywhere continuous (there are not even moving discontinuities).

If $\varphi(\kappa)$ is i.d. and T is a time constant, then $[\varphi(\kappa)]^{\Delta t/T}$ is the ch.f. of $\Delta X(t)$ of a Lévy process with stationary transition pdf

$$p(x, t|y, s) = \frac{1}{2\pi} \,\mathrm{PV} \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\kappa(x-y)} [\varphi(\kappa)]^{t-s/T} \,\mathrm{d}\kappa. \tag{30}$$

Almost all trajectories are continuous with the exception of a countable set of moving jumps. If $L_t(x)$ is the Lévy–Khintchin function of the i.d. law of the increment $X(s+t) - X(s)$, and $v_t(x)$ is the random number of the jumps in $[s, s+t)$ of height in absolute value larger than $x > 0$, then $|L_t(x)| = \mathbf{E}(v_t(x))$, namely: the Lévy–Khintchin function is a measure of the frequency and height of the trajectory jumps.

The ch.f. of a Student law $\Sigma(\nu, a^2)$ is

$$\varphi(\kappa) = 2 \,\frac{|a\kappa|^{\nu/2} \mathrm{K}_{\nu/2}(|a\kappa|)}{2^{\nu/2} \Gamma(\nu/2)}, \tag{31}$$

where $\mathrm{K}_\alpha(z)$ is a modified Bessel function. These laws are i.d., but in general not stable. That notwithstanding we get two advantages:

- all Student laws with $\nu > 2$ have a finite variance, while every stable, non-Gaussian law has divergent variance;
- the stable, non-Gaussian laws decay as $|x|^{-\alpha-1}$ with $\alpha < 2$, while the Student laws go as $|x|^{-\nu-1}$ with $\nu > 0$; this allows the Student laws to approximate the Gaussian behavior as well as we want.

A Lévy process defined by the ch.f. (31) will be called in the following a Lévy–Student process. Its transition pdf $p(x, t|y, s)$ is obtained from (30) and (31): its knowledge is enough to calculate everything of our process, but in practice the integral must be treated numerically. For $t - s = T$ $p(x, t|y, s)$ is a Student $\Sigma(\nu, a^2)$: we can then produce sample trajectory simulations by taking T as the time step, since the increments are exactly Student distributed when observed at the (arbitrary) time scale T.

We produce a simplified model which compares the simulated solutions of the following two SDEs:

$$\mathrm{d}X(t) = v(X(t))\,\mathrm{d}t + \mathrm{d}W(t), \tag{32}$$

$$\mathrm{d}Y(t) = v(Y(t))\,\mathrm{d}t + \mathrm{d}S(t), \tag{33}$$

where $W(t)$ is a Wiener process, $S(t)$ is a Lévy–Student process, and $v(x) = -bxH(q - |x|)$ for given $b > 0$ and $q > 0$ with H the Heaviside function. This flux will attract the trajectory toward the origin when $|x| \leqslant q$, and will allow the movement to be completely free for $|x| > q$. In our simulations $\Delta Y(t)$ of (33) is Student $\Sigma(4, 1)$ with $\sigma = 0.71$, while $\Delta W(t)$ of (32) is Gaussian with the same σ. Their pdfs look not very different; but that notwithstanding the process $Y(t)$ differs in several respects from $X(t)$. For $b = 0.35$ and $q = 10$ the Figs. 6–9 show the typical trajectories of a 10^4 steps solution $X(t)$ and $Y(t)$. In the Gaussian case with σ smaller than q the trajectories always stay inside the beam core, and the process is essentially an Ornstein–Uhlenbeck position process. In the Student case the trajectories: show a wider dispersion and a few larger spikes; have the propensity to make occasional excursions far away from the beam core; and finally seldom they also definitely drift away from the core. In fact the trajectories of a non-Gaussian Lévy process are only stochastically, and not pathwise continuous and hence they contain occasional jumps. This feature of a Lévy–Student process suggests to adopt this model to describe the rare escape of particles away from the beam core.

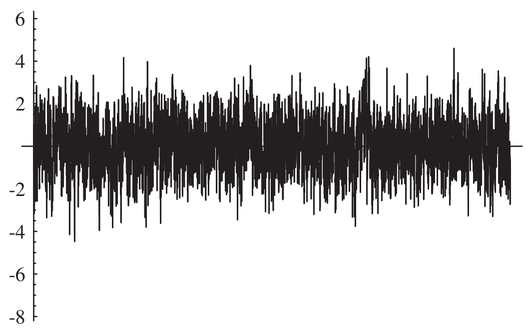

Fig. 6. Typical trajectory of a stationary, Gaussian (Ornstein–Uhlenbeck) process. To compare it with the Student trajectory, the vertical scale has been set equal to that of Fig. 7.

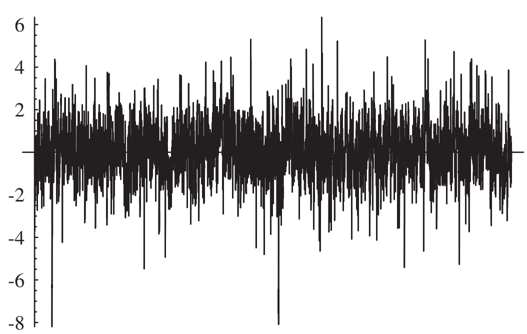

Fig. 7. Typical trajectory of a stationary, Student process ($\nu = 4$ and $a = 1$).

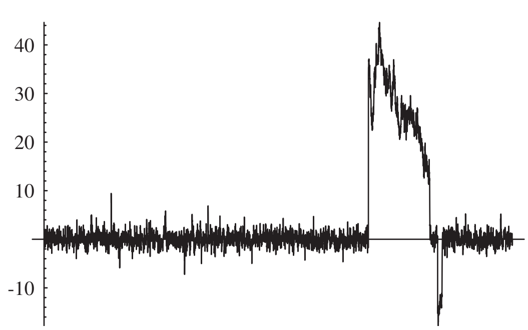

Fig. 8. Occasional trajectory of a stationary, Student process with a temporary excursion out of the core ($\nu = 4$).

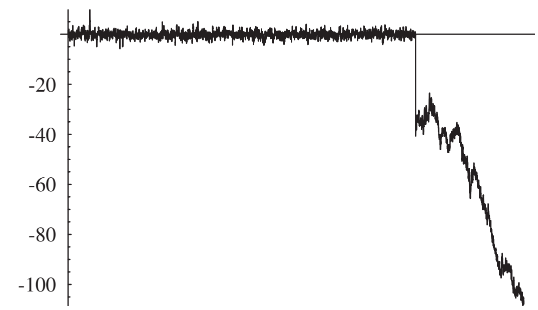

Fig. 9. Rare, but possible trajectory of a stationary, Student process: here the particle definitely drifts away from the core ($\nu = 4$).

6. Conclusions

Several problems are open along this line of research. First, we should find both the Lévy–Khintchin function of the Student laws to fine tune the frequency and the size of the jumps, and the increment laws of the Student process at different time scales. Second, it is important to have the integro-differential form of the Chapman–Kolmogorov equation to analyze the time evolution of the process. Then it is necessary to add a dynamics to have controlled diffusions: namely to build a generalized SM for the Lévy–Student processes. Finally, we must search for empirical or numerical evidence to support the hypothesis that the path increments of a beam are in fact distributed according to a Student law. About this last problem it is interesting to point out that a few numerical evidences begin to emerge [12] which confirm our conjecture.

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