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STATIONARY DISTRIBUTIONS OF NON GAUSSIAN ORNSTEIN-UHLENBECK PROCESSES FOR BEAM HALOS*

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Abstract

Beam halos are studied in a dynamical stochastic model with enhanced transverse dispersion and transverse emittance growth. We explore the effects of non-Gaussian noises with larger variances and possible jumps. The stationary distribution of Ornstein-Uhlenbeck processes with particular Lévy noises is then calculated. For Student processes the asymptotic spatial behavior coincides with that found in other independent numerical halo models.

INTRODUCTION

The charged particle beam dynamics and the possible halo formation are described in this paper in terms of stochastic processes. To have time reversal invariance a dynamics is added; but since our Markov process is not derivable, we are obliged to drop the momentum equation and to work in a configuration space. Consequently the dynamics is introduced by means of a *stochastic variational principle*. This scheme, the stochastic mechanics (*S.M.*), is known for its application to classical stochastic models for quantum mechanics [1], but is suitable for a large number of other systems [2, 3]. This leads to a linearized theory summarized in a phenomenological Schrödinger equation [4], and the space charge effects can be introduced by coupling it with the Maxwell equations [5, 6].

A new role in this stochastic model can be played by non–Gaussian, Lévy distributions [6]. Their today's popularity, however, is mainly confined to the *stable* laws [2]. We use instead non–Gaussian Lévy laws which are *infinitely divisible* (*i.d.*), but not stable as Student and Variance-Gamma (*V.G.*) laws [6, 7]. This has two advantages: first, at variance with stable non–Gaussian laws, the *i.d.* laws can have *finite variances*; second, they can incrementally *approximate the Gaussian laws*. On the other hand an *i.d.* laws is all that is required to build the *Lévy processes* used here to represent the evolution of our particle beam.

STOCHASTIC BEAM DYNAMICS

In the *S.M.* model the position $\mathbf{Q}(t)$ of a representative particle in the beam is a process ruled by the Itô stochastic differential equation (*S.D.E.*)

$$d\mathbf{Q}(t) = \mathbf{v}_{(+)}(\mathbf{Q}(t),t)\,dt + \sqrt{D}\,d\mathbf{W}(t)\,,$$

where $\mathbf{v}_{(+)}(\mathbf{r},t)$ is the forward velocity, $d\mathbf{W}(t)$ is the increment process of a standard Wiener noise, D is a constant diffusion coefficient, and $\alpha = 2mD$ is an action connected to the emittance of the beam. To add a dynamics we introduce a stochastic least action principle and we get a Nelson process [1]. If $\rho(\mathbf{r},t)$ is the *p.d.f.* of $\mathbf{Q}(t)$, and we define the backward velocity $\mathbf{v}_{(-)} = \mathbf{v}_{(+)} - 2D\nabla\rho/\rho$ and the current and osmotic velocities $\mathbf{v} = (\mathbf{v}_{(+)} + \mathbf{v}_{(-)})/2$ and $\mathbf{u} = (\mathbf{v}_{(+)} - \mathbf{v}_{(-)})/2$, from the stochastic least action principle we get that the current velocity is irrotational, $m\mathbf{v}(\mathbf{r},t) = \nabla S(\mathbf{r},t)$, and the Lagrange equations of motion for ρ and S are

$$\partial_t\rho = -\nabla\cdot(\rho\nabla S/m) \qquad (1)$$
$$\partial_t S = -\nabla S^2/2m + 2mD^2\rho^{-1/2}\nabla^2\rho^{1/2} - V \qquad (2)$$

where V is an external potential. Here the forward velocity $\mathbf{v}_{(+)}(\mathbf{r},t)$ is not given *a priori*, but it is dynamically determined by the evolution equation (2). With the representation

$$\Psi(\mathbf{r},t) = \sqrt{\rho(\mathbf{r},t)}\,\mathrm{e}^{iS(\mathbf{r},t)/\alpha}\,, \qquad \alpha = 2mD$$

the coupled equations (1) and (2) become a single linear equation of the form of a phenomenological Schrödinger equation, with the Planck action constant replaced by α:

$$i\alpha\partial_t\Psi = -\frac{\alpha^2}{2m}\nabla^2\psi + V\Psi\,. \qquad (3)$$

We analyzed in several papers [6] both the stationary and the non stationary solutions of this equation.

In this *S.M.* scheme $|\Psi(\mathbf{r},t)|^2$ is the *p.d.f.* of a Nelson process; hence when the N–particles are a pure ensemble, $N|\Psi(\mathbf{r},t)|^2\,d^3\mathbf{r}$ is the number of particles in a small neighborhood of $\mathbf{r}$. Our N particles, however, are not a pure ensemble due to their mutual *e.m.* interaction: we hence take into account the space charge effects by coupling the equation (3) with the Maxwell equations [5, 6]. We studied this system of differential equations within a cylindrical symmetry and we found that for a given external potential no simple analytical solutions are available and we must resort to numerical solutions [5]. On the other hand for a given radial distribution we can analytically solve the system to find the external and the space charge potentials: this can be easily done for Gaussian transverse distributions [6].

If, however, a halo is produced by large deviations from the beam axis, we can suppose that the the stationary transverse distributions are non–Gaussian: consider the family of the *i.d.* Student laws $\Sigma(\lambda, a^2)$ with *p.d.f.*'s

$$f(x) = a^\lambda B(1/2\,,\,\lambda/2)^{-1}\left(1 + x^2/a^2\right)^{-(\lambda+1)/2} \qquad (4)$$

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where $\lambda > 0$ and $B(x, y)$ is the Beta function. The potentials associated to the radial distribution of the circularly symmetric, bivariate Student laws $\Sigma_2(\lambda, a^2)$ can now be determined [6], and we can calculate the probability $P(c)$ of being beyond a distance $c\sigma$ from the beam axis: in the Gauss case we get $P(10) \simeq 1.9 \times 10^{-22}$, while in the Student case with $10 \leq \lambda \leq 22$ we get $2.8 \times 10^{-9} \leq P(10) \leq 2.2 \times 10^{-6}$. Hence, for $\mathcal{N} = 10^{11}$ particle per meter of beam, we find practically no particle beyond 10σ in the Gaussian case, but between 10^3 and 10^5 in the Student case.

LÉVY PROCESSES

The present interest – from physics to finance [2] – about non–Gaussian Lévy laws is mostly confined to the *stable* laws, a sub–family of the *i.d.* laws (see [7] for details). The *i.d.* laws constitute both the more general form of possible limit laws for the Central Limit Theorem, and the class of all the laws of the increments for every stationary, stochastically continuous, independent increments process (Lévy process). Non–Gaussian Lévy processes have trajectories with moving discontinuities (e.g. a Poisson process): a possible model for the relatively rare escape of particles from the beam core. In the following we will limit ourselves to 1–dimensional systems. Remark that there is another important subclass of *i.d.* laws which also contains all the stable laws: that of the selfdecomposable (*s.dec.*) laws. Not only they are the larger subclass of absolutely continuous, *i.d.* laws allowing the construction of selfsimilar processes, but they are also connected to non Gaussian Ornstein–Uhlenbeck (*O.U.*) processes. In fact every *O.U.* process with a Lévy noise has a stationary distribution which is *s.dec.* (see [7] and references quoted therein). We are interested in the processes generated by particular classes of *s.dec.* laws as the Student and the *V.G.* [7].

The general *i.d.* laws are strictly connected to the definition of the independent increments of a Lévy process. In fact it is apparent that the increments $\Delta X(t) = X(t + \Delta t) - X(t)$ of these processes must be distributed according to *i.d.* laws. Let now $\varphi(u)$ be the characteristic function (*ch.f.*) of an *i.d.* law, and T a time constant; then $[\varphi(u)]^{(t-s)/T}$ is the *ch.f.* of the increment $X(t) - X(s)$ of a Lévy process with stationary transition *p.d.f.*

$$p(x,t|y,s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iu(x-y)} [\varphi(u)]^{(t-s)/T} \mathrm{d}u. \quad (5)$$

According to the Lévy–Khintchin formula [7] a non Gaussian $\varphi(u)$ can be given in terms of a characteristic Lévy measure $\nu(B)$ on $(-\infty, +\infty)$ which represents the average number of jumps of non vanishing size belonging to B. The *ch.f.* of the law $\Sigma(\lambda, a^2)$ in equation (4) is

$$\varphi(u) = 2^{(1-\lambda)/2}\Gamma(\lambda/2)^{-1}|au|^{\lambda/2}\mathrm{K}_{\lambda/2}(|au|) \quad (6)$$

where $\mathrm{K}_\alpha(z)$ is a modified Bessel function. These laws are *i.d.*, but in general not stable.

A Lévy process defined by the *ch.f.* (6) will be called a Student process. Its transition pdf $p(x, t|y, s)$ is obtained from (5) and (6). In general this integral must be treated numerically, but for particular cases we can get exact results. In particular [7] for $\lambda = 3$ we have a closed, non elementary form of the transition *p.d.f.*, its spatial asymptotic behavior which is x^{-4} all along the evolution, and an elementary form of the transition *p.d.f.* for $t - s = T, 2T, \ldots$ In particular, for $t - s = T$, $p(x, t|y, s)$ is a Student $\Sigma(3, a^2)$ and we can then produce sample trajectory simulations by taking T as the time step, since the increments are exactly Student distributed when observed at the (arbitrary) time scale T. We also introduce the *V.G.* laws $\mathcal{VG}(\lambda, a)$ for $\lambda > 0$, with

$$f(x) = (2\pi a^2)^{-\frac{1}{2}} 2^{1-\lambda}\Gamma(\lambda)^{-1}(|x|/a)^{\lambda-\frac{1}{2}} K_{\lambda-\frac{1}{2}}(|x|/a)$$
$$\varphi(u) = \left(1 + a^2u^2\right)^{-\lambda};$$

since their spatial asymptotic behavior is exponential their variance $2\lambda a^2$ is always finite. Remark that $\mathcal{VG}(1, a)$ is the usual Laplace double exponential law. Also here from (5) we can calculate the transition *p.d.f.* of a Variance–Gamma process. At variance with the Student process which has a power law as asymptotic behavior, the Variance–Gamma process always shows exponential decays so that its average jumps will be shorter.

ORNSTEIN–UHLENBECK PROCESSES

At present we do not have yet a dynamical theory of Lévy processes leading to a generalized *S.M.* allowing a stochastic control of the beam size. On the other hand the simple Lévy processes based on *i.d.* laws with finite variance behave as ordinary diffusions and, as the Brownian motion, have no stationary distributions. We can however produce a simplified model by means of *O.U.* processes where a suitable velocity field will keep the process confined in a small region. First of all we can compare the different, simulated solutions of the following *S.D.E.*

$$\mathrm{d}X(t) = v(X(t))\,\mathrm{d}t + \mathrm{d}Z(t) \quad (7)$$

where $Z(t)$ is a Lévy process and $v(x) = -bx\,H(q - |x|)$ for given $b > 0$ and $q > 0$, with H the Heaviside function. This velocity field will attract the trajectory toward the origin when $|x| \leq q$, and will allow the movement to be completely free for $|x| > q$. Hence for $|x| \leq q$ we have *O.U.* processes, while for $|x| > q$ we have free diffusions. The process $Z(t)$ in (7) can be either a Gaussian, or a non Gaussian noise. Figures 1 show the typical 10^4 steps trajectories respectively (a) for Gaussian, (b) Laplace and (c), (d) Student noises with comparable variances. The processes in (b), (c) and (d) differ in several respects from that in (a). For $b = 0.35$, $q = 10$ and variances smaller than q the Gaussian trajectories (a) always stay inside the beam core, and the process is essentially an Ornstein–Uhlenbeck position process. In the Laplace (b) and Student (c) case the trajectories are still kept in the beam core, but show a wider

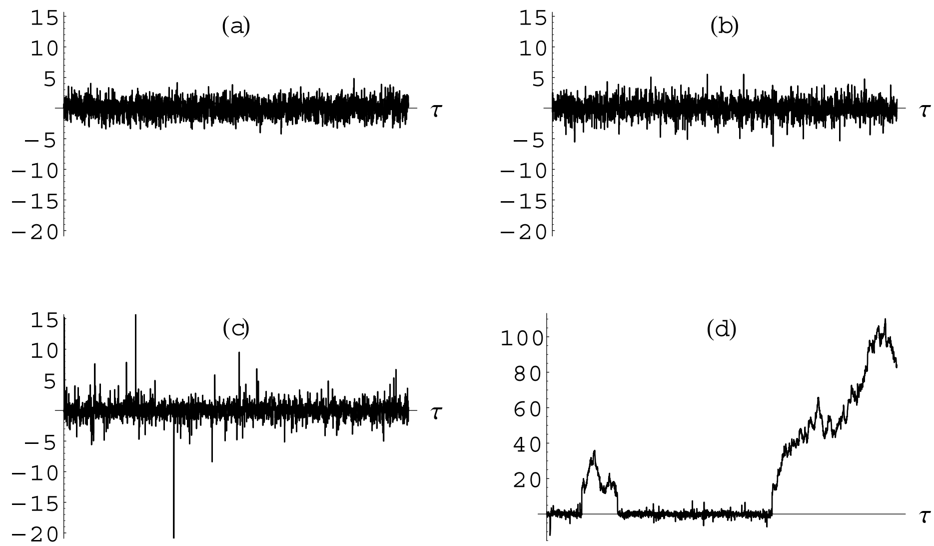

Figure 1: *O.U.* processes $X(t)$ from (7) with $v(x)$ of limited range ($\tau = t/T$): (a) *O.U.* process with Gaussian noise; (b) *O.U.* process with Laplace noise; (c) *O.U.* process with Student noise; (d) occasional Student trajectory leaving the beam.

dispersion and a few larger spikes (jumps). They also have, as the Student case (d) shows, the propensity to make occasional excursions far away from the beam core; and finally, seldom they also definitely drift away from the core. This feature of a Lévy processes, due to their jumping behavior, suggests to adopt this model to describe the rare escape of particles away from the beam core.

When on the other hand $v(t)$ in (7) simply is $-bx$, we have full *O.U.* processes and it is possible to calculate the law of the stationary distributions from the form of the noise. In particular for the Student $\Sigma(3, a^2)$ noise it is possible to show that

$$\psi(u) = \ln \varphi(u) = -b^{-1}\left[a|u| + \mathrm{Li}_2(-a|u|)\right]$$

where $\mathrm{Li}_2(x)$ is the *dilogarithm function*. The form of the corresponding *p.d.f.* is not known analytically, but it can be assessed numerically by calculating the inverse Fourier transform of the *ch.f.*. It comes out from these calculations that the stationary *O.U.* law for the Student $\Sigma(3, a^2)$ noise still shows the same x^{-4} behavior which characterizes the transition *p.d.f.*'s all along the process evolution. Finally it is important to remark that this same asymptotic behavior x^{-4} has been independently found in other different numerical simulations [8] of halo formation in particle beams. This seems to substantiate the conclusion that behind the dynamics of a charged particle beam there is some sort of Lévy–Student process.

CONCLUSIONS

Several problems are open along this line of research. First, we should find the Lévy–Khintchin functions of the Student laws to fine tune the frequency and the size of the jumps, and the increment laws of the Student process at different time scales [7]. Second, it is important to have the integro–differential form of the Chapman–Kolmogorov equation to analyze the time evolution of the process: a forthcoming paper will be devoted to this topics. Then it is necessary to add a dynamics to have controlled diffusions: namely to build a generalized *S.M.* for the Lévy processes. Finally we must search for empirical or numerical evidence beyond what is already known [8] to support the hypothesis that the path increments of a beam are in fact distributed according either to a Student, or to some other Lévy law.

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