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Markov processes and generalized Schrödinger equations

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Starting from the forward and backward infinitesimal generators of bilateral, time-homogeneous Markov processes, the self-adjoint Hamiltonians of the generalized Schrödinger equations are first introduced by means of suitable Doob transformations. Then, by broadening with the aid of the Dirichlet forms, the results of the Nelson stochastic mechanics, we prove that it is possible to associate bilateral, and time-homogeneous Markov processes to the wave functions stationary solutions of our generalized Schrödinger equations. Particular attention is then paid to the special case of the Lévy-Schrödinger (LS) equations and to their associated Lévy-type Markov processes, and to a few examples of Cauchy background noise. © *2011 American Institute of Physics*. [doi:10.1063/1.3663205]

I. INTRODUCTION

In a few recent papers,[1] it has been proposed to broaden the scope of the well-known relation between the Wiener process and the Schrödinger equation[2–5] to other suitable Markov processes. This idea – already introduced elsewhere, but essentially only for stable processes[6,7] – led to a LS (Lévy–Schrödinger) equation containing additional integral terms which take into account the possible jumping part of the background noise. This equation has been presented in the framework of *stochastic mechanics*[2,5] as a model for systems more general than just the usual quantum mechanics: namely, as a true *dynamical theory of Lévy processes* that can find applications in several physical fields.[8] However in the previous papers,[1] our discussion was essentially heuristic and rather oriented to discuss the basic ideas and to show a number of explicit examples of wave packets solutions of these LS equations in the free case, by pointing out the new features as, for instance, their time-dependent *multi-modality*. In particular, the derivation of the LS equation consistently followed a time-honored[9] formal procedure consisting in the replacement of t by an imaginary time variable it. While this usually leads to correct results; however, it is apparent that it can only be a heuristic, handpicked tactics implemented just in order to see where it leads, and if the results are reasonable, then – as already claimed in our previous papers – a more solid foundation must be found to give substance to these findings. The aim of the present paper is, in fact, to pursue this enquiry by giving a rigorous presentation of the relations between the LS equations and their background Markov processes.

In the original Nelson papers,[2] the Schrödinger equation of quantum mechanics was associated with the diffusion processes weak solutions of the stochastic differential equations (SDE),

$$dX_t = b(X_t, t)dt + dW_t, \tag{1}$$

where W_t is a Wiener process. Our aim is then to analyze how this Nelson approach can be generalized when a wider class of Markov processes is considered instead of the diffusion processes (1), and what kind of equations are involved, in particular, when Lévy processes are considered instead of W_t.

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The Lévy processes[10–13] can indeed be considered as the most natural generalization of the Wiener process: they have stationary, independent increments, and they are stochastically continuous. The Wiener process itself is a Lévy process, but it essentially differs from the others because it is the unique with *a.s.* (almost surely) continuous paths: the other Lévy processes, indeed, typically show random jumps all along their trajectories. In the recent years, we have witnessed a considerable growth of interest in non-Gaussian stochastic processes – and in particular into Lévy processes – in domains ranging from statistical mechanics to mathematical finance. In the physical field, the research scope is presently focused mainly on the stable processes and on the corresponding fractional calculus,[6,7,14] but in the financial domain a vastly more general type of processes is at present in use,[15] while interesting generalizations seem to be at hand.[16] Here, we suggest that the stochastic mechanics should be considered as a dynamical theory of the entire gamut of the *infinitely divisible* (not only stable) Lévy processes with time reversal invariance, and that the horizon of its applications should be widened even to cases different from the quantum systems.

This approach presents several advantages: on the one hand the use of general infinitely divisible processes lends the possibility of having realistic, *finite variances*, a situation ruled out for non-Gaussian, stable processes; on the other there are examples of non-stable Lévy processes which are connected with the simplest form of the quantum, *relativistic Schrödinger equation*: a link with important physical applications that was missing in the original Nelson model and was recognized only several years later.[17] This last remark shows, among others, that the present inquiry is not only justified by the desire of formal generalization, but is required by the need to attain physically meaningful cases that otherwise would not be contemplated in the narrower precinct of the stable laws. Of course, it is well known that the types of general infinitely divisible laws are not closed under convolution: when this happens, the role of the scale parameters becomes relevant since a change in their values cannot be compensated by reciprocal changes in other parameters, and the process no longer is scale invariant, at variance with the stable processes. This means that, to a certain extent, a scale change produces different processes, so that for instance we are no longer free to look at the process at different time scales by presuming to see the same features. Since, however, the infinitely divisible distributions can have a finite variance, it is easy to prove that the Lévy processes generated by these infinitely divisible laws will always have a finite variance which grows linearly with the time: a feature typical of the ordinary (non-anomalous) diffusions, while the stable non-Gaussian processes are bound to show typical (anomalous) super- and sub-diffusive behavior.[15]

To give a rigorous justification of the L-S equation, essentially introduced in Ref. 1 by means of an analogy, let us first remark that the original Nelson approach for deriving the Schrödinger equation was based on a deep understanding of the *dynamics* of the stochastic processes, while this reckoned on new definitions of the kinematical quantities – *forward and backward mean velocities* and *mean accelerations* – that anyway safely revert to the ordinary ones when the processes degenerate in deterministic trajectories. For the time being, however, our approach will be rather different: we will not resort openly to an underlying dynamics, but starting instead with the infinitesimal generators L of a semigroup in a Hilbert space, we will explore on the one hand under what conditions we can associate it with a suitable Markov process $X_t \in \boldsymbol{R}^n$ with *pdf* (probability density function) ρ_t and on the other the formal procedures leading from L to a self-adjoint, bounded from below operator H on $\mathcal{L}^2_{\boldsymbol{C}}(\boldsymbol{R}^n, d^n x)$ and to a wave function $\Psi_t \in \mathcal{L}^2_{\boldsymbol{C}}(\boldsymbol{R}^n, d^n x)$ which turns out to be a solution of the *generalized Schrödinger equation*

$$i\partial_t \Psi_t = H\Psi_t \tag{2}$$

with $|\Psi_t|^2 = \rho_t,\ \forall t \in \boldsymbol{R}$. While the first task will be accomplished by resorting to the properties of the Dirichlet forms $\mathcal{E}$ (Refs. 18 and 19) that can be defined from L, the second result will be obtained by following the path of the Doob transformations.[9,20]

The paper is then organized as follows: while in Sec. II we will first recall the less usual features of the Markov processes of our interest, in Sec. III we will introduce their associated infinitesimal generators and Dirichlet forms, and in the subsequent Sec. IV we will briefly summarize the essential notations about the Lévy processes. In Sec. VI we will then recall how, by means of the Doob transformations previously defined in Sec. V, the stationary solutions of the usual Schrödinger

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equation actually admit a stochastic representation in terms of the diffusion processes (1). Finally, in Sec. VII we will focus our attention on our main result about the Lévy-type,[21] Markov processes associated with the stationary solutions of the LS equation

$$i\partial_t \Psi_t = -L_0 \Psi_t + V\Psi_t \tag{3}$$

which is a particular form of (2). Here, V is an suitable real function, while for an infinitely derivable function on $\boldsymbol{R}^n$ with compact support $f \in C_0^\infty(\boldsymbol{R}^n)$, L_0 explicitly operates in the following way:

$$[L_0 f](x) = \alpha_{ij}\partial^2_{ij} f(x) + \int_{y\neq 0} \left[f(x+y) - f(x) - \mathbf{1}_{B_1}(y) y_i \partial_i f(x) \right] \ell(dy),$$

where α_{ij} is a symmetric, positive definite matrix, $\mathbf{1}_{B_1}(y)$ is the indicator of the set $B_1 = \{y \in \boldsymbol{R}^n : |y| \leq 1\}$, and $\ell(dy)$ is a *Lévy measure*.[10,11] The name of Eq. (3) is due to the fact that L_0 turns out to be the infinitesimal generator of a symmetric Lévy process, while V plays the role of a potential, so that (3) closely resembles the usual Schrödinger equation that one obtains when L_0 is the infinitesimal generator of a Wiener process. In Sec. VIII, a few examples of stationary states of Cauchy-Schrödinger equations with their associated Lévy-type processes are explicitly discussed.

II. MARKOV PROCESSES

A stochastic processes X_t is usually defined for $t \geq 0$, but it will be important for us to consider also processes defined for every $t \in \boldsymbol{R}$: we will call them *bilateral processes*. This will allow us to introduce *forward* and *backward* representations that will be instrumental to define the suitable symmetric operators and the self-adjoint Hamiltonians needed in our generalized Schrödinger equations. It will be useful, moreover, to recall that we will call a process X_t *stationary* when all its joint distributions (for a Markov process, those at one and two times are enough) are invariant for a change of the time origin. In this case, the distributions at one time are invariants, and the conditional (transition) distributions depend only on the time differences. On the other hand, we will call it *time-homogeneous* when just the conditional (transition) distributions are independent from the time origin and depend only on the time differences. In this case, however, the process can possibly be non-stationary when the one-time distributions are not constant, and as a consequence the joint distributions depend on the changes of the time origin.

Let then $X = (X_t)_{t\in\boldsymbol{R}}$ be a bilateral, time-homogeneous Markov process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with its natural filtration, and taking values on $(\boldsymbol{R}^n, \mathcal{B}(\boldsymbol{R}^n))$. We will first of all denote, respectively, by p_t and $\tilde{p}_t$ its *forward and backward transition functions* defined as

$$p_t(x, B) := \mathbb{P}\{X_{s+t} \in B \mid X_s = x\}, \qquad \tilde{p}_t(x, B) := \mathbb{P}\{X_{s-t} \in B \mid X_s = x\} \tag{4}$$

for $s \in \boldsymbol{R}$, $t \geq 0$, $x \in \boldsymbol{R}^n$, and $B \in \mathcal{B}(\boldsymbol{R}^n)$. We will say that μ is an invariant measure for X when

$$\mu(B) = \int p_t(x, B)\,\mu(dx) = \int \tilde{p}_t(x, B)\,\mu(dx), \qquad t > 0$$

for $B \in \mathcal{B}(\boldsymbol{R}^n)$. Remark that here μ is not necessarily supposed to be a *probability* measure. We will indeed keep open the possibility of X being a stationary process with a general σ-finite measure as one time marginal, rather than a strictly probabilistic one. In this case, X actually is an *improper process*, namely, a process which is properly defined as a measurable application from an underlying probabilizable space into a trajectory space and is adapted to a filtration, but which is endowed with a measure which is not finite. In particular, we will find instrumental the use of the Lebesgue measure on $(\boldsymbol{R}^n, \mathcal{B}(\boldsymbol{R}^n))$ that turns out to be invariant for many of our semigroups and can consequently be adopted as the overall measure of the process. This is on the other hand not new if we think to the case of plane waves solutions of the Schrödinger equation in quantum mechanics.

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Given an invariant measure μ, it is possible to prove[22] that, always for $t \geq 0$, we can define on $\mathcal{L}^2(\boldsymbol{R}^n, d\mu)$ endowed with the usual scalar product $\langle f, g\rangle_\mu$ the two semigroups

$$[T_t f](x) := \int f(y)\, p_t(x, dy) = \mathbb{E}\{f(X_{s+t}) \,|\, X_s = x\},$$
$$[\tilde{T}_t f](x) := \int f(y)\, \tilde{p}_t(x, dy) = \mathbb{E}\{f(X_{s-t}) \,|\, X_s = x\} = \mathbb{E}\{f(X_s) \,|\, X_{s+t} = x\},$$

respectively, called *forward and backward semigroups*. We also denote by $(L, D(L))$ and $(\tilde{L}, D(\tilde{L}))$, with the specification of their domains of definition, the corresponding infinitesimal generators. For these semigroups, it is possible to prove that

$$T_t^\dagger = \tilde{T}_t, \qquad t \geq 0. \tag{5}$$

In particular, when T_t is self-adjoint so that $T_t^\dagger = \tilde{T}_t = T_t$ the Markov process X_t is also called μ-*symmetric*, while we will say that the process is simply *symmetric* when $\mathbb{P}_{X_t}(B) = \mathbb{P}_{X_t}(-B)$ for every $B \in \mathcal{B}(\boldsymbol{R}^n)$: these two notions are, however, strictly related.[11] We will, moreover, call the process *rotationally invariant* if $P_{X_t}(B) = P_{Z_t}(\mathbb{O}B)$ for every Borel set B and for every given orthogonal matrix $\mathbb{O}$.

We finally introduce also the *space-time version*[23] Y of X, namely, the process

$$Y_t = (X_t, \tau_t) \tag{6}$$

on $(\boldsymbol{R}^{n+1}, \mathcal{B}(\boldsymbol{R}^{n+1}))$, with just one more *degenerate* component: $\tau_t = t\, a.s.$ It is easy to prove then that the Y forward and backward semigroups and generators – now denoted by T_t^Y, $\tilde{T}_t^Y$, L^Y, and $\tilde{L}^Y$ – are defined on the space $\mathcal{L}^2(\boldsymbol{R}^{n+1}, d\mu\, dt)$ and verify relations similar to (5), namely, $(T_t^Y)^\dagger = \tilde{T}_t^Y$. This space-time version Y will be useful for two reasons: first Y is always time-homogeneous,[22] even when X it is not; second the Doob transforms of combinations of their generators (see the subsequent Sec. V) will give rise exactly to the space-time operators needed to recover our generalized Schrödinger equations.

III. DIRICHLET FORMS

Up to now we have defined semigroups starting from suitable, given Markov processes, but in this paper we will be mainly concerned with the reverse question: under which conditions can we define a Markov process from a given semigroup $(T_t)_{t\geq 0}$ on a real Hilbert space $\mathfrak{H}$ with scalar product $\langle\,\cdot, \cdot\,\rangle$ and norm $\|\,\cdot\,\|$? This well-known problem can be faced in several ways, and we will choose to approach it from the standpoint of the Dirichlet forms. We refer the reader to classical monographs[18,19] for an extensive discussion about this argument. Let $(\mathcal{E}, D(\mathcal{E}))$ be a positive definite, bilinear form on $\mathfrak{H}$, endowed with the norm on $D(\mathcal{E})$,

$$\|u\|_1^2 := \mathcal{E}(u, u) + \|u\|^2.$$

Some bilinear forms can naturally be associated with linear operators L in the Hilbert space $\mathfrak{H}$, for instance as $\mathcal{E}(u, v) = -\langle Lu, v\rangle$, and we look for operators which are the generators of a semigroup T_t, because this could produce the required link between Markov processes and bilinear forms. In particular, it is well known that semigroup generators are always associated with *coercive, closed* bilinear forms.[19]

Theorem 3.1: *Let $(\mathcal{E}, D(\mathcal{E}))$ be a coercive, closed form on $\mathfrak{H}$: then there exist a pair of operators $(L, D(L))$ and $(\tilde{L}, D(\tilde{L}))$ on $\mathfrak{H}$, with*

$$D(L) := \{u \in D(\mathcal{E})|\ v \to \mathcal{E}(u, v) \textit{ is continuous with respect to } \|\cdot\|_1 \text{ on } D(\mathcal{E})\},$$
$$D(\tilde{L}) := \{v \in D(\mathcal{E})|\ u \to \mathcal{E}(u, v) \textit{ is continuous with respect to } \|\cdot\|_1 \text{ on } D(\mathcal{E})\}$$

and

$$\mathcal{E}(u, v) = -\langle Lu, v\rangle, \qquad u \in D(L),\ v \in D(\mathcal{E}), \tag{7}$$

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$$\mathcal{E}(u, v) = -\langle u, \tilde{L}v\rangle, \qquad u \in D(\mathcal{E}),\ v \in D(\tilde{L}) \tag{8}$$

which are the infinitesimal generators of two strongly continuous, contraction semigroups $(T_t)_{t\geq 0}$ *and* $(\tilde{T}_t)_{t\geq 0}$ *such that*

$$T_t^{\dagger} = \tilde{T}_t, \qquad \forall\, t \geq 0.$$

According to the previous theorem, coercive, closed forms generate pairs of strongly continuous, contractive semigroups. In order to be sure, however, that these semigroups are associated with some Markov process we must be able to implement them by means of explicit transition functions. The following result[19] shows that a way to realize this program is to deal with *regular Dirichlet* forms, which are particular cases of coercive, closed forms. The subsequent exposition could well be proposed for more general Hilbert spaces, but to settle our notation from now on we will rather limit ourselves just to $\mathfrak{H} = \mathcal{L}^2(\boldsymbol{R}^n, d\mu)$.

Theorem 3.2: *Take a regular Dirichlet form* $(\mathcal{E}, D(\mathcal{E}))$ *on* $\mathcal{L}^2(\boldsymbol{R}^n, d\mu)$, *with its associated strongly continuous semigroups* $(T_t)_{t\geq 0}$ *and* $(\tilde{T}_t)_{t\geq 0}$*: then there are two time-homogeneous transition functions* p_t *and* $\tilde{p}_t$ *on* $(\boldsymbol{R}^n, \mathcal{B}(\boldsymbol{R}^n))$ *such that* μ*-a.s.,*

$$[T_t f](x) = \int f(y)p_t(x, dy), \qquad [\tilde{T}_t f](x) = \int f(y)\tilde{p}_t(x, dy)$$

for every $t \geq 0$ *and* $f \in \mathcal{L}^2(\boldsymbol{R}^n, d\mu)$.

By means of Theorem 3.2, we can then associate with a regular Dirichlet form, two Markov processes $(X_t)_{t\geq 0}$ and $(\tilde{X}_t)_{t\geq 0}$ defined on $\boldsymbol{R}^n$ – but for an arbitrary initial distribution – respectively, by p_t and $\tilde{p}_t$, and enjoying several useful properties such as right continuity with left limit and strong Markov property (for details see Refs. 18, 19, and 23). The transition functions p_t and $\tilde{p}_t$, however, could in general be *sub-Markovian*, namely, we could have $p_t(x, \boldsymbol{R}^n) \leq 1$ for some $x \in \boldsymbol{R}^n$. To avoid this, it can be easily proved, by a general property of strongly continuous semigroups,[25] that if the constant function $u_1 = 1$ belongs to $D(L)$ and $D(\tilde{L})$, then we find $p_t(x, \boldsymbol{R}^n) = \tilde{p}_t(x, \boldsymbol{R}^n) = 1$ for every $x \in \boldsymbol{R}^n$ if and only if $Lu_1 = \tilde{L}u_1 = 0$. In this case, moreover, we also have that μ is an invariant measure for both p_t and $\tilde{p}_t$. Indeed, since p_t and $\tilde{p}_t$ are *in duality* with respect to μ, namely,

$$\iint f(x)g(y)\, p_t(x, dy)\mu(dx) = \iint f(y)g(x)\, \tilde{p}_t(x, dy)\mu(dx) \tag{9}$$

as we easily deduce from $\tilde{T}_t = T_t^{\dagger}$, by taking $f(x) = 1$ and $g(y) = \mathbf{1}_B(y)$ for $B \in \mathcal{B}(\boldsymbol{R}^n)$, it is easy to prove that

$$\int p_t(x, B)\mu(dx) = \mu(B),$$

namely, that μ is invariant. The same proof can be adapted to $\tilde{p}_t$.

Finally, under special condition it is also possible to associate with $(\mathcal{E}, D(\mathcal{E}))$ a *single* bilateral Markov process $X = (X_t)_{t\in\boldsymbol{R}}$ obtained by sewing together $(X_t)_{t\geq 0}$ and $(\tilde{X}_t)_{t\geq 0}$. This occurs, in particular, when μ is an invariant probability measure. In this case, in fact, we first define the canonical process X_t with $t \in \boldsymbol{R}$ and its distribution $\mathbb{P}_\mu$ as the Kolmogorov extension of

$$\begin{aligned}\mathbb{P}_\mu(X_{t_1} \in B_1) &= \mu(B_1),\\ \mathbb{P}_\mu(X_{t_1} \in B_1, \ldots, X_{t_k} \in B_k) &= \int_{B_1\times\ldots\times B_k} p_{t_k - t_{k-1}}(x_{k-1}, dx_k)\ldots\\ &\qquad\qquad \ldots p_{t_2-t_1}(x_1, dx_2)\mu(dx_1)\end{aligned} \tag{10}$$

for $B_k \in \mathcal{B}(\boldsymbol{R}^n)$, $k \in \boldsymbol{N}$, and $t_1 \leq t_2 \leq \ldots \leq t_k \leq \ldots$, and then we show that $(X_t)_{t\in\boldsymbol{R}}$ is a Markov process with respect to $\mathbb{P}_\mu$, having p_t and $\tilde{p}_t$, respectively, as its forward and backward transition

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functions. Actually, it is straightforward to prove that $(X_t)_{t\in\boldsymbol{R}}$ is a Markov process and that p_t is the forward transition function for it. As for $\tilde{p}_t$, we have instead to check that

$$\mathbb{P}_\mu(X_{s-t} \in B | X_s = x) = \mathbb{E}_\mu\{\mathbf{1}_B(X_{s-t}) | X_s = x\} = \tilde{p}_t(x, B) \tag{11}$$

for every Borel set B, $s \in \boldsymbol{R}$, and $t \geq 0$. To this effect, it will be enough to remark that, for every B_1 and B_2, from (10) and (9) it is easy to show that

$$\mathbb{E}_\mu\{\mathbf{1}_{B_2}(X_s)\mathbf{1}_{B_1}(X_{s-t})\} = \mathbb{E}_\mu\{\mathbf{1}_{B_2}(X_s)\tilde{p}_t(X_s, B_1)\}$$

which proves (11). It can be seen, moreover,[22] that this unifying procedure can be adopted even when the invariant measure is not a probability. As a consequence, according to Theorem 3.2, the semigroup generators L and $\tilde{L}$ on $\mathcal{L}^2(\boldsymbol{R}^n, d\mu)$ derived through (7) and (8) from a regular Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ can be considered as the forward and backward generators of a single, bilateral Markov process $(X_t)_{t\in\boldsymbol{R}}$ when $Lu_1 = \tilde{L}u_1 = 0$.

Let us conclude this survey with a few remarks about how to check that a bilinear form $(\mathcal{E}, D(\mathcal{E}))$ actually is a regular Dirichlet form. We first recall that, if the space $C_0^\infty(\boldsymbol{R}^n)$ of the infinitely derivable real functions with compact support is contained in $D(\mathcal{E})$, it is possible to prove[19] that for a symmetric Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on $\mathcal{L}^2(\boldsymbol{R}^n, d\mu)$ the following *Beurling-Deny formula* holds for every $f, g \in C_0^\infty(\boldsymbol{R}^n)$:

$$\begin{aligned}\mathcal{E}(f, g) = \sum_{i,j=1}^n \int \partial_i f(x)\partial_j g(x)\mu^{ij}(dx) + \int f(x)g(x)k(dx) \\ + \iint_{x\neq z} [f(x) - f(z)][g(x) - g(z)]J(dx, dz),\end{aligned} \tag{12}$$

where $k(dx)$ is a positive Radon measure on $\boldsymbol{R}^n$ (*killing measure*), $J(dx, dz)$ is a symmetric, positive Radon measure defined on $\boldsymbol{R}^n \times \boldsymbol{R}^n$ for $x \neq z$ (*jump measure*) and such that for every $f \in C_0^\infty(\boldsymbol{R}^n)$,

$$\iint_{x\neq z} |f(x) - f(z)|^2 J(dx, dz) < \infty \tag{13}$$

and $\mu^{ij}(dx)$ for $i, j = 1, \dots, n$ are positive Radon measures on $\boldsymbol{R}^n$ (*diffusion measures*) such that for every compact subset $B \subseteq \boldsymbol{R}^n$ and $b \in \boldsymbol{R}^n$, it results

$$\mu^{ij}(B) = \mu^{ji}(B), \qquad \sum_{i,j=1}^n b_i b_j \mu^{ij}(B) \geq 0. \tag{14}$$

For us, however, it is important to recall that, if our form is also a *closable* one,[19] it is possible to prove a sort of reverse statement so that we will practically be able use (12) to check that $\mathcal{E}$ is a regular Dirichlet form.

Proposition 3.3: Take a closable, bilinear form $(\mathcal{E}, D(\mathcal{E}))$ on $\mathcal{L}^2(\boldsymbol{R}^n, d\mu)$ with $D(\mathcal{E}) = C_0^\infty(\boldsymbol{R}^n)$: if $\mathcal{E}$ satisfies the Beurling-Deny formula (12), then its closure $(\bar{\mathcal{E}}, D(\bar{\mathcal{E}}))$ is a regular Dirichlet form.

Remark, moreover, that closability can often be verified in a very simple way as shown by the following result.[19]

Proposition 3.4: Let $(L, D(L))$ be a symmetric, negative definite linear operator on $\mathfrak{H}$, and define the bilinear form $\mathcal{E}(u, v) := -\langle Lu, v\rangle$ with $D(\mathcal{E}) := D(L)$. Then $(\mathcal{E}, D(\mathcal{E}))$ is closable.

Taken together Theorem 3.2 and Proposition 3.3 imply that when we want to know if there is a Markov process associated with a given generator L, essentially we must first consider the bilinear forms (7) and (8), and then check that the Beurling-Deny formula (12) holds.

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IV. LÉVY PROCESSES

The processes with independent increments constitute an important class of Markov processes, and among them the Lévy processes[10–13] are of particular relevance and are today widely applied in a variety of fields;[8,15] they have also been introduced in the larger context of quantum probability.[24] Actually, these processes are also time-homogeneous and hence, they can be identified by means of a transition function p_t only. They are stochastically continuous with independent and stationary increments, and they include many families of well-known processes as the Poisson, the Cauchy, the Student, the Variance-Gamma and, of course, the Wiener process which is the unique Gaussian process, with *a.s.* continuous paths. The Lévy processes are Feller processes and – if integrable with zero expectation – they are also martingales. In general, however, they always are semi-martingales, so that in any case they can be used as integrators in Itō integrals.[12] It is finally important to remark that between the Lévy processes and the infinite divisible distributions,[10] there is a one-to-one correspondence.

Given a filtered, complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, let us consider a Lévy process $(Z_t)_{t\,\geq\,0}$ with Lévy measure ℓ. It is well known then that the following *Lévy-Khintchin formula*[10] holds

Theorem 4.1: *Let* $(Z_t)_{t\,\geq\,0}$ *a Lévy process. Then its logarithmic characteristic* $\eta(u) := \log \mathbb{E}\{e^{iu\cdot Z_1}\}$ *satisfies*

$$\eta(u) = -\frac{1}{2} u\cdot Au + i\gamma\cdot u + \int_{y\neq 0}\left[e^{iu\cdot y} - 1 - i(u\cdot y)\,\mathbf{1}_{B_1}(y)\right]\ell(dy), \tag{15}$$

where $A = \|\alpha_{ij}\|$ *is a symmetric non-negative-definite* $n\times n$ *matrix,* $\gamma\in \boldsymbol{R}^n$, $B_1 = \{y\in\boldsymbol{R}^n : |y| \leq 1\}$ *and* $\ell(dy)$ *is a Lévy measure. This representation of* η *by means of the triplet* (A, ℓ, γ) *is unique. Vice versa, if* A *is a symmetric non-negative-definite* $n\times n$ *matrix,* $\gamma\in\boldsymbol{R}^n$ *and* ℓ *is a Lévy measure, then (15) is the logarithmic characteristic associated with a unique (in distribution) Lévy process* $(Z_t)_{t\,\geq\,0}$.

The triplet (A, ℓ, γ) is usually called the *generating triplet* of $(Z_t)_{t\,\geq\,0}$. The laws $\mathbb{P}_{Z_1}$ with characteristic functions $\varphi(u) = e^{\eta(u)} = \mathbb{E}\{e^{iu\cdot Z_1}\}$ turn out to be *infinitely divisible (id)*,[10] and as a consequence φ^t still is the characteristic function of some other *id* law for every $t > 0$. More precisely, if φ_t is the characteristic function of $\mathbb{P}_{Z_t}$ for $t \geq 0$, namely, $\varphi_t(u) = \mathbb{E}\{e^{iu\cdot Z_t}\}$, then it is easy to show that $\varphi_t = \varphi^t$. This result provides a one-to-one correspondence between the Lévy processes and the *id* laws[10,13] in such a way that every Lévy process is in fact uniquely determined by its *id* distribution at a unique instant, usually at $t = 1$.

We conclude this section with the explicit expressions of the infinitesimal generator $(L_0, D(L_0))$ and of the Dirichlet form $(\mathcal{E}_0, D(\mathcal{E}_0))$ associated with a symmetric Lévy process Z_t in $\mathcal{L}^2(\boldsymbol{R}^n, dx)$. The generator L_0 is a pseudo-differential operator with symbol η,[11,12] namely,

$$[L_0 f](x) = \frac{-1}{(2\pi)^n}\int e^{iux}\hat{f}(u)\eta(u)du, \tag{16}$$

where $\hat{f}$ is the Fourier transform of f, and $D(L_0)$ is the set of the $f\in\mathcal{L}^2(\boldsymbol{R}^n, dx)$ such that

$$\int |\hat{f}(u)|^2|\eta(u)|^2 du < \infty. \tag{17}$$

From (17), it can be proved that actually the Schwarz space $\mathcal{S}(\boldsymbol{R}^n)$ is a subset of $D(L_0)$, and when $f\in\mathcal{S}(\boldsymbol{R}^n)$ the infinitesimal generator of Z_t takes the more explicit form

$$[L_0 f](x) = \frac{1}{2}\nabla\cdot A\nabla f(x) + \int_{y\neq 0}[\delta_y^2 f](x)\,\ell(dy), \tag{18}$$

where $A = \|\alpha_{ij}\|$ is the symmetric, positive definite matrix, and ℓ is the Levy measure of the generating triplet of Z_t. We also adopted the shorthand notations

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$$[\delta_y f](x) := f(x+y) - f(x),$$
$$[\delta_y^2 f](x) := f(x+y) - f(x) - y \cdot \nabla f(x) \mathbf{1}_{B_1}(y)$$

with $B_1 = \{y \in \boldsymbol{R}^n : |y| \leq 1\}$. It is also easy to see then that

$$\delta_y(fg) = g\delta_y f + f\delta_y g + \delta_y f \delta_y g, \tag{19}$$

$$\delta_y^2(fg) = g\delta_y^2 f + f\delta_y^2 g + \delta_y f \delta_y g. \tag{20}$$

If the Lévy process is also rotationally invariant, then we have $\alpha_{ij} = \alpha\delta_{ij}$ and (18) is reduced to

$$[L_0 f](x) = \frac{\alpha}{2}\nabla^2 f(x) + \int_{y\neq 0} [\delta_y^2 f](x)\,\ell(dy). \tag{21}$$

As for the bilinear form associated with our symmetric Lévy process Z_t with generator $(L_0, D(L_0))$, namely,

$$\mathcal{E}_0(f,g) = -\langle L_0 f, g\rangle = -\int g(x)[L_0 f](x)\,dx,$$

we have[18] on $D(L_0)$,

$$\mathcal{E}_0(f,g) = -\int \nabla g(x)\cdot A\nabla f(x)\,dx - \frac{1}{2}\int\int_{y\neq 0} [\delta_y f](x)[\delta_y g](x)\,\ell(dy)dx. \tag{22}$$

This last expression can also be extended to a $D(\mathcal{E}_0) \supseteq D(L_0)$, the set of the $f \in \mathcal{L}^2(\boldsymbol{R}^n, dx)$ such that

$$\int \nabla f(x)\cdot A\nabla f(x)\,dx + \int\int_{y\neq 0} \left|[\delta_y f](x)\right|^2 \ell(dy)dx < \infty. \tag{23}$$

V. DOOB TRANSFORMATIONS

We turn now to the discussion of the association of a generalized Schrödinger equation to our Markov processes. Let $(X_t)_{t\in\boldsymbol{R}}$ be a time-homogeneous, bilateral Markov process with the infinitesimal forward and backward generators $(L, D(L))$ and $(\tilde{L}, D(\tilde{L}))$. We suppose that

1. X_t has an *a.c.* invariant measure $\mu(dx) = \rho(x)dx$;
2. $\rho(x) > 0$ *a.s.* with respect to the Lebesgue measure, so that $\mu(dx)$ is equivalent to the Lebesgue measure;
3. the set $D(L) \cap D(\tilde{L})$ is dense in $\mathcal{L}^2(\boldsymbol{R}^n, d\mu)$.
   As already stated in Sec. II, the invariant measure μ is not necessarily required to be a probability measure: it will be made clear in a few subsequent examples about the plane waves that we will indeed also consider cases where the invariant measure is rather σ-finite, as the Lebesgue measure on $\boldsymbol{R}^n$. From our Hypothesis 3, it also follows[25] that both $(T_t)_{t\geq 0}$ and $(\tilde{T}_t)_{t\geq 0}$ are *strongly continuous* semigroups in $\mathcal{L}^2(\boldsymbol{R}^n, d\mu)$, and that

$$\tilde{L} = L^\dagger. \tag{24}$$

If then Y_t is the space-time version of X_t, its infinitesimal forward and backward generators L^Y and $\tilde{L}^Y$ are defined on $\mathcal{L}^2(\boldsymbol{R}^{n+1}, d\mu\,dt)$, and it can be easily shown that

$$L^Y = L + \partial_t, \qquad \tilde{L}^Y = \tilde{L} - \partial_t \tag{25}$$

with $D(L^Y) = D(L)\otimes\mathcal{H}^1$ and $D(\tilde{L}^Y) = D(\tilde{L})\otimes\mathcal{H}^1$, where $\mathcal{H}^1$ is the space of the absolutely continuous functions of t with a square integrable derivative. The result (25) is apparently connected to the fact that the infinitesimal generator of the translation semigroup – namely,

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the semigroup of the degenerate Markov process τ_t – is the time derivative.[25] Remark that if Hypothesis 3 is satisfied by L and $\tilde{L}$, then it will hold also for L^Y and $\tilde{L}^Y$, and we will have

$$\tilde{L}^Y = (L^Y)^\dagger. \tag{26}$$

All our generators $L, \tilde{L}, L^Y$, and $\tilde{L}^Y$ can now be also naturally extended to the respective spaces $\mathcal{L}^2_{\boldsymbol{C}}(\boldsymbol{R}^n, d\mu)$ and $\mathcal{L}^2_{\boldsymbol{C}}(\boldsymbol{R}^{n+1}, d\mu\, dt)$ of the complex valued functions, where they are still densely defined and closed, while the relations (24) and (26) are preserved. Moreover, in order to make sure that the Hamiltonian operators that we will introduce later are bounded from below, we also add a fourth hypothesis:

4. $\exists\, C \in \boldsymbol{R}$ such that for every ϕ in $D(L)$,

$$-\Re\langle\phi, L\phi\rangle_\mu + \Im\langle\phi, L\phi\rangle_\mu \geq C\,\|\phi\|_\mu\,. \tag{27}$$

Remark, however, that if X_t is a μ-symmetric process, namely, if L turns out to be self-adjoint in $\mathcal{L}^2_{\boldsymbol{C}}(\boldsymbol{R}^n, d\mu)$, then $\Im\langle\phi, L\phi\rangle_\mu$ vanishes and the condition (27) becomes

$$-\langle\phi, L\phi\rangle_\mu \geq C\,\|\phi\|_\mu$$

which is automatically satisfied because L is the generator of a contraction semigroup so that[25] $\langle\phi, L\phi\rangle_\mu \leq 0$, and it will be enough to take $C = 0$. As a consequence, Hypothesis 4 is always satisfied by μ-symmetric processes. The relevance of this fourth condition will be made clear in the following, and it lies mainly in the fact that for a given symmetric, bounded from below operator $(H_0, D(H_0))$ on a Hilbert space $\mathfrak{H}$, there always exists[26] a *self-adjoint* operator $(H, D(H))$ bounded from below such that $D(H_0) \subset D(H)$, and that for every $v \in D(H_0)$, $Hv = H_0 v$. This operator H is usually called the *Friedrichs extension* of H_0.

Proposition 5.1: If X_t is a time-homogeneous Markov process satisfying Hypotheses 1–4, then the operator in $\mathcal{L}^2_{\boldsymbol{C}}(\boldsymbol{R}^{n+1}, d\mu\, dt)$,

$$K^Y := -\frac{\tilde{L}^Y + iL^Y}{i+1} = -\frac{L^Y + \tilde{L}^Y}{2} + \frac{L^Y - \tilde{L}^Y}{2i} \tag{28}$$

defined on $D(K^Y) := D(L^Y) \cap D(\tilde{L}^Y)$, is symmetric.

Proof: From Hypothesis 3, we deduce that K^Y is densely defined, while the symmetry easily follows from (26). □

We can also introduce in $\mathcal{L}^2_{\boldsymbol{C}}(\boldsymbol{R}^n, d\mu)$, the reduced operator

$$K := -\frac{\tilde{L} + iL}{i+1} = -\frac{L + \tilde{L}}{2} + \frac{L - \tilde{L}}{2i} \tag{29}$$

defined on $D(K) := D(L) \cap D(\tilde{L})$, and prove in a similar way that it is symmetric. Remark how the existence of both a forward and a backward generator is instrumental here in the definition of our symmetric operators K and K^Y. It should also be said, however, that starting from our generators we could define several different symmetric operators, our present choice being dictated mainly by an analogy with the Gaussian case, and by our interest in preserving the presence of the time derivatives ∂_t in the final operators.

Given now an arbitrary real function $S(x)$ (a change in it would simply be tantamount to a gauge transformation) and a constant $E \in \boldsymbol{R}$, we first define the *wave functions*

$$\psi(x) := \sqrt{\rho(x)}\, e^{iS(x)}, \qquad \Psi(x,t) := \psi(x)\, e^{-iEt} \tag{30}$$

and then we remark that the operators

$$\begin{aligned} U_\Psi &: f \in \mathcal{L}^2_{\boldsymbol{C}}(\boldsymbol{R}^{n+1}, d\mu dt) \longrightarrow f\Psi \in \mathcal{L}^2_{\boldsymbol{C}}(\boldsymbol{R}^{n+1}, dx dt), \\ U_\psi &: f \in \mathcal{L}^2_{\boldsymbol{C}}(\boldsymbol{R}^n, d\mu) \longrightarrow f\psi \in \mathcal{L}^2_{\boldsymbol{C}}(\boldsymbol{R}^n, dx) \end{aligned} \tag{31}$$

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are unitary with

$$U_\Psi^\dagger = U_\Psi^{-1} = U_{\frac{1}{\Psi}}, \qquad U_\psi^\dagger = U_\psi^{-1} = U_{\frac{1}{\psi}}.$$

By means of these we can now introduce the operators

$$K_\Psi^Y = U_\Psi K^Y U_\Psi^{-1}, \qquad K_\psi = U_\psi K U_\psi^{-1} \tag{32}$$

acting, respectively, on $\mathcal{L}^2_{\boldsymbol{C}}(\boldsymbol{R}^{n+1}, dxdt)$ and $\mathcal{L}^2_{\boldsymbol{C}}(\boldsymbol{R}^n, dx)$ with $D(K_\Psi^Y) = U_\Psi D(K^Y)$ and $D(K_\psi) = U_\psi D(K)$. These unitary transformations are reminiscent of the well-known *Doob transformation*[23,27,28] which is applied to the infinitesimal generators L of Markov processes for a real, positive Ψ in the domain of L with $L\Psi = 0$. Our transformation could also be defined in a more general way,[23] but in fact (32) turns out to be well suited to our purposes so that we will continue to call it Doob transformation, while K_Ψ^Y and K_ψ will be called the Doob transforms of K^Y and K.

Proposition 5.2: The operator K_Ψ^Y can be written as

$$K_\Psi^Y = H_0 \otimes I_t - i I_0 \otimes \partial_t,$$

where I_0 and I_t are the identity operators, respectively, on the x and *t variables, while*

$$H_0 = K_\psi + E \tag{33}$$

turns out to be a symmetric and bounded from below operator in $\mathcal{L}^2_{\boldsymbol{C}}(\boldsymbol{R}^n, dx)$. We also have that ψ is an eigenvector with eigenvalue E of the Friedrichs extension H of H_0,

$$H\psi(x) = E\psi(x) \tag{34}$$

and that the function $\Psi(x, t)$ is a strong solution of

$$i\partial_t \Psi(x, t) = H\Psi(x, t) \tag{35}$$

being for every $t > 0$ also a solution of (34).

Proof: From (25), we have for $\phi \in D(K_\Psi^Y)$,

$$\begin{aligned} K_\Psi^Y \phi &= -\Psi \frac{L + \tilde{L}}{2} \frac{\phi}{\Psi} - i\Psi\partial_t \frac{\phi}{\Psi} + \Psi \frac{L - \tilde{L}}{2i} \frac{\phi}{\Psi} \\ &= -\psi \left(\frac{L + \tilde{L}}{2} - \frac{L - \tilde{L}}{2i} \right) \frac{\phi}{\psi} + E\phi - i\partial_t\phi \;=\; (H_0 \otimes I_t)\phi - i\,(I_0 \otimes \partial_t)\,\phi. \end{aligned}$$

From Hypothesis 4, we can see now that H_0 is bounded from below, while from Hypothesis 3 we deduce that it is densely defined, and from (24) that it is symmetric: then its Friedrichs extension H exists and is self-adjoint. Moreover, the constant function $\phi_1 = 1$ is an element of $D(L) \cap D(\tilde{L})$ and from (9) and Hypothesis 1 it is easy to see that $L\phi_1 = \tilde{L}\phi_1 = 0$. As a consequence, we have from (33) that $\psi \in D(H_0)$, and that $H_0\psi = E\psi$: namely, ψ is an eigenfunction of H_0 corresponding to the eigenvalue E. It is then straightforward to prove (35). □

The Friedrichs extension H of H_0 will be called in the following the *Hamiltonian operator* associated with the Markov process X_t, and Eq. (35) will take the name of *generalized Schrödinger equation*. Remark that if, in particular, X_t is a μ-symmetric process, namely, if L is self-adjoint, then we simply have

$$K^Y = -L - i\partial_t, \qquad K = -L, \qquad H_0 = -U_\psi L U_\psi^{-1} + E, \tag{36}$$

so that H_0 itself is self-adjoint and hence coincides with H. This, however, is not the case for every Markov process $X(t)$ that we can consider within our initial hypotheses.

In conclusion, we have shown that from every Markov process X_t obeying our four initial hypotheses, we can always derive a self-adjoint Hamiltonian H and a corresponding generalized Schrödinger equation (35). In this scheme, the process X_t is associated with a particular stationary solution Ψ of (35) in such a way that $|\Psi|^2$ coincides with the invariant measure of X_t. Vice versa it

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would be interesting to be able to trace back a suitable Markov process X_t from a solution Ψ – at least from a *stationary* solution – of (35) with a self-adjoint Hamiltonian. In fact, even when from a given Hamiltonian H and a stationary solution Ψ of (35) we can manage – by treading back along the path mapped in this section – to get a semigroup L, we are still left with the problem of checking that the minimal conditions are met in order to be sure that there is a Markov process X_t associated with L. In this endeavor, the previous discussion about the Dirichlet forms developed in Sec. III will turn out to be instrumental as it will be made clear in the following.

VI. STOCHASTIC MECHANICS

Let us begin by remembering that the names we gave to H and to Eq. (35) at the end of the previous section are justified by the fact that when X_t is a solution of the Itō SDE,

$$dX_t = b(X_t)dt + dW_t, \tag{37}$$

where W_t is a Wiener process, then H turns out to be the Hamiltonian operator appearing in the usual Schrödinger equation of quantum mechanics. To show how this works, we will briefly review here this well-known result in the less familiar framework of the Doob transformations[9,20] because – at variance with the original Nelson *stochastic mechanics* – this procedure allows a derivation of the Schrödinger equation without introducing an explicit *dynamics* that, at present, is still not completely ironed out in the more general context of the Lévy processes.

Let us start by considering the operator

$$L_0 := \frac{1}{2}\nabla^2$$

with $D(L_0)$ the set of all the functions f that, along with their first and second generalized derivatives, belong $\mathcal{L}^2(\boldsymbol{R}^n, dx)$. It is well known that $(L_0, D(L_0))$ is the infinitesimal generator of a Wiener process. If now we take $\phi \in D(L_0)$ such that

$$\int_{\boldsymbol{R}^n} \phi^2(x)dx = 1, \qquad \phi \neq 0 \quad a.s. \text{ in } dx, \tag{38}$$

we can define in $\mathcal{L}^2(\boldsymbol{R}^n, d\mu)$ with $\mu(dx) = \rho(x)dx = \phi^2(x)dx$ a second operator

$$Lf := \frac{L_0(\phi f) - f L_0\phi}{\phi}, \tag{39}$$

where $D(L) := C_0^\infty(\boldsymbol{R}^n)$ is the set of infinitely derivable functions on $\boldsymbol{R}^n$ with bounded support. It is straightforward now to see that L is correctly defined,[22] and that (39) can be recast in the form

$$Lf = \frac{\nabla\phi}{\phi}\cdot\nabla f + \frac{1}{2}\nabla^2 f = b\cdot\nabla f + \frac{1}{2}\nabla^2 f, \qquad b = \frac{\nabla\phi}{\phi} \tag{40}$$

which on the other hand is typical for the generators of a process satisfying the SDE (37). At first sight, this seems to imply directly that to every given ϕ, we can always associate a Markov process (weak) solution of the SDE (37) with b defined as in (40), but this could actually be deceptive because this association is indeed contingent on the properties of the function b, and hence of ϕ. To be more precise: if we know that Eq. (37) has a solution X_t, then its generator certainly has the form (40); but vice versa if an operator (40) – or a SDE (37) – is given with an arbitrary b, we cannot, in general, be sure that a corresponding Markov process X_t solution of (37) does in fact exist, albeit in a weak sense. In the light of these remarks, it is important then to be able to prove, by means of the

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Dirichlet forms, that for L and b defined as in (40) we can always find a Markov process solution of (37).

Theorem 6.1: *The operator* $(L, D(L))$ *defined in (39) is closable and its closure* $(\bar{L}, D(\bar{L}))$ *is a self-adjoint, negative definite operator which is the infinitesimal generator of a Markov process* $(X_t)_{t\in\boldsymbol{R}}$ *(weak) solution of (37). The measure* $\mu(dx)$ *is invariant for this Markov process.*

Proof: Let us consider on $\mathcal{L}^2(\boldsymbol{R}^n, d\mu)$ the bilinear form

$$\mathcal{E}(f, g) := -\langle Lf, g\rangle_\mu$$

with $D(\mathcal{E}) := C_0^\infty(\boldsymbol{R}^n)$. For $f, g \in D(\mathcal{E})$ with an integration by parts we get

$$\begin{aligned}\mathcal{E}(f, g) &= -\frac{1}{2}\int_{\boldsymbol{R}^n} g(x)\phi(x)\nabla^2(\phi f)(x)dx + \frac{1}{2}\int_{\boldsymbol{R}^n} g(x)f(x)\phi(x)\nabla^2\phi(x)dx\\ &= -\frac{1}{2}\int_{\boldsymbol{R}^n} g(x)\phi^2(x)\nabla^2 f(x)dx - \int_{\boldsymbol{R}^n} g(x)\phi(x)\nabla f(x)\cdot\nabla\phi(x)dx\\ &= \int_{\boldsymbol{R}^n}\phi^2(x)\nabla g(x)\cdot\nabla f(x)dx,\end{aligned}$$

namely, our form satisfies the Beurling-Deny formula (12) with vanishing jump and killing measures, and with the condition (14) trivially satisfied. As a consequence $\mathcal{E}$ is symmetric and positive definite, so that also $-L$ is symmetric and positive definite and then is also closable.[26] Hence by Proposition 3.4, $(\mathcal{E}, D(\mathcal{E}))$ is closable and from Proposition 3.3 its closure $(\bar{\mathcal{E}}, D(\bar{\mathcal{E}}))$ is a symmetric, regular Dirichlet form which from Theorem 3.1 is associated with a self-adjoint, infinitesimal generator $(\bar{L}, D(\bar{L}))$. Of course $(\bar{L}, D(\bar{L}))$ itself turns out[26] to be the closure of $(L, D(L))$.

This $\bar{L}$ generates now a unique, bilateral Markov process $(X_t)_{t\in\boldsymbol{R}}$ on $(\boldsymbol{R}^n, \mathcal{B}(\boldsymbol{R}^n))$ having μ as its invariant measure when the conditions discussed in the remarks following Theorem 3.2 are met. In this context, we are then left just with the task of showing that the constant function $f_1(x) = 1$ belongs to $D(\bar{L})$ and that $\bar{L}f_1 = 0$. In fact for every $f \in C_0^\infty$ dense in $\mathcal{L}^2(\boldsymbol{R}^n, d\mu)$, with an integration by parts we have

$$\begin{aligned}\langle f_1, \bar{L}f\rangle_\mu &= \int_{\boldsymbol{R}^n}[\bar{L}f]\phi^2 dx = \int_{\boldsymbol{R}^n}[Lf]\phi^2 dx\\ &= \frac{1}{2}\int_{\boldsymbol{R}^n}\phi\nabla^2(f\phi)dx - \frac{1}{2}\int_{\boldsymbol{R}^n} f\phi\nabla^2\phi dx\\ &= \frac{1}{2}\int_{\boldsymbol{R}^n} f\phi\nabla^2\phi dx - \frac{1}{2}\int_{\boldsymbol{R}^n} f\phi\nabla^2\phi dx = 0.\end{aligned}$$

Being $\bar{L}$ self-adjoint this implies first that $f_1 \in D(\bar{L})$, and that $\langle\bar{L}f_1, f\rangle_\mu = 0$ for every $f \in C_0^\infty$ dense in $\mathcal{L}^2(\boldsymbol{R}^n, d\mu)$, and then that $\bar{L}f_1 = 0$. □

Since the process $(X_t)_{t\in\boldsymbol{R}}$ obtained from L_0 in Theorem 6.1 satisfies all Hypotheses 1–4 with $\rho(x) = \phi^2(x)$, we can now go on looking for the Hamiltonians produced by the Doob transformations. Take first the wave functions (30) with arbitrary $E \in \boldsymbol{R}$ and $S(x) = 0$, namely,

$$\psi(x) = \phi(x), \qquad \Psi(x, t) = e^{-iEt}\phi(x). \tag{41}$$

Since $\bar{L}$ is self-adjoint, by applying (36) we easily find as Hamiltonian associated with X_t by the Doob transformation, the operator

$$H := -\frac{1}{2}\nabla^2 + V(x), \tag{42}$$

where the potential function (defined up to a constant additive factor) is

$$V(x) = \frac{\nabla^2\phi(x)}{2\phi(x)} + E, \tag{43}$$

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namely, the potential of a Schrödinger equation admitting ϕ as eigenvector with eigenvalue E as it is immediately seen by rewriting (43) as

$$-\frac{1}{2}\nabla^2\phi + V\phi = E\phi.$$

In this way, the Doob transformation associates a Markov process to every stationary solution (41) of the Schrödinger equation

$$i\partial_t\Psi = H\Psi \tag{44}$$

with Hamiltonian (42). When on the other hand, we consider a non-vanishing $S(x)$ – namely, a *gauge transformation* with respect to the previous case – then our wave functions show the complete form (30), and starting again from (36) a slightly longer calculation shows that the Hamiltonian now is

$$H = \frac{1}{2}(i\nabla + \nabla S)^2 + V \tag{45}$$

with $V(x)$ always defined as in (43): in other words, in this case from ϕ and S we get both a scalar and a vector potential. Remark that, despite the presence of the term ∇S in (45), no physical electromagnetic field is actually acting on the particle, as is well known from the gauge transformation theory. The study of a stochastic description of a particle subjected to an electromagnetic field is not undertaken here: readers interested in this argument can usefully refer to Ref. 29.

Similar results can be obtained by initially choosing a constant function $\phi(x) = 1$ and $S(x) = p \cdot x$ so that

$$\psi(x) = e^{ip\cdot x}, \qquad \Psi(x,t) = e^{ip\cdot x - iEt},$$

namely, the wave function of a plane wave. In this case, however, instead of (39) we have to take

$$Lf := L_0 f + p\cdot\nabla f = \frac{1}{2}\nabla^2 f + p\cdot\nabla f$$

which is still of the form (40), albeit with a constant $b(x) = p$. Since this L is no longer self-adjoint in $\mathcal{L}(\boldsymbol{R}^2, dx)$ because, with an integration by parts, we find

$$L^\dagger = \tilde{L} = \frac{1}{2}\nabla^2 f - p\cdot\nabla f,$$

we now get from (29) and (32)

$$Kf = -\frac{1}{2}\nabla^2 f - ip\cdot\nabla f, \qquad K_\psi f = -\frac{1}{2}\nabla^2 f - \frac{p^2}{2}f$$

and then finally – by choosing, as usual, $E = p^2/2$ – we obtain from (33),

$$H = \frac{1}{2}\nabla^2$$

which is the Hamiltonian of the Schrödinger equation (44) in its free form. Remark that since $b(x) = p$ the resulting Markov process associated with Ψ now simply is a Wiener process plus a constant drift, but, at variance with the previous cases, we no longer have normalizable stationary solutions of (44), because $\phi^2(x)dx = dx$ defines an invariant measure which is the Lebesgue measure and not a probability: in other words our Markov process X_t will be an improper one in the sense outlined in Sec. II.

VII. LÉVY-SCHRÖDINGER EQUATION

In this section, we will focus our attention on a form of the generalized Schrödinger equation (35) which, without being the most general one, is less particular than that discussed in Sec. VI: namely, the Lévy-Schrödinger equation already introduced in a few previous papers,[1] where the Hamiltonian operator was found to be

$$H = -L_0 + V, \qquad D(H) = D(L_0) \cap D(V) \tag{46}$$

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with $(L_0, D(L_0))$ infinitesimal generator of a symmetric Lévy process taking values in $\boldsymbol{R}^n$, and V is a measurable real function defined on $\boldsymbol{R}^n$ that makes the operator $(H, D(H))$ self-adjoint and bounded from below. We have already seen in (16) and (18) how the generator $(L_0, D(L_0))$ of a Lévy process actually operates. We add here that when the Lévy process is symmetric its logarithmic characteristic $\eta(u)$ is a real function, and hence from (16) we easily have that $(L_0, D(L_0))$ is self-adjoint in $\mathcal{L}^2(\boldsymbol{R}^n, dx)$, while from the Lévy-Khintchin formula we also deduce that it is negative definite.[10] In the quoted papers, however, the choice (46) was essentially dictated by an analogy argument and there was no real attempt to deduce it: here instead we will try to extend this idea, and to justify it within the framework of the Doob transformations.

To show the way, we here consider first the case $\phi(x) = 1$ associated with the Lebesgue measure $\mu(dx) = dx$. Every Lévy process is a time-homogeneous Markov process, and this μ acts as its invariant measure.[10] It is apparent, moreover, that here we are not required to introduce a further operator L – as we did in (39) for the Wiener case – because L_0 itself plays this role. As a consequence, we can skip to prove a statement as Theorem 6.1, for L_0 is by hypothesis the generator of a Lévy process. On the other hand, even if here μ is only σ-finite and cannot be considered as a probability measure, we can apply the Doob transformation defined in Sec. V because all the Hypotheses 1–4 are met. Since L_0 is self-adjoint in $\mathcal{L}^2(\boldsymbol{R}^n, dx)$ (and hence for the backward generator we have $\tilde{L}_0 = L_0^\dagger = L_0$) from (28) we find

$$K = -L_0, \qquad\qquad K^Y = -L_0 - i\,\partial_t$$

and to implement a Doob transformation – by taking the Lebesgue measure as invariant measure, $S(x) = 0$ and $E = 0$ for simplicity – we choose

$$\Psi(t, x) = \psi(x) = 1, \tag{47}$$

namely, the simplest possible form of a plane wave. As a consequence in this first case, we finally get

$$H = -L_0,$$

namely, we find the case $V(x) = 0$ of (46). Since this Hamiltonian essentially coincides with our initial generator L_0 of a Lévy process, it is straightforward to conclude – as in Proposition 5.2 and the subsequent remarks – that to a plane wave (47) solution of a free generalized Schrödinger equation (35), namely, of the free Lévy-Schrödinger equation

$$i\,\partial_t \Psi = H\Psi = -L_0\Psi, \tag{48}$$

we can simply associate the Lévy process corresponding to L_0. As a matter of fact, this will be an improper process with generator L_0 and with the Lebesgue measure μ as initial – and invariant – measure. The free equation (48) is the case that has already been discussed at length – albeit in a more heuristic framework – in the previous papers.[1] If our Lévy process is also rotationally invariant, then from (21), and within the notations of Sec. IV, the Hamilton operator H becomes

$$[Hf](x) = -\frac{\alpha}{2}\nabla^2 f(x) - \int_{y\neq 0}\left[\delta_y^2 f\right](x)\,\ell(dy) \tag{49}$$

for any complex Schwarz function f and $x \in \boldsymbol{R}^n$. Note that if the jump term vanishes (namely, if L_0 is the generator of a Wiener process) and $\alpha = 1$ we have

$$H = -\frac{1}{2}\,\nabla^2,$$

i.e., the free Hamilton operator (42) of the stochastic mechanics presented in Sec. VI. As a consequence, we see that (49) can be considered as the generalization of the usual quantum mechanical Hamiltonian by means of a jump term produced by the possible non-Gaussian nature of our background Lévy process.

We turn then our attention to the more interesting case of a non-constant ϕ leading to a non-vanishing V in (46), and we suppose that $\alpha_{ij} = 0$ in (18), namely, that L_0 is the generator of a pure

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jump process without an unessential Gaussian component (we can always add it later). With this L_0 we take now a $\phi \in D(L_0)$ such that

$$\int_{\boldsymbol{R}^n} \phi^2(x)\, dx = 1, \qquad \phi > 0, \text{ a.s. in } dx \tag{50}$$

and in analogy with (39) we introduce the new operator $(L, D(L))$ in $\mathcal{L}^2(\boldsymbol{R}^n, d\mu)$,

$$Lf := \frac{L_0(\phi f) - f L_0\phi}{\phi} \tag{51}$$

with $D(L) := C_0^\infty(\boldsymbol{R}^n)$ and $\mu(dx) = \phi^2(x)dx$.

Proposition 7.1: If $\phi \in D(L_0)$, then for every $f \in C_0^\infty(\boldsymbol{R}^n)$ we have $\phi f \in D(L_0)$ and

$$L_0(\phi f) = f L_0\phi + \phi\, L_0 f + \int_{y\neq 0} \delta_y\phi\, \delta_y f\, \ell(dy). \tag{52}$$

Proof: See Appendix. □

This statement – which generalizes an *integration by parts* rule – proves first that our definition (51) is consistent in the sense that $\phi f \in D(L_0)$; then from (51) and (52) it also gives the jump version of (40),

$$Lf = L_0 f + \int_{y\neq 0} \frac{\delta_y\phi}{\phi}\, \delta_y f\, \ell(dy)$$

so that, by taking into account Eq. (21) with $a = 0$, we find

$$\begin{aligned}
[Lf](x) &= \int_{y\neq 0} \left(\delta_y^2 f + \frac{\delta_y\phi}{\phi}\, \delta_y f\right) \ell(dy) \\
&= \int_{y\neq 0} \left(\delta_y f - \frac{\phi \mathbf{1}_{B_1}(y)}{\phi + \delta_y\phi}\, y\cdot\nabla f\right) \frac{\phi + \delta_y\phi}{\phi}\, \ell(dy) \\
&= \int_{y\neq 0} [f(x+y) - f(x) - \gamma(x,y)\, y\cdot\nabla f(x)]\, \lambda(x;dy),
\end{aligned} \tag{53}$$

where

$$\gamma(x,y) = \frac{\phi(x)}{\phi(x+y)}\, \mathbf{1}_{B_1}(y), \qquad \lambda(x;dy) = \frac{\phi(x+y)}{\phi(x)}\, \ell(dy). \tag{54}$$

Equation (53) explicitly shows that the generator L introduced in (51) is a *Lévy-type* operator.[11,21] We then state our main result on the existence of a (Lévy-type) Markov process associated with L and, through a subsequent Doob transformation, to the (stationary) solutions of a Lévy–Schrödinger equation.

Theorem 7.2: *The operator $(L, D(L))$ defined in (51) is closable and its closure $(\bar{L}, D(\bar{L}))$ is a self-adjoint, negative definite operator which is the infinitesimal generator of a Markov process $(X_t)_{t\in\boldsymbol{R}}$. The measure $\mu(dx)$ is invariant for this Markov process.*

Proof: As in Theorem 6.1 from $(L, D(L))$, we first define in $\mathcal{L}^2(\boldsymbol{R}^n, d\mu)$ the bilinear form

$$\mathcal{E}(f,g) := -\langle Lf, g\rangle_\mu$$

with $D(\mathcal{E}) := C_0^\infty(\boldsymbol{R}^n)$ and we remark that from (51) we have

$$\mathcal{E}(f,g) = -\int \frac{L_0(\phi f) - f L_0\phi}{\phi}\, g\phi^2\, dx = -\int \phi\, g L_0(f\phi)\, dx + \int \phi\, g f L_0\phi\, dx.$$

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Then from (22) with $\alpha_{ij} = 0$, we obtain

$$\mathcal{E}(f,g) = \frac{1}{2}\int\int_{y\neq 0} [\delta_y(g\phi)](x)\,[\delta_y(f\phi)](x)\,\ell(dy)dx \\ -\frac{1}{2}\int\int_{y\neq 0} [\delta_y(fg\phi)](x)\,[\delta_y\phi](x)\,\ell(dy)dx$$

and from (19) after some tiring but simple algebra, we get

$$\mathcal{E}(f,g) = \frac{1}{2}\int\int_{y\neq 0} [\delta_y f](x)[\delta_y g](x)\phi(x+y)\phi(x)\,\ell(dy)dx. \tag{55}$$

This expression for $\mathcal{E}$ has the required form (12) with vanishing killing and diffusion measures, and $J(dz, dx)$ given by $\phi(x+y)\phi(x)\ell(dy)dx$ with $z = x + y$, which is positive because of (50). Since the condition (13) is satisfied, by reproducing the same argument previously adopted in Theorem 6.1 we get that there exists a self-adjoint, infinitesimal generator $(\bar{L}, D(\bar{L}))$ which turns out[26] to be the closure of $(L, D(L))$.

This $\bar{L}$ generates a unique, bilateral Markov process $(X_t)_{t\in\boldsymbol{R}}$ on $(\boldsymbol{R}^n, \mathcal{B}(\boldsymbol{R}^n))$ having μ as its invariant measure when the conditions discussed in the remarks following Theorem 3.2 are met. Hence, as in Theorem 6.1, we should only check that the constant element $f_1(x) = 1$ of $\mathcal{L}^2(\boldsymbol{R}^n, d\mu)$ belongs to $D(\bar{L})$ and that $\bar{L}f_1 = 0$. In fact, for every $f \in C_0^\infty(\boldsymbol{R}^n)$ dense in $\mathcal{L}^2(\boldsymbol{R}^n, d\mu)$ we have again from (22) with $\alpha_{ij} = 0$ that

$$\begin{aligned}\langle f_1, \bar{L}f\rangle_\mu &= \int_{\boldsymbol{R}^n}[\bar{L}f]\phi^2 dx = \int_{\boldsymbol{R}^n}[Lf]\phi^2 dx \\ &= \int_{\boldsymbol{R}^n}\phi^2\frac{L_0(\phi f) - fL_0(\phi)}{\phi}\,dx = \int_{\boldsymbol{R}^n}[\phi L_0(\phi f) - \phi f L_0(\phi)]\,dx \\ &= -\frac{1}{2}\int\int_{y\neq 0}\left[\delta_y\phi\,\delta_y(\phi f) - \delta_y(\phi f)\delta_y\,\phi\right]dx = 0.\end{aligned}$$

Being $\bar{L}$ self-adjoint this implies first that $f_1 \in D(\bar{L})$, and that $\langle \bar{L}f_1, f\rangle_\mu = 0$ for every $f \in C_0^\infty$ dense in $\mathcal{L}^2(\boldsymbol{R}^n, d\mu)$, and then that $\bar{L}f_1 = 0$. □

This result will now put us in condition to perform a suitable Doob transformation with the confidence that we can also associate a Lévy-type, Markov process X to the chosen wave functions. In fact, being $\mu(dx) = \rho(x)dx = \phi^2(x)dx$ an invariant measure for X_t, taking as before $S(x) = 0$ and $E \in \boldsymbol{R}$, namely,

$$\psi(x) = \phi(x), \qquad \Psi(t,x) = e^{-iEt}\phi(x)$$

and by applying (36), we immediately get the following expression for the Hamiltonian operator associated with X_t of Theorem 7.2,

$$[Hf](x) := [-L_0 f + Vf](x) = -\int_{y\neq 0}[\delta_y^2 f](x)\,\ell(dy) + V(x)f(x), \tag{56}$$

where the potential function now is

$$V(x) = \frac{[L_0\phi](x)}{\phi(x)} + E = \frac{1}{\phi(x)}\int_{y\neq 0}[\delta_y^2\phi](x)\,\ell(dy) + E \tag{57}$$

showing also that Ψ is a stationary solution of

$$i\partial_t\Psi = H\Psi = (-L_0 + V)\Psi = -\int_{y\neq 0}\delta_y^2\Psi\,\ell(dy) + V\Psi, \tag{58}$$

namely, of our Lévy-Schrödinger equation with the potential (57).

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VIII. CAUCHY NOISE

We will conclude the paper by proposing (in the one-dimensional case $n = 1$) two examples for the simplest stable, non-Gaussian Lévy background noise produced by the generator L_0 of a Cauchy process, namely, an operator of the form (21) without Gaussian term ($a = 0$) and with Lévy measure

$$\ell(dx) = \frac{dx}{\pi x^2}. \tag{59}$$

Remark that in general for this Cauchy background noise with Lévy measure (59), the generator (53) becomes

$$Lf = \frac{1}{\pi}\int_{y\neq 0}\left(\frac{\delta_y^2 f}{y^2} + \frac{1}{y}\frac{\delta_y\phi}{\phi}\frac{\delta_y f}{y}\right)dy \tag{60}$$

and that the convergence of this integral in $y = 0$ is a consequence of the fact that, for $y \to 0$, $\delta_y^2 f$ vanishes at the second order, while $\delta_y\phi$ and $\delta_y f$ are infinitesimal of the first order. The corresponding pure jump Cauchy-Schrödinger equation

$$i\partial_t\Psi = -\int_{y\neq 0}\frac{\delta_y^2\Psi}{\pi y^2}\,dy + V\Psi \tag{61}$$

has already been discussed in various disguises in several previous papers[1,6] and we will show here two examples of its stationary solutions for potentials of the form (43).

To define our invariant measure $\mu(dx) = \rho(x)dx = \phi^2(x)dx$ let us take first of all the functions

$$\phi(x) = \sqrt{\frac{2a}{\pi}}\frac{a}{a^2 + x^2}, \qquad \rho(x) = \phi^2(x) = \frac{2}{a\pi}\left(\frac{a^2}{a^2+x^2}\right)^2, \tag{62}$$

so that the stationary *pdf* will be that of a $\mathfrak{T}_a(3)$ Student law.[1] A direct calculation of (43) with these entries will then show that, by choosing the energy origin so that $E = -1/a$ and $V(\pm\infty) = 0$, we have

$$V(x) = -\frac{2a}{x^2 + a^2}. \tag{63}$$

In other words, the wave function

$$\Psi(x,t) = \sqrt{\frac{2a}{\pi}}\frac{a}{a^2 + x^2}\,e^{-it/a} \tag{64}$$

turns out to be a stationary solution of Eq. (61) with potential (63) corresponding to the eigenvalue $E = -1/a$. This result is summarized in Figure 1. On the other hand, we see from Theorem 7.2 that to the wave function (64) we can also associate a Lévy-type, Markov process X_t completely defined by the generator (53) with

$$\gamma(x,y) = \frac{a^2 + (x+y)^2}{a^2 + x^2}\,\mathbf{1}_{[-1,1]}(y), \qquad \lambda(x;dy) = \frac{a^2 + x^2}{a^2 + (x+y)^2}\frac{dy}{\pi y^2}$$

as can be deduced from (54) and (62).

In a similar vein, and always for the same equation (61), we can take as a second example the starting functions

$$\phi(x) = \sqrt{\frac{a}{\pi}}\frac{1}{\sqrt{a^2 + x^2}}, \qquad \rho(x) = \phi^2(x) = \frac{1}{\pi}\frac{a}{a^2 + x^2},$$

namely, the *pdf* of a $\mathfrak{C}(a) = \mathfrak{T}_a(1)$ Cauchy law. Treading the same path as before, a slightly more laborious calculation will show that, by choosing now $E = 0$ to have again $V(\pm\infty) = 0$, we find

$$V(x) = -\frac{2}{\pi}\left[\frac{1}{\sqrt{a^2+x^2}} + \frac{x}{a^2+x^2}\log\left(\sqrt{1+\frac{x^2}{a^2}} - \frac{x}{a}\right)\right], \tag{65}$$

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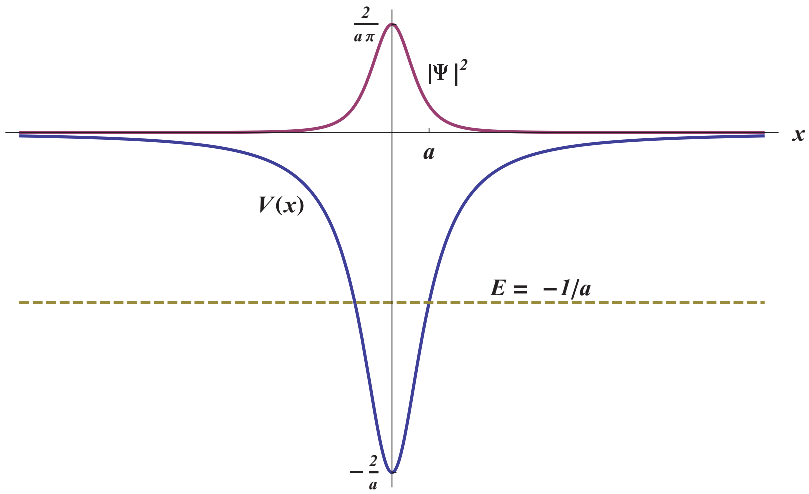

FIG. 1. (Color online) Potentials $V(x)$ and square modulus of the stationary wave functions $|\Psi|^2$ for the pair (63) and (64).

so that now the wave function

$$\Psi(x, t) = \sqrt{\frac{a}{\pi}}\frac{1}{\sqrt{a^2 + x^2}} \tag{66}$$

is stationary solution with eigenvalue $E = 0$ of Eq. (61) with potential (65). The potential and the corresponding *pdf* are depicted in Figure 2. The Lévy-type, Markov process X_t associated with this wave function is again defined by the generator (53) with

$$\gamma(x, y) = \sqrt{\frac{a^2 + (x + y)^2}{a^2 + x^2}}\,\mathbf{1}_{[-1,1]}(y), \qquad\qquad \lambda(x; dy) = \sqrt{\frac{a^2 + x^2}{a^2 + (x + y)^2}}\,\frac{dy}{\pi y^2}.$$

The two generators introduced here completely determine the two Markov processes associated with our stationary solutions of the Cauchy-Schrödinger equation (61).

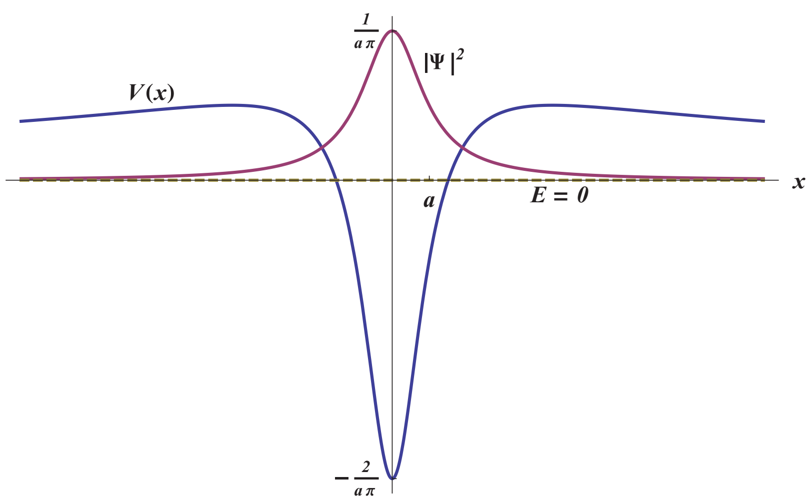

FIG. 2. (Color online) Potentials $V(x)$ and square modulus of the stationary wave functions $|\Psi|^2$ for the pair (65) and (66).

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IX. CONCLUSIONS

The adoption, proposed in a few previous papers,[1] of the LS equation – a generalization of the usual Schrödinger equation associated with the Wiener process – amounts in fact to suppose that the behavior of the physical systems in consideration is based on an underlying Lévy process that can have both Gaussian (continuous) and non-Gaussian (jumping) components. The consequent use of all the gamut of the *id*, even non-stable, processes on the other hand turns out to be important and physically meaningful because there are significant cases that fall in the domain of the LS picture, without being in that of a stable (fractional) Schrödinger equation.[7] In particular, the simplest form of a relativistic, free Schrödinger equation[1,6,17] can be associated with a peculiar type of self-decomposable, non-stable process acting as background noise. Moreover, in many instances of the LS equation the resulting energy-momentum relations can be seen as small corrections to the classical relations for small values of certain parameters.[1] It must also be remembered that – in discordance with the stable, fractional case – our models are not tied to the use of background noises with infinite variances: these can indeed be finite even for purely non-Gaussian noises – as for instance in the case of the relativistic, free Schrödinger equation – and can then be used as a legitimate measure of the dispersion. Finally, let us recall that a typical non-stable, Student Lévy noise seems to be suitable for applications, as for instance in the models of halo formation in intense beam of charged particles in accelerators.[8,13,30]

In view of all that it was then important to explicitly give more rigorous details about the formal association between LS wave functions and the underlying Lévy processes, namely, a true generalized stochastic mechanics. And it was urgent also to explore this Lévy–Nelson stochastic mechanics by adding suitable potentials to the free LS equation, and by studying the corresponding possible stationary and coherent states. To this end, we found expedient to broaden the scope of our enquiry to the field of Markov processes more general than the Lévy processes.

From this standpoint in the present paper, we have studied – with the aid of the Dirichlet forms – under what conditions the generalized Schrödinger equation (35), with a fairly general self-adjoint Hamiltonian H, admits a stochastic representation in terms of Markov processes: a conspicuous extension of the well known, older results of the stochastic mechanics.[2] More precisely, it has been shown how we can associate with every stationary wave functions Ψ, of the form (30) and solutions of Eq. (35), a bilateral, time-homogeneous Markov process $(X_t)_{t\in\boldsymbol{R}}$ whose generator L in its turn plays the role of the starting point to produce exactly the Hamiltonian H of Eq. (35). This association moreover is defined in such a way that, in analogy with the well-known Born postulate of quantum mechanics, $|\Psi|^2$ always coincides with the *pdf* of $(X_t)_{t\in\boldsymbol{R}}$.

The whole procedure adopted here is inherently based on the Doob transformation (31) previously adopted[9,20] in the particular case of the Wiener process to get the usual Schrödinger equation, as we have summarized in Sec. VI. This choice allows us, among other things, to sidestep for the time being the problem of the explicit definition of a dynamics for jump processes that would pave the way to recover our association along a more traditional path, either by means of the Newton law, as originally done,[2] or through a variational approach, as in later advances.[5] The definition of these structures, whose preliminary results have already been presented in Ref. 22, seems indeed at present to require more ironing and will be the object of future enquiries.

We have then focused our attention on the case of the LS equation (58), a particular kind of generalized Schrödinger equation characterized by an Hamiltonian (56) derived from a Lévy process generator. This equation was previously suggested only in an heuristic way,[1] while in the present paper we succeeded in proving a few rigorous results: first, we showed how the stationary wave functions of this LS equation actually satisfy the conditions required to be associated with Markov processes; then we pointed out that the LS Hamiltonian H is composed of a kinetic part (the generator L_0 of our background, symmetric Lévy process) plus a potential V, a term that was lacking of a justification in our previous formulations. Third, we also proved that the bilateral, time-homogeneous Markov processes associated with a stationary wave function Ψ turn out in fact to be Lévy-type processes, a generalization of the Lévy processes which is at present under intense scrutiny.[21] Finally, we presented a few examples of stationary solutions of LS equations with Cauchy background noise.

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These were much needed advances conspicuously absent in the previous papers, as already explicitly remarked there, It would be important now first to extend these results even to the non-stationary wave packets solutions of the generalized Schrödinger equation (35), at least in its LS form (58), that have been extensively studied in a recent paper,[1] where their inherent muli-modality has been put in evidence. Then to give a satisfactory formulation of the Nelson dynamics of the jump processes involved: a much needed advance that would constitute an open window on the true nature of these special processes. And finally a detailed study of the characteristics of the Lévy-type processes associated with the LS wave functions: this too will be the subject of future papers.

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APPENDIX: PROOF OF PROPOSITION 7.1

We begin by proving (52) for $\phi \in C_0^\infty(\boldsymbol{R}^n) \subseteq \mathcal{S}(\boldsymbol{R}^n) \subseteq D(L_0)$. In fact in this case we apparently have $\phi f \in C_0^\infty(\boldsymbol{R}^n)$, while from (18) and (20) with $A = 0$, it is

$$\begin{aligned} L_0(\phi f) &= \int_{y\neq 0} \delta_y^2(\phi f)\,\ell(dy) \\ &= f\int_{y\neq 0} \delta_y^2\phi\,\ell(dy) + \phi\int_{y\neq 0}\delta_y^2 f\,\ell(dy) + \int_{y\neq 0}\delta_y\phi\,\delta_y f\,\ell(dy) \\ &= f L_0\phi + \phi L_0 f + \int_{y\neq 0}\delta_y\phi\,\delta_y f\,\ell(dy) \end{aligned} \tag{A1}$$

because we can see that the third term of (A1) belongs to $\mathcal{L}^2_{\boldsymbol{R}}(\boldsymbol{R}^n, dx)$. Being indeed f and ϕ bounded, and $\phi \in C_0^\infty(\boldsymbol{R}^n) \subseteq D(L_0) \subseteq D(\mathcal{E}_0)$, from (23) we get

$$\begin{aligned} \int\left|\int_{y\neq 0}\delta_y f\delta_y\phi\,\ell(dy)\right|^2 dx &\leq \iint_{y\neq 0}\left|\delta_y f\,\delta_y\phi\right|^2\ell(dy)dx \\ &\leq 4\kappa^2\iint_{y\neq 0}|\delta_y\phi|^2\,\ell(dy)dx < \infty \end{aligned}$$

with $\kappa = \sup|f|$.

If instead we suppose that $\phi \in D(L_0) \subseteq D(\mathcal{E}_0)$, we have $\phi f \in \mathcal{L}^2_{\boldsymbol{R}}(\boldsymbol{R}^n, dx)$ since f is bounded. As a first step, let us show that $\phi f \in D(\mathcal{E}_0) \supseteq D(L_0)$: because of (23) to do that we have just to prove that

$$\iint_{y\neq 0}|\delta_y(\phi f)|^2\,\ell(dy)dx < \infty. \tag{A2}$$

Remark that from (19) we have

$$\begin{aligned} \iint_{y\neq 0}|\delta_y(\phi f)|^2\,\ell(dy)dx \leq 2\Bigg[&\iint_{y\neq 0}|\phi\,\delta_y f|^2\,\ell(dy)dx + \iint_{y\neq 0}|f\delta_y\phi|^2\,\ell(dy)dx \\ &+ \iint_{y\neq 0}|\delta_y\phi\delta_y f|^2\,\ell(dy)dx\Bigg]. \end{aligned} \tag{A3}$$

The second and the third integrals on the right of (A3) are finite because f is bounded and $\phi \in D(\mathcal{E}_0)$; as for the first integral, instead, since $\phi \in \mathcal{L}^2_{\boldsymbol{R}}(\boldsymbol{R}^n, dx)$, it is enough to remark that, from the typical property of Lévy measures

$$\int_{y\neq 0}\left(|y|^2\wedge 1\right)\,\ell(dy) < \infty$$

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and since $f \in C_0^\infty(\boldsymbol{R}^n)$, we have

$$\begin{aligned}\int_{y\neq 0} |\delta_y f|^2(x)\,\ell(dy) &= \int_{|y|\le 1,\, y\neq 0} |\delta_y f|^2(x)\,\ell(dy) + \int_{|y|\ge 1} |\delta_y f|^2(x)\,\ell(dy)\\ &\le C\int_{|y|\le 1,\, y\neq 0} |y|^2 |\nabla f(x)|^2\,\ell(dy) + C'\int_{|y|\ge 1}\ell(dy)\\ &\le C''\int_{|y|\le 1,\, y\neq 0} |y|^2\,\ell(dy) + C'\int_{|y|\ge 1}\ell(dy) = M < \infty\end{aligned}$$

with $C'' = C \sup|\nabla f(x)|^2$. In a similar way, we can prove that also $L_0 f$ is a bounded function, a remark that will be useful in the following.

In order to complete the proof of the proposition, we will show now that it exists a $\psi \in \mathcal{L}^2_{\boldsymbol{R}}(\boldsymbol{R}^n, dx)$ with the form of the second member of (52), and such that for every $g \in D(L_0)$,

$$\langle L_0 g, \phi f\rangle = \langle g, \psi\rangle.$$

If this is true, we can indeed deduce form the self-adjointness of L_0 that $f\phi \in D(L_0)$, and that (52) is verified. On the other hand, since Proposition 3.3 actually states that $C_0^\infty(\boldsymbol{R}^n)$ is a *core* for $(L_0, D(L_0))$, we can restrict our discussion to $g \in C_0^\infty(\boldsymbol{R}^n)$, so that we have

$$\begin{aligned}\langle L_0 g, \phi f\rangle &= \langle f L_0 g, \phi\rangle\\ &= \langle L_0(fg), \phi\rangle - \langle g L_0 f, \phi\rangle - \int\phi\left(\int_{y\neq 0}\delta_y g\,\delta_y f\,\ell(dy)\right)dx\\ &= \langle g, f L_0\phi\rangle - \langle g, \phi L_0 f\rangle - \iint_{y\neq 0}\phi\,\delta_y g\,\delta_y f\,\ell(dy)dx, \end{aligned}\qquad \text{(A4)}$$

where the second equality follows from the fact that (52) has been already proved for $f, g \in C_0^\infty(\boldsymbol{R}^n)$, while the third equality comes from the previously quoted $L_0 f$ boundedness (so that $\phi L_0 f \in \mathcal{L}^2_{\boldsymbol{R}}(\boldsymbol{R}^n, dx)$ and the scalar products exist) and the L_0 self-adjointness. Take now the third term of (A4): according to (19), we have

$$\phi\,\delta_y g\,\delta_y f = \delta_y(\phi g)\,\delta_y f - g\,\delta_y\phi\,\delta_y f - \delta_y\,\phi\,\delta_y f\,\delta_y g. \qquad \text{(A5)}$$

Since we have seen that $\phi g \in D(\mathcal{E}_0)$, from (22) we first find

$$\iint_{y\neq 0}\delta_y(\phi g)\,\delta_y f\,\ell(dy)dx = -2\mathcal{E}_0(\phi g, f) = -2\langle\phi g, L_0 f\rangle = -2\langle g, \phi L_0 f\rangle. \qquad \text{(A6)}$$

Then, taking now into account that

$$\iint_{y\neq 0}\left|\delta_y\phi\,\delta_y f\,\delta_y g\right|^2\ell(dy)dx \le C\iint_{y\neq 0}|\delta_y\phi|^2\,\ell(dy)dx < \infty$$

from the Fubini's theorem and the symmetry of $\ell(dy)$ it results with the change of variables $x \to x + y$ and $y \to -y$ that

$$\iint_{y\neq 0}\delta_y\phi\,\delta_y f\,\delta_y g\,\ell(dy)dx = -\iint_{y\neq 0}\delta_y\phi\,\delta_y f\,\delta_y g\,\ell(dy)dx = 0. \qquad \text{(A7)}$$

Finally, by observing that f is bounded and $\phi \in D(\mathcal{E}_0)$, we have

$$\int_{y\neq 0}[\delta_y\phi](x)[\delta_y f](x)\,\ell(dy) \in \mathcal{L}^2_{\boldsymbol{R}}(\boldsymbol{R}^n, dx).$$

So, by collecting (A4), (A5), (A6), and (A7), we get

$$\langle L_0 g, \phi f\rangle = \left\langle g\,,\ f L_0\phi + \phi L_0 f + \int_{y\neq 0}\delta_y\phi\,\delta_y f\,\ell(dy)\right\rangle$$

which completes the proof.

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