Lévy–Schrödinger wave packets

Nicola Cufaro Petroni

Dipartimento di Matematica and TIRES, Università di Bari, INFN Sezione di Bari,
via E. Orabona 4, 70125 Bari, Italy

E-mail: cufaro@ba.infn.it

Abstract

We analyze the time-dependent solutions of the pseudo-differential Lévy–
Schrödinger wave equation in the free case, and compare them with the
associated Lévy processes. We list the principal laws used to describe the
time evolutions of both the Lévy process densities and the Lévy–Schrödinger
wave packets. To have self-adjoint generators and unitary evolutions we will
consider only absolutely continuous, infinitely divisible Lévy noises with laws
symmetric under change of sign of the independent variable. We then show
several examples of the characteristic behavior of the Lévy–Schrödinger wave
packets, and in particular of the multimodality arising in their evolutions:
a feature at variance with the typical diffusive unimodality of both the
corresponding Lévy process densities and usual Schrödinger wavefunctions.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction and notations

In a recent paper [1], it was shown how to extend the well-known relation between the Wiener
process and the Schrödinger equation [2–5] to other suitable Lévy processes. This idea—
discussed elsewhere only in the stable case [6, 7]—leads to a LS (Lévy–Schrödinger) equation
containing additional integral terms, which take into account the possible jumping part of the
background noise and has been presented in the framework of *stochastic mechanics* [2, 5]
as a model for systems more general than just the usual quantum mechanics: namely a true
dynamical theory of Lévy processes that can be applied to several physical problems [8]. The
aim of this paper is to show a number of explicit examples of wave packet solutions of these
LS equations in the free case.

   In recent years, we have witnessed a considerable growth of interest in non-Gaussian
stochastic processes—and in particular into Lévy processes—in domains ranging from
statistical mechanics to mathematical finance. In the physical field, however, the research
scope is presently rather confined to the stable processes and corresponding fractional calculus

[6, 7, 9], while in the financial domain a vastly more general type of process is at present in use. Here, we suggest that a Lévy stochastic mechanics should be considered as a dynamical theory of the entire gamut of the *infinitely divisible* processes with time reversal invariance, and that the horizon of its applications should be widened even to cases different from the quantum systems.

This approach has several advantages: first of all the use of general infinitely divisible processes lends the possibility of having realistic, finite variances, a situation ruled out for non-Gaussian stable processes. Second, the presence of a Gaussian component and the wide spectrum of decay velocities of the increment densities will give the possibility of having models with differences from the usual Brownian (and quantum mechanical, Schrödinger) case as small as we want. Last but not least, there are examples of non-stable Lévy processes which are connected with the simplest form of the quantum, *relativistic Schrödinger equation*: a link with important physical applications that was missing in the original Nelson model [10, 11]. This final remark, on the other hand, shows that this inquiry is not only justified by the desire for formal generalization but is also required by the need to attain physically meaningful cases that otherwise would not be contemplated in the narrower precinct of the stable laws.

In this paper, we will show practical examples for the behavior of the evolving wave packet solutions of particular kinds of (non-Wiener) LS equations, and we will put in evidence their characteristics: in particular the *multimodality* arising in many of these evolutions which has a correspondence neither in the associated process diffusions nor in the usual Schrödinger evolutions with the same initial wavefunctions: an effect which has already been observed only in confined Lévy flights [12]. As we will discuss in section 5, this seems to be coherent with the usual stochastic mechanics scheme, insofar as in this theory the Schrödinger equation is recovered by introducing a dynamics modeled by means of a quantum potential [2, 5]. In the following exposition, laws and processes will always be one dimensional. An extensive analysis of the topics discussed in this first section is available in the two monographs [13] and [14], while a short introduction can be found in [15].

In this paper, the law of a *rv* (random variable) X with law $\mathfrak{F}$ is characterized either by its *pdf* (probability density function) f, when—as it is generally supposed—the law is *ac* (absolutely continuous) or by its *chf* (characteristic function) φ with the usual reciprocity relations

$$\varphi(u) = \int_{-\infty}^{+\infty} f(x)\, \mathrm{e}^{\mathrm{i}ux}\, \mathrm{d}x, \qquad f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \varphi(x)\, \mathrm{e}^{-\mathrm{i}ux}\, \mathrm{d}u. \qquad (1)$$

In order to have background noises with generators self-adjoint in L^2—an essential requirement for our purposes—we will consider only *symmetric* laws, namely we will require $f(-x) = f(x)$ and $\varphi(-u) = \varphi(u)$ so that the *chf* φ will also be real. This also means that, when it exists, the expectation vanishes ($\mathbf{E}[X] = 0$), namely the law is also *centered*. For our purposes it will also be expedient to introduce a dimensional *scale parameter* $a > 0$ which, to fix the ideas, will be supposed to be a *length*. If a *rv* X with law $\mathfrak{F}$ is dimensionless, then $X_a = aX$ will be a length and will follow a law $\mathfrak{F}_a$ with

$$f_a(x)\mathrm{d}x = f\left(\frac{x}{a}\right) \frac{\mathrm{d}x}{a}, \qquad \varphi_a(u) = \varphi(au).$$

We could now think to $\mathfrak{F}_a$ as the parametric family of the rescaled *rv*s aX: these parametric families spanned just by one scale parameter a are here entire *types of laws*. A type of laws [16] is a family of laws that only differ among themselves by a centering and a rescaling: in other words, if $\varphi(u)$ is the *chf* of a law, all the laws of the same type have *chf*s $\mathrm{e}^{\mathrm{i}bu}\varphi(au)$ with a centering parameter $b \in \mathbb{R}$ and a scaling parameter $a > 0$. Since here we only deal with centered laws, our types are indeed spanned by means of the scale parameter a only.

In this paper, however, we will also consider other parametric families of laws with some dimensionless parameter λ which is not in general coincident with the scale parameter a. We could then have two-parameter families $\mathfrak{F}_a(\lambda)$, and in general we are interested in finding which families are closed under convolution (namely under addition of the corresponding independent *rv*s). When a type of laws is closed under convolution (as in the normal case) its laws are said to be *stable*: the convolution would produce another law of the same type, namely a law with only a different *scale* parameter (in our notation: same λ, but different a). If instead the convolution produces a law of the same family, but not of the same type (different λ), then the family is closed under convolution, but its laws are not stable.

Since we will restrict our analysis to background noises driven by Lévy processes, we will be interested almost exclusively in *id* (infinitely divisible) laws. We remember that a law φ is *id* if for every n there exists a *chf* φ_n such that $\varphi = \varphi_n^n$, while it is *stable* when for every $c > 0$ it is always possible to find $a > 0$ and $b \in \mathbf{R}$ such that $e^{ibu}\varphi(au) = [\varphi(u)]^c$. It is possible to show that every stable law is also *id* [13–15]. The *lch* (logarithmic characteristic) of the *id* laws $\eta = \ln \varphi$, with $\varphi = e^\eta$, satisfy the Lévy–Khintchin formula [13, 14]

$$\eta(u) = i\alpha u - \frac{1}{2}\beta^2 u^2 + \int_{y \neq 0} [e^{iuy} - 1 - iuy\, I_D(y)]\, \ell(y)\, dy, \tag{2}$$

where $D = \{y : |y| < 1\}$ and is then specified by a Lévy triplet $\mathcal{L} = (\alpha, \beta, \ell)$. The measure $\nu(dy) = \ell(y)\, dy$ is also called the Lévy measure. In particular, when the law is symmetric we have $\alpha = 0$ and $\ell(-x) = \ell(x)$, so that the Lévy–Khintchin formula will be reduced to the symmetric real expression

$$\eta(u) = -\frac{1}{2}\beta^2 u^2 + \int_{y \neq 0} (\cos uy - 1)\, \ell(y)\, dy \tag{3}$$

and hence the *chf* φ will not only be real but also non-negative: $\varphi(u) \geqslant 0$.

The Markov processes dealt with in this paper are stationary, independent increments processes and are then defined by means of the *chf* $\varphi^{\Delta t/\tau}$ of their Δt-increments, where τ is a dimensional, time scale parameter. Here too we can introduce a dimensionless formulation through a coordinate $s = t/\tau$, but to simplify the notation we can continue to use the same symbol t for this dimensionless time. In this case, the stationary *chf* is $\varphi^{\Delta t}$, and the dimensional formulation will be recovered by simple substitution of t/τ to t. A stochastically continuous process with stationary and independent increments is called a *Lévy process* when $X(0) = 0$, **P**-a.s., but this paper will mostly be about the same kind of processes for arbitrary initial conditions $X(0) = X_0$, **P**-a.s. with law $f_0(x)$ and $\varphi_0(u) = e^{\eta_0(u)}$. All these processes, independently from their initial conditions, will share both the same evolution equations and the same transition *pdf*s

$$f_{X(t)}(x|X(s) = y) = p(x, t|y, s).$$

To avoid confusion we will then adopt different notations for their respective marginal *pdf*s: for a Lévy process (namely with initial condition $X_0 = 0$) we will write

$$f_{X(t)}(x) = q(x, t), \qquad \varphi_{X(t)}(u) = \chi(u, t)$$

with $q(x, 0) = \delta(x)$ and $\chi(x, 0) = 1$, while for the general stationary and independent increments process (with arbitrary initial condition X_0) we will write

$$f_{X(t)}(x) = p(x, t), \qquad \varphi_{X(t)}(u) = \phi(u, t)$$

with $p(x, 0) = f_0(x)$ and $\phi(x, 0) = \varphi_0(x)$. It is also easy to show that

$$p(x, t|y, s) = q(x - y, t - s). \tag{4}$$

The infinitesimal generator $A = \eta(\partial)$ (here ∂ stands for the derivation with respect to the variable of a test function v) of the semigroup of a Lévy process will be a pseudo-differential operator with symbol η [1, 14], namely from (2)

$$[Av](x) = [\eta(\partial)v](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}ux} \eta(u)\hat{v}(u)\,\mathrm{d}u$$

$$= \alpha\,\partial_x v(x) + \frac{\beta^2}{2}\,\partial_x^2 v(x) + \int_{y \neq 0} [v(x+y) - v(x) - yI_D(y)\,\partial_x v(x)]\,\ell(y)\,\mathrm{d}y, \qquad (5)$$

where $\hat{v}$ denotes the FT (Fourier transform) of the test function v according to

$$\hat{v}(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} v(x)\,\mathrm{e}^{-\mathrm{i}ux}\mathrm{d}x, \qquad v(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{v}(u)\,\mathrm{e}^{\mathrm{i}ux}\,\mathrm{d}u.$$

The generator A will be self-adjoint in $L^2(\mathbb{R}, \mathrm{d}x)$ when the law is symmetric, and in this case (5) reduces to

$$[Av](x) = \frac{\beta^2}{2}\,\partial_x^2 v(x) + \int_{y \neq 0} [v(x+y) - v(x)]\,\ell(y)\,\mathrm{d}y \qquad (6)$$

so that it is determined only by two elements of our Lévy triplet: β and ℓ. Given the process stationarity, in a dimensionless formulation the transition law degenerate in $x = 0$ at $t = 0$ will have as *chf* $\chi = \varphi^t = \mathrm{e}^{t\eta}$ and as *pdf*

$$q(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \chi(u, t)\,\mathrm{e}^{-\mathrm{i}ux}\,\mathrm{d}u = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \varphi(u)^t\,\mathrm{e}^{-\mathrm{i}ux}\,\mathrm{d}u. \qquad (7)$$

This transition law plays an important role in the evolution of an arbitrary initial law f_0, φ_0: the process *chf* will indeed now be $\phi(u, t) = \chi(u, t)\varphi_0(u)$, and the corresponding *pdf* will be calculated as

$$p(x, t) = [q(t) * f_0](x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \phi(u, t)\,\mathrm{e}^{-\mathrm{i}ux}\,\mathrm{d}u.$$

This *pdf* will also be a solution of the evolution pseudo-differential equation [1, 14]

$$\partial_t p = \eta(\partial)p, \qquad p(x, 0) = f_0(x) \qquad (8)$$

which—for a centered, symmetric noise—from (6) takes the form

$$\partial_t p(x, t) = \eta(\partial_x)p(x, t)$$

$$= \frac{\beta^2}{2}\partial_x^2 p(x, t) + \int_{y \neq 0} [p(x+y, t) - p(x, t)]\,\ell(y)\,\mathrm{d}y. \qquad (9)$$

We finally remember that since (8) and (9) are given in terms of process *pdf*s, these equations are supposed to hold only for *ac* processes. It is then advisable to recall that [13–15] any non-degenerate, *sd* (self-decomposable) distribution is *ac*. We remember that a law $\varphi(u)$ is *sd* when for every $a \in (0, 1)$ we can always find another *chf* $\varphi_a(u)$ such that $\varphi(u) = \varphi(au)\varphi_a(u)$. Every stable law is also *sd*; every *sd* law is also *id*. Such a property moreover holds for the corresponding processes for every t [13] and hence we can always explicitly write down the evolution equations (9) in terms of the process *pdf*s at least for the *sd* case. We remark, however, that there are also non-*sd* processes which are *ac*: the *ac* compound Poisson processes of appendix A are an example in point.

We list in table 1 the properties of a few basic, symmetric, dimensionless laws: degenerate (Dirac) $\mathfrak{D}$, normal (Gauss) $\mathfrak{N}$, Cauchy $\mathfrak{C}$, Laplace $\mathfrak{L}$, uniform $\mathfrak{U}$, and doubly degenerate in $+1, -1$ (symmetric Bernoulli) $\mathfrak{D}_1$. Here $\Theta(x)$ is the Heaviside function. These laws are also relevant to particular cases of the families that we will introduce in section 2. We remark that

Table 1. List of the properties of a few basic, dimensionless laws discussed in this paper: degenerate (Dirac) $\mathfrak{D}$, normal (Gauss) $\mathfrak{N}$, Cauchy $\mathfrak{C}$, Laplace $\mathfrak{L}$, uniform $\mathfrak{U}$, and doubly degenerate in $+1, -1$ (symmetric Bernoulli) $\mathfrak{D}_1$. Θ is the Heaviside function.

law	f	φ	β	ℓ	**E**	**V**
$\mathfrak{D}$	$\delta(x)$	1	0	0	0	0
$\mathfrak{N}$	$\frac{e^{-x^2/2}}{\sqrt{2\pi}}$	$e^{-u^2/2}$	1	0	0	1
$\mathfrak{C}$	$\frac{1}{\pi}\frac{1}{1+x^2}$	$e^{-|u|}$	0	$\frac{1}{\pi x^2}$	$-$	$+\infty$
$\mathfrak{L}$	$\frac{e^{-|x|}}{2}$	$\frac{1}{1+u^2}$	0	$\frac{e^{-|x|}}{|x|}$	0	2
$\mathfrak{U}$	$\frac{\Theta(x+1)-\Theta(x-1)}{2}$	$\frac{\sin u}{u}$	$-$	$-$	0	$\frac{1}{3}$
$\mathfrak{D}_1$	$\frac{\delta_1(x)+\delta_{-1}(x)}{2}$	$\cos u$	$-$	$-$	0	1

Table 2. Families of *sd*, dimensionless laws: the stable $\mathfrak{S}(\lambda)$, the variance-Gamma $\mathfrak{VG}(\lambda)$, the Student $\mathfrak{T}(\lambda)$ and the relativistic *qm* (quantum mechanics) $\mathfrak{R}(\lambda)$.

law	f	φ	β	ℓ	$0<\lambda$
	$H_\lambda(|x|)$	$e^{-|u|^\lambda/\lambda}$	0	$\frac{|x|^{-1-\lambda}}{-2\lambda\Gamma(-\lambda)\cos(\lambda\pi/2)}$	$\lambda\leqslant 2$
$\mathfrak{S}(\lambda)$	$\frac{1}{\pi}\frac{1}{1+x^2}$	$e^{-|u|}$	0	$\frac{1}{\pi x^2}$	$\lambda=1$
	$\frac{e^{-x^2/2}}{\sqrt{2\pi}}$	$e^{-u^2/2}$	1	0	$\lambda=2$
$\mathfrak{VG}(\lambda)$	$\frac{|x|^{\lambda-1/2}K_{\lambda-1/2}(|x|)}{2^{\lambda-1}\Gamma(\lambda)\sqrt{2\pi}}$	$\left(\frac{1}{1+u^2}\right)^\lambda$	0	$\frac{\lambda e^{-|x|}}{|x|}$	$0<\lambda$
	$\frac{e^{-|x|}}{2}$	$\frac{1}{1+u^2}$	0	$\frac{e^{-|x|}}{|x|}$	$\lambda=1$
$\mathfrak{T}(\lambda)$	$\frac{1}{B\left(\frac{1}{2},\frac{\lambda}{2}\right)}\left(\frac{1}{1+x^2}\right)^{\frac{\lambda+1}{2}}$	$\frac{2|u|^{\lambda/2}K_{\lambda/2}(|u|)}{2^{\lambda/2}\Gamma(\lambda/2)}$	0	$\ldots$	$0<\lambda$
	$\frac{1}{\pi}\frac{1}{1+x^2}$	$e^{-|u|}$	0	$\frac{1}{\pi x^2}$	$\lambda=1$
$\mathfrak{R}(\lambda)$	$\frac{\lambda e^\lambda K_1\left(\sqrt{\lambda^2+x^2}\right)}{\pi\sqrt{\lambda^2+x^2}}$	$e^{\lambda(1-\sqrt{1+u^2})}$	0	$\frac{\lambda K_1(|x|)}{\pi|x|}$	$0<\lambda$

in table 1 there is no value for the expectation of $\mathfrak{C}$ because it does not exist ($\mathfrak{C}$ is centered on the median), and no values for the Lévy triplet of $\mathfrak{U}$ and $\mathfrak{D}_1$ since these are not *id* laws. Moreover, in general our laws are not necessarily standard (their variance **V** is not forcibly 1).

The paper is organized as follows: in section 2 we recall the essential properties of the law families of our interest; then, in section 3, the LS equation is introduced with its connections to the Lévy processes. In section 4, our examples are elaborated, and finally in sections 5 and 6 the results are discussed. Further technical details are collected in the appendices in order to avoid excessively burdening the text.

2. Families of *id* laws

We will introduce here the principal families of *id* laws considered in this paper. For a graphical synthesis of the relations among them see figure 1. We limit ourselves here to dimensionless laws in order to have one-parameter families that can be easily represented. We then list in table 2 the properties of the principal families of dimensionless, *sd* laws that will be discussed. K_ν, B and Γ respectively are the modified Bessel functions of the second kind, and the Euler Beta and Gamma functions, while H_λ stands for the Fox *H*-functions representing the *pdf*
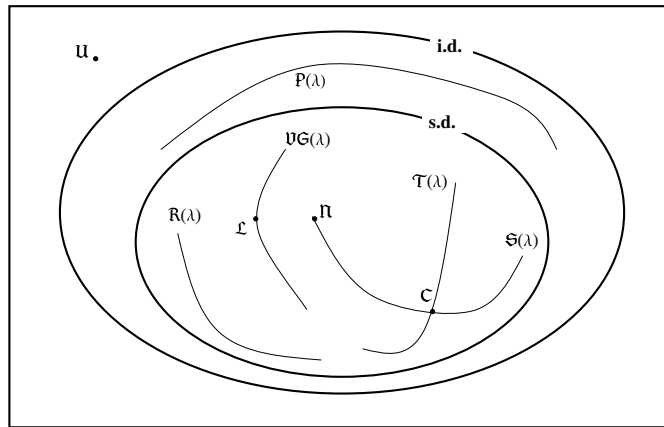
Figure 1. Graphical synthesis of the relations among the families of laws discussed in section 2. The uniform $\mathfrak{U}$ is our unique example beyond the pale of the *id* laws, while the laws of the (simple) Poisson family $\mathfrak{P}(\lambda)$ are *id* but not *sd*. Notable cases ($\mathfrak{N}$, $\mathfrak{L}$, $\mathfrak{C}$) within the *sd* families are put in evidence; the Cauchy $\mathfrak{C}$ law lies at the intersection of the stable $\mathfrak{S}(\lambda)$ and Student $\mathfrak{T}(\lambda)$ families.

of stable laws [17]. From tables 1 and 2, we can on the other hand immediately see that $\mathfrak{S}(1) = \mathfrak{T}(1) = \mathfrak{C}$, $\mathfrak{S}(2) = \mathfrak{N}$ and $\mathfrak{VG}(1) = \mathfrak{L}$, as also evidenced in figure 1.

2.1. The stable laws $\mathfrak{S}(\lambda)$

This is the more widely studied family of *id* laws, even if among them only the normal $\mathfrak{S}(2) = \mathfrak{N}$ enjoys a finite variance. But for the $\mathfrak{N}$, the $\mathfrak{C}$ and precious few other cases the *pdf*s of the stable laws exist only in the form of Fox *H*-functions. To see why these laws are *stable*, take the family $\mathfrak{S}_a(\lambda)$ with two parameters, $0 < \lambda \leqslant 2$, $a > 0$, and

$$\varphi(u) = e^{-a^\lambda |u|^\lambda / \lambda}.$$

For a given fixed λ, the family $\mathfrak{S}_a(\lambda)$ is closed under convolution, and since $\mathfrak{S}_a(\lambda)$ for a given λ is a *type* of laws, its laws are stable. This has far reaching consequences. In particular, it is at the root of the well-known fact that the stable Lévy processes are *self-similar*: a property not extended to other, non-stable Lévy processes [15]. The generators of the stable Lévy processes are for $0 < \lambda < 2$ and $\lambda \neq 1$:

$$[Av](x) = \frac{-1}{2\lambda \Gamma(-\lambda) \cos \frac{\lambda \pi}{2}} \int_{y \neq 0} \frac{v(x+y) - v(x)}{|y|^{1+\lambda}} \, dy,$$

while for $\lambda = 1$ ($\mathfrak{C}$ law) and $\lambda = 2$ ($\mathfrak{N}$ law) we respectively have

$$[Av](x) = \int_{y \neq 0} \frac{v(x+y) - v(x)}{\pi y^2} \, dy \qquad [Av](x) = \frac{1}{2} \partial_x^2 v(x).$$

2.2. The variance-Gamma laws $\mathfrak{VG}(\lambda)$

It is apparent from table 2 that the family of the variance-Gamma laws $\mathfrak{VG}(\lambda)$ is closed under convolution in the sense that $\mathfrak{VG}(\lambda_1) * \mathfrak{VG}(\lambda_2) = \mathfrak{VG}(\lambda_1 + \lambda_2)$. That notwithstanding,

however, the variance-Gamma laws are not stable: take indeed the two-parameter family $\mathfrak{VG}_a(\lambda)$

$$\varphi(u) = \left(\frac{1}{1 + a^2 u^2}\right)^\lambda.$$

Every sub-family with a given, fixed a is closed under convolution, but at variance with the stable case the parameter describing the sub-family is λ, rather than a. As a consequence, the closed subfamilies do not constitute types of laws and hence the laws are not stable. The *pdf*s of the variance-Gamma laws can also be given as finite combinations of elementary functions but only in particular instances. All our dimensionless $\mathfrak{VG}(\lambda)$ laws are endowed with expectations (which vanish by symmetry) and finite variances 2λ. The generator of the corresponding Lévy process is

$$[Av](x) = \lambda \int_{y \neq 0} \frac{v(x + y) - v(x)}{|y|} \, e^{-|y|} \, dy, \qquad \lambda > 0$$

which coincides with that of $\mathfrak{L}$ for $\lambda = 1$.

2.3. The Student laws $\mathfrak{T}(\lambda)$

But for the Cauchy $\mathfrak{C}$ case, the laws of the Student family are not stable, and $\mathfrak{T}(\lambda)$ itself is not closed under convolution: convolutions of Student laws are not Student laws. As can be seen from table 2, the variance-Gamma and the Student families enjoy a sort of duality since their *pdf*s and *chf*s are essentially exchanged. This has been discussed at length in a few recent papers [18–20]. We remark that to evidence this correspondence, we have chosen the Student laws of $\mathfrak{T}(\lambda)$ without introducing the usual parametric scaling x^2/λ of its variable that would have put equal to $\lambda/(\lambda - 2)$ all their variances for $\lambda > 2$. In particular, this means that for $\lambda \to +\infty$ we will not get a standard $\mathfrak{N}$ law, as also shown in figure 1. The following remarks are however virtually untouched by this choice. While the *pdf*s and *chf*s of the Student laws are known, differently from the variance-Gamma laws, their Lévy measures and generators do not have a known general expression and can be explicitly given only in particular instances [18]. Of course $\mathfrak{T}(1) = \mathfrak{C}$ is the well-known Cauchy law (see section 2.1 on the stable laws), while $\mathfrak{T}(3)$ given in (B.5) will play in the following the role of a possible initial law. The existence of the moments of the $\mathfrak{T}(\lambda)$ laws depends on the value of the parameter λ: the nth moment exists if $n < \lambda$. In particular the expectation exists (and vanishes) for $\lambda > 1$, while the variance exists finite for $\lambda > 2$ and its value is $(\lambda - 2)^{-1}$.

2.4. The compound Poisson laws $\mathfrak{N}_\sigma * \mathfrak{P}(\lambda, \mathfrak{H})$

Within the notations of appendix A, take the compound Poisson laws $\mathfrak{N}_\sigma * \mathfrak{P}(\lambda, \mathfrak{H})$ with the Lévy triplet $\mathcal{L} = (0, \sigma, \lambda h)$ and generator

$$[Av](x) = \frac{\sigma^2}{2} \partial_x^2 v(x) + \lambda \int_{y \neq 0} [v(x + y) - v(x)] h(y) \, dy.$$

When in particular $\mathfrak{H} = \mathfrak{N}_a$, then the Lévy triplet of $\mathfrak{N}_\sigma * \mathfrak{P}(\lambda, \mathfrak{N}_a)$ is

$$\mathcal{L} = \left(0, \sigma, \lambda \frac{e^{-x^2/2a^2}}{\sqrt{2\pi a^2}}\right)$$

and we obtain a law with the following *pdf* and *lch*:

$$f(x) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \frac{e^{-x^2/2(ka^2 + \sigma^2)}}{\sqrt{2\pi(ka^2 + \sigma^2)}}, \qquad \eta(u) = \lambda(e^{-a^2 u^2/2} - 1) - \frac{\sigma^2 u^2}{2},$$

namely a Poisson mixture of centered normal laws $\mathfrak{N}(0, ka^2 + \sigma^2)$. The self-adjoint generator then is

$$[Av](x) = \frac{\sigma^2}{2}\partial_x^2 v(x) + \lambda \int_{-\infty}^{+\infty} [v(x+y) - v(x)] \frac{e^{-y^2/2a^2}}{\sqrt{2\pi a^2}} dy,$$

and we could look at it as to a Poisson correction of a Wiener generator.

If instead we suppose, as another example, that $\mathfrak{H} = \mathfrak{D}_a$ the Lévy triplet of $\mathfrak{N}_\sigma * \mathfrak{P}(\lambda, \mathfrak{D}_a)$ will now be

$$\mathcal{L} = \left(0, \sigma, \lambda \frac{\delta_1(x/a) + \delta_{-1}(x/a)}{2a}\right),$$

while *pdf* and *lch* become

$$f(x) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} \frac{e^{-[x-(k-2j)a]^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}$$

$$\eta(u) = \lambda(\cos au - 1) - \frac{\sigma^2 u^2}{2}.$$

Here the law is again a mixture of normal laws $\mathfrak{N}(na, \sigma^2)$, $n = 0, \pm 1, \ldots$, but with a non-zero expectation which is an integer multiple of a. The generator finally is

$$[Av](x) = \frac{\sigma^2}{2}\partial_x^2 v(x) + \lambda \frac{v(x+a) - 2v(x) + v(x-a)}{2}$$

because the integral jump term reduces itself to a finite difference term.

2.5. The relativistic qm laws $\mathfrak{R}(\lambda)$

The family of the relativistic *qm* (quantum mechanics) laws on the other hand is a particular case of the well-known (centered and symmetric) generalized-hyperbolic family $\mathfrak{GH}(\delta, \alpha, \lambda)$ [18]: in fact we have $\mathfrak{R}(\lambda) = \mathfrak{GH}\left(-\frac{1}{2}, 1, \lambda\right)$, as can be seen by direct inspection of their *pdf*s and *chf*s. This family owes its name to the fact that (for $\lambda = mc^2\tau/\hbar$ and $a = \hbar/mc$, in terms of the time scale τ, the particle mass m, the velocity of light c and the Planck constant $\hbar$) its pseudo-differential generator

$$A = \eta(\partial_x) = \frac{\tau}{\hbar}\left(mc^2 - \sqrt{m^2c^4 - c^2\hbar^2\partial_x^2}\right) \tag{10}$$

essentially coincides with the Hamiltonian operator of the simplest form of a free relativistic Schrödinger equation [1, 10, 14] (see section 3). $\mathfrak{R}(\lambda)$ is closed under convolution, as can be seen from the form of the *chf*s, but the laws are not stable for the same reasons as the variance-Gamma: the parameter λ is not a scale parameter. The *pdf*s and *chf*s are explicitly known (see table 2), and all their moments exist: the odd moments (in particular the expectation) vanish by symmetry, while the even moments are always finite and its variance is λ. Since the Lévy measure is explicitly known (see table 2) the Lévy dimensionless generator has the integral form

$$[Av](x) = \lambda \int_{y\neq 0} [v(x+y) - v(x)] \frac{K_1(|y|)}{\pi|y|} dy,$$

where K_α is a modified Bessel function.

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3. The Lévy–Schrödinger equation

It has been shown in [1] that the evolution equation (9) of a centered, symmetric Lévy process can be formally turned into a LS equation: in fact the pseudo-differential generator $\eta(\partial)$ of our processes is a self-adjoint operator in L^2 and hence can correctly play the role of a Hamiltonian. We summarize in the following the formal steps leading to the LS equation; this will also establish the notation for the subsequent sections.

Take as background noise a centered, symmetric, *id* law with f, $\varphi = e^\eta$, $\mathcal{L} = (0, \beta, \ell)$ so that (3) holds, and then define the (dimensionless) transition *chf* $\chi(u, t) = \varphi^t(u)$ and the reduced transition *pdf*

$$q(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \varphi^t(u)\, e^{-iux}\, du$$

of the corresponding Lévy process. With an initial law f_0, $\varphi_0 = e^{\eta_0}$ the *chf* and the *pdf* of the process will then be

$$\phi(u, t) = \chi(u, t)\varphi_0(u)$$

$$p(x, t) = [q(t) * f_0](x) = \int_{-\infty}^{+\infty} q(x - y, t) f_0(y)\, dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \phi(u, t)\, e^{-iux}\, du. \tag{11}$$

The *pdf* $p(x, t)$ must also be a solution of the evolution equation (9) and in principle we could find p also by directly solving this equation.

We pass then to the LS propagators by means of the formal substitution $t \to it$:

$$\gamma(u, t) = \chi(u, it) = \varphi^{it}(u) = e^{it\eta(u)}, \qquad g(x, t) = q(x, it)$$

so that g and γ will still verify the same reciprocity relations (7) of q and χ

$$g(x, t) = q(x, it) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \chi(u, it)\, e^{-iux}\, du = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \gamma(u, t)\, e^{-iux}\, du.$$

We remark that if the law of the background noise is centered, symmetric and *id* then η is real, symmetric and positive and hence we always have $|\gamma| = 1$. This implies first that γ is not normalizable in L^2, and hence that also g is not normalizable in L^2. This is not surprising since, as is well known, the propagators are not supposed to be normalizable *wf*s. On the other hand, as we will see soon, this also entails that an initial normalized *wf* will stay normalized all along its evolution. We choose now an initial LS *wf*: to compare the evolutions of the *wf*s with that of the process *pdf*s, we will take—whenever we can—a law f_0, $\varphi_0 = e^{\eta_0}$ and a *wf* ψ_0 such that $|\psi_0|^2 = f_0$. By choosing f_0 and φ_0 centered and symmetric we will have (for vanishing initial phases) real φ_0 and ψ_0, with

$$\psi_0(x) = \sqrt{f_0(x)} \qquad\qquad \varphi_0 = \hat{\psi}_0 * \hat{\psi}_0 \tag{12}$$

where $\hat{\psi}_0$ is the *FT* of ψ_0. Now the LS *wf*s will obey the following evolution scheme:

$$\hat{\psi}(u, t) = \gamma(u, t)\hat{\psi}_0(u)$$

$$\psi(x, t) = [g(t) * \psi_0](x) = \int_{-\infty}^{+\infty} g(x - y, t)\psi_0(y)\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{\psi}(u, t)\, e^{iux}\, du. \tag{13}$$

Here, we can see the relevance of having $|\gamma|^2 = 1$ (namely of having a centered, symmetric background Lévy noise, and hence a self-adjoint generator): we have indeed

that $|\hat{\psi}(t)|^2 = |\gamma|^2 |\psi_0|^2 = |\psi_0|^2$, so that if $\|\psi_0\|^2 = 1$ then also $\|\hat{\psi}(t)\|^2 = 1$, and as a consequence (by Parseval and Plancherel theorems) $\|\psi(t)\|^2 = 1$ at every t. In other words, we can say that the non-normalizability of the propagator is the counterpart of the unitarity of the LS evolution. Finally it is easy to see that the *wf*s $\psi(x, t)$ introduced in the previous steps must satisfy the free LS equation

$$
i\partial_t \psi(x, t) = -\eta(\partial_x)\psi(x, t)
$$
$$
= -\frac{\beta^2}{2}\partial_x^2 \psi(x, t) - \int_{y \neq 0} [\psi(x + y, t) - \psi(x, t)]\,\ell(y)\,\mathrm{d}y. \quad (14)
$$

As already remarked in section 2.5, when in particular the background noise follows an $\Re(\lambda)$ law, the generator $\eta(\partial_x)$ takes the form (10) and (by restoring the dimensional constants, and by taking out of the *wf* an unessential phase factor $\mathrm{e}^{\mathrm{i}mc^2 t/\hbar}$) the LS equation (14) becomes

$$
i\partial_t \psi(x, t) = \sqrt{m^2 c^4 - c^2\hbar^2 \partial_x^2}\,\psi(x, t)
$$
$$
= mc^2 \left[\psi(x, t) - \int_{y \neq 0} \frac{\psi(x + y, t) - \psi(x, t)}{\pi |y|} K_1\left(\frac{mc|y|}{\hbar}\right) \mathrm{d}y \right], \quad (15)
$$

which coincides with the well-known relativistic Schrödinger equation [1, 10, 14]. This represents then an important example, because it shows first of all that the broadening of the scope of our inquiry to the entire gamut of the *id* laws is not suggested just by a desire of generalization. In fact, by confining ourselves to just the stable processes, we would have precluded the possibility of obtaining (15). On the other hand, its connection to a probabilistic, pathwise model as the stochastic mechanics inevitably raises many questions about the possibility of associating trajectories with quantum mechanics. In the case of the relativistic Schrödinger equation, moreover, this problem is compounded with that of coming to terms with the paradoxes due to the non-local effects that seem to be associated with the quantum mechanics: effects that should be in sheer contrast with the existence of a limiting velocity supposed in a relativistic model. This, of course, is not the place to thoroughly discuss this time honored problem by tapping into an already huge literature: we will limit ourselves to remarking that the non-locality implied either by the EPR correlations or rate of propagation of the probability flows fails in fact to materialize in observable, measurable superluminal effects actually violating either relativity or causality. On the other hand, the quoted phenomena are essential ingredients of a relativistic quantum theory independently from its possible interpretation in terms of trajectories. This is not, of course, to pretend that this problem simply does not exist, but only to suggest that at present—our aim here is not to give a convincing explanation of these paradoxes—we can confidently live with it without feeling obliged to abandon other essential principles we are comfortable with.

4. Processes and wave packets

We will give now several examples of LS *wf*s compared with the corresponding purely Lévy evolutions. We classify these examples first by choosing the laws of the background noises: this will be done by picking up the *id* laws that allow a reasonable knowledge both of the Lévy process transition *pdf* and of the LS propagator. Besides the usual Wiener case (that will be considered just to show the way), this will indeed allow us to calculate the evolutions by means of integrations, without being obliged to solve pseudo-differential equations. These equations will be used instead—when it is possible—as a check on the solutions found from transition *pdf*s and propagators. We will compare then the typical evolutions of the Lévy process *pdf*s and of the *wf*s solutions of a free LS equation: for details, notations and formulas about both

the initial laws and *wf*s, and the transition *pdf*s and propagators, we will make due references to appendix B and to appendix C. Remark also that in the following we will reintroduce the dimensional parameters a, b and τ.

4.1. Gauss

Take a Wiener process with transition law (C.1). For a normal initial law (B.1) $\mathfrak{N}_b$ we have

$$\phi(u, t) = \chi(u, t)\varphi_0(u) = e^{-(2Dt+b^2)u^2/2}$$

so that the evolution is always Gaussian $\mathfrak{N}(2Dt + b^2)$: it starts with a non-degenerate normal distribution of variance b^2 and then widens as the usual diffusions do with variance $2Dt + b^2$. The LS evolution of the *wf*s on the other hand is here the usual quantum mechanical one. Take as initial *wf* the Gaussian (B.2): then from (C.2) we have as wave packets

$$\hat{\psi}(u, t) = \gamma(u, t)\hat{\psi}_0(u) = \sqrt[4]{\frac{2b^2}{\pi}} \, e^{-(b^2+iDt)u^2}$$

$$\psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{\psi}(u, t) \, e^{iux} du = \sqrt[4]{\frac{b^2}{2\pi}} \frac{e^{-x^2/4(b^2+iDt)}}{\sqrt{b^2 + iDt}}.$$

It is well known that in this case, $|\psi(x, t)|^2$ has a widening, Gaussian shape all along its evolution. We neglect to display pictures of these well-known unimodal evolutions.

4.2. Cauchy and Student

The Cauchy process is one of the most studied non-Gaussian, Lévy processes [6], first because it is stable, and then because the calculations are relatively accessible. For example, if the initial law is a Cauchy $\mathfrak{C}_b$ with $\chi(u, t) = e^{-ct|u|}$, from (C.3) and (B.3) we immediately have for the transition *chf*

$$\phi(u, t) = e^{-(b+ct)|u|},$$

namely the process law remains a Cauchy $\mathfrak{C}_{b+ct}$ at every t with a typical broadening for $t \rightarrow +\infty$:

$$p(x, t) = \frac{1}{\pi} \frac{b + ct}{(b + ct)^2 + x^2}. \tag{16}$$

Of course this behavior (which is in common with the Gaussian Wiener process) comes out from the fact that the Cauchy laws are stable, and we neglect to display the corresponding figure. Even when the initial *pdf* is a $\mathfrak{T}_b(3)$ with $\varphi_0(u) = (1+b|u|)e^{-b|u|}$ calculations are easy: now the transition law is again $\mathfrak{C}_{ct}$, and the one-time process law $\mathfrak{C}_{ct} * \mathfrak{T}_b(3)$ will have as *chf*

$$\phi(u, t) = \chi(u, t)\varphi_0(u) = (1 + b|u|) \, e^{-(b+ct)|u|},$$

while the *pdf* is recovered by the *chf* inversion

$$p(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \phi(u, t) \, e^{-iux} du = \frac{(b + ct)^2(2b + ct) + vtx^2}{\pi[(b + ct)^2 + x^2]^2}. \tag{17}$$

It would be easy to check that this is again a normalized, unimodal, bell-shaped, broadening *pdf* (see figure 2), with neither an expectation nor a finite variance for $t > 0$. For this example, we can also show by direct calculation that the *pdf*s (16) and (17) are both solutions of the pseudo-differential Cauchy equation (C.4).
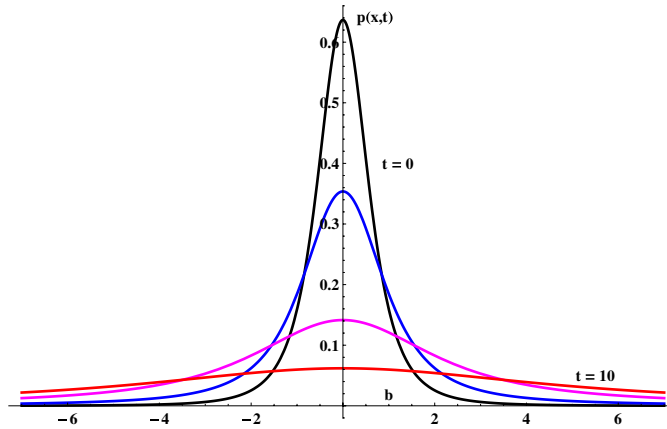
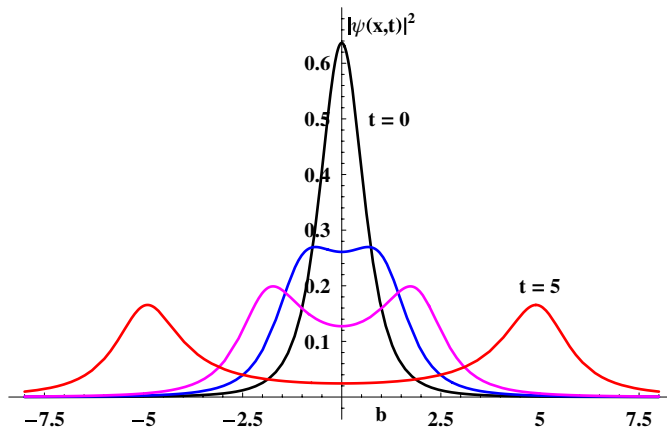**Figure 2.** The *pdf* (17) for a Cauchy process with a Student $\mathfrak{T}_b(3)$ initial distribution.

Figure 3. The square modulus of the Cauchy–Schrödinger *wf* (18) for a Student $\mathfrak{T}_b(3)$ initial distribution.

The Cauchy–Schrödinger evolutions, on the other hand, show a more interesting structure. The simplest case is found when we take as $|\psi_0|^2$ the Student $\mathfrak{T}_b(3)$ law (B.6): from (C.5) we indeed have

$$\hat{\psi}(u, t) = \gamma(u, t)\hat{\psi}_0(u) = \sqrt{b}\, e^{-(b+\mathrm{i}ct)|u|}$$

and hence

$$\psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{\psi}(u, t)\, e^{\mathrm{i}ux}\, \mathrm{d}u = \sqrt{\frac{2b}{\pi}}\, \frac{b + \mathrm{i}ct}{(b + \mathrm{i}ct)^2 + x^2}. \tag{18}$$

This *wf* (see figure 3) is correctly normalized in L^2 but shows an apparent *bimodality*. In fact $|\psi|^2$ has now two well-defined maxima smoothly drifting away from the center as $t \to +\infty$. It is also possible to show—by direct calculation—that our *wf* is a solution of the Cauchy–
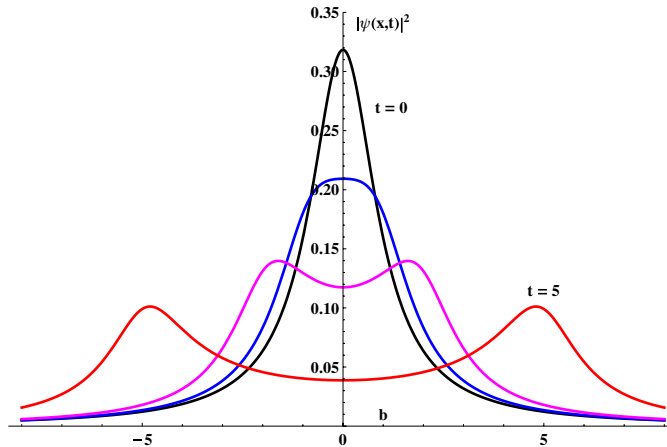
Figure 4. The square modulus of the Cauchy–Schrödinger *wf* (19) for a Cauchy $\mathfrak{C}_b$ initial distribution.

Schrödinger equation (C.6). Similar results are found in the case of a Cauchy $\mathfrak{C}_b$ initial *wf* (B.4): from the propagator (C.5) we have

$$\hat{\psi}(u, t) = \frac{\sqrt{2b}}{\pi} K_0(b|u|) e^{-ict|u|}$$

and hence by inverting the *FT*

$$\psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\sqrt{2b}}{\pi} K_0(b|u|) \, e^{-ict|u|} \, e^{iux} \, du$$

$$= \frac{1}{\pi\sqrt{b\pi}} \left[A\left(\frac{x + ct}{b}\right) + \overline{A\left(\frac{x - ct}{b}\right)} \right], \tag{19}$$

where we defined

$$A(z) = \frac{\frac{\pi}{2} - i \operatorname{arcsinh} z}{\sqrt{1 + z^2}}.$$

The *wf* (19) is normalized in L^2 and shows (see figure 4) a behavior similar to that of (18): its *pdf* $|\psi|^2$ starts as a Cauchy $\mathfrak{C}_b$ distribution and then widens with two well-defined maxima drifting away from the center. Here too, hence, we have bimodality: remark the difference with the Cauchy process *pdf*s $\mathfrak{C}_{b+ct}$ and $\mathfrak{C}_{ct} * \mathfrak{T}_b(3)$ which instead broaden by remaining strictly unimodal.

4.3. Laplace

This multimodality of the LS wave packets can also be found in other examples. Take first the variance-Gamma process introduced in appendix C. At variance with the Cauchy process, this is an example of a non-stable, *sd* process and hence has a certain interest as a non-typical case. At present, we will limit our discussion to initial states of the same variance-Gamma family of the background noise, and also always choose coincident scale parameters $a = b$ for the background noise and the initial states.
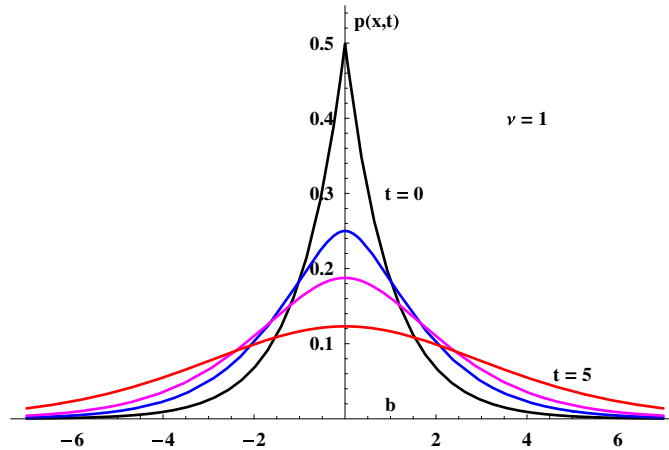
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Figure 5. The *pdf* (20) for a variance-Gamma process with Laplace $\mathfrak{VG}_b(1) = \mathfrak{L}_b$ initial distribution.

For a variance-Gamma process with transition law (C.7) and initial *pdf* (B.8) we immediately have

$$\phi(u,t) = \chi(u,t)\varphi_0(u) = \left(\frac{1}{1+b^2u^2}\right)^{\nu+\omega t}$$

and hence the process law simply is $\mathfrak{VG}_b(\nu + \omega t)$ with *pdf*

$$p(x,t) = \frac{2}{2^\nu \Gamma(\nu)\sqrt{2\pi}\,b}\left(\frac{|x|}{b}\right)^{\nu+\omega t-\frac{1}{2}} K_{\nu+\omega t-\frac{1}{2}}\left(\frac{|x|}{b}\right), \tag{20}$$

namely always a variance-Gamma but with a growing parameter $\nu + \omega t$. On the one hand, this explains why it would be delusory to think of simplifying the example by starting, for instance, with a simple Laplace $\mathfrak{L}_b = \mathfrak{VG}_b(1)$ initial law: in fact at every time $t > 0$ the process law would in any case no longer be a Laplace law, but a more general variance-Gamma with $\nu + \omega t \neq 1$. On the other hand, this apparently explains why at every t the *pdf* will appear as a broadening, unimodal distribution as shown in figures 5 and 6 respectively for $\nu = 1$ and $\nu = 2$.

For a LS evolution, on the other hand, we have from (C.9) and (B.10)

$$\hat{\psi}(u,t) = \sqrt{\frac{b}{\sqrt{\pi}}\frac{\Gamma(2\nu)}{\Gamma\left(2\nu-\frac{1}{2}\right)}}\left(\frac{1}{1+b^2u^2}\right)^{\nu+i\omega t}$$

so that the inverse *FT* will be

$$\begin{aligned}
\psi(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{\psi}(u,t)e^{iux}\,du \\
&= \sqrt{\frac{b}{\sqrt{\pi}}\frac{\Gamma(2\nu)}{\Gamma\left(2\nu-\frac{1}{2}\right)}}\frac{2}{2^{\nu+i\omega t}\Gamma(\nu+i\omega t)\sqrt{2\pi}} \\
&\quad \times \frac{1}{b}\left(\frac{|x|}{b}\right)^{\nu+i\omega+1/2}K_{\nu+i\omega+1/2}\left(\frac{|x|}{b}\right).
\end{aligned} \tag{21}$$

Numerical calculations and plotting then show that the *wf* (21) always is normalized, and that $|\psi|^2$ has several maxima, with the first two more prominent symmetrically drifting away
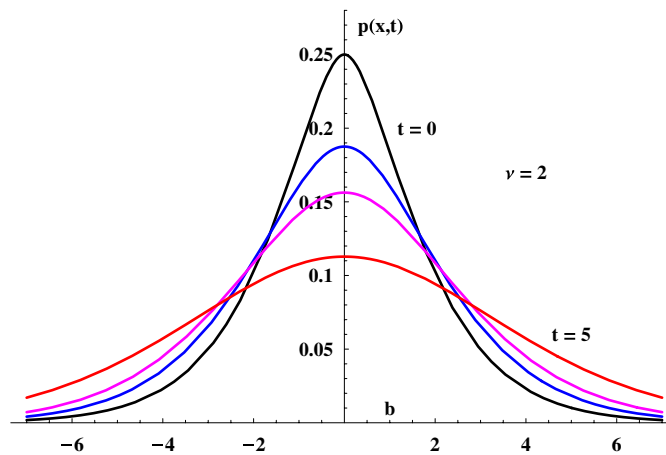
Figure 6. The *pdf* (20) for a variance-Gamma process with variance-Gamma $\mathfrak{VG}_b(2)$ initial distribution.
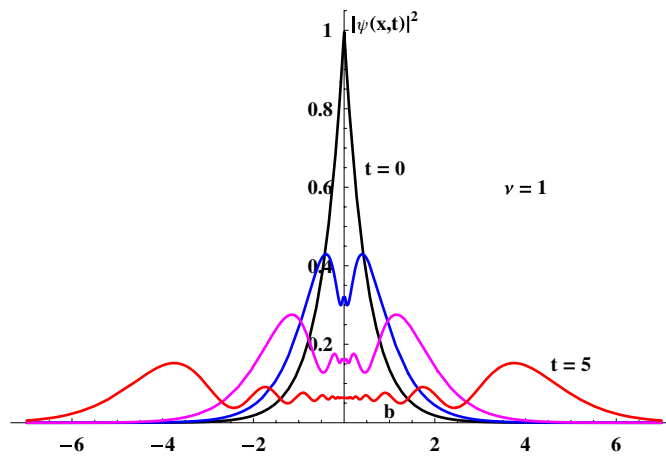

Figure 7. The square modulus of the variance-Gamma–Schrödinger *wf* (21) with a Laplace $\mathfrak{VG}_b(1) = \mathfrak{L}_b$ initial *wf*.

from the center (see figure 7 for $\nu = 1$ and figure 8 for $\nu = 2$). The behavior in $x = 0$ is indeed rapidly oscillating, but with infinitesimal amplitude as we approach $x = 0$: in fact the singular behavior of the Bessel function is here competing with an infinitesimal $|x|^\nu$ factor. The distribution also shows a slowly decreasing, flat plateau (with micro-oscillations) in the central region, while the diverging maxima can be rather dull as in figure 8.

4.4. Poisson

The following examples come from two *ac*, but not *sd* background noises: the compound Wiener–Poisson processes introduced in appendix A. First take the process with the transition
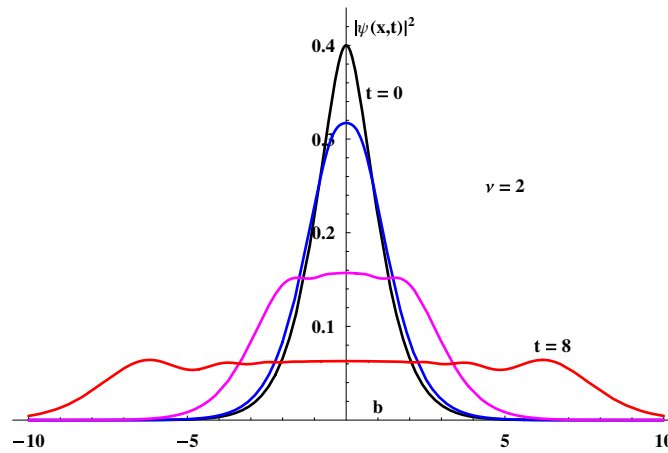
Figure 8. The square modulus of the variance-Gamma–Schrödinger *wf* (21) with a variance-Gamma $\mathfrak{VG}_b(2)$ initial *wf*.

law $\mathfrak{N}(2Dt) * \mathfrak{P}(\omega t, \mathfrak{N}_a)$ in (C.11): with a normal initial law (B.1) the marginal law of the process becomes $\mathfrak{N}(2Dt + b^2) * \mathfrak{P}(\omega t, \mathfrak{N}_a)$ namely

$$p(x, t) = \mathrm{e}^{-\omega t} \sum_{k=0}^{\infty} \frac{(\omega t)^k}{k!} \frac{\mathrm{e}^{-x^2/2(ka^2 + 2Dt + b^2)}}{\sqrt{2\pi(ka^2 + 2Dt + b^2)}}, \tag{22}$$

which apparently is a Poisson mixture of centered, normal *pdf*s of different variances, and hence has the usual bell-like, unimodal, diffusing shape that we will not bother to show. For the other transition law $\mathfrak{N}(2Dt) * \mathfrak{P}(\omega t, \mathfrak{D}_a)$ in (C.15) with the same normal initial distribution the marginal law instead is $\mathfrak{N}(2Dt + b^2) * \mathfrak{P}(\omega t, \mathfrak{D}_a)$ namely

$$p(x, t) = \mathrm{e}^{-\omega t} \sum_{k=0}^{\infty} \frac{(\omega t)^k}{k!} \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} \frac{\mathrm{e}^{-[x-(k-2j)a]^2/2(2Dt + b^2)}}{\sqrt{2\pi(2Dt + b^2)}}. \tag{23}$$

In other words, we always have generalized Poisson mixtures, but of non-centered normal *pdf*s. Even in this case, however, the shape of the overall *pdf* will be that of a bell-like, unimodal, diffusing curve (see figure 9).

For the LS equation on the other hand, consider first the propagator $\mathfrak{N}(2\mathrm{i}Dt) * \mathfrak{P}(\mathrm{i}\omega t, \mathfrak{N}_a)$ in (C.13) applied to an initial Gaussian *wf* (B.2); we then have

$$\hat{\psi}(u, t) = \mathrm{e}^{\mathrm{i}\omega t\left(\mathrm{e}^{-a^2 u^2/2} - 1\right)} \sqrt[4]{\frac{2b^2}{\pi}} \mathrm{e}^{-(b^2 + \mathrm{i}Dt)u^2}$$

and, by inverting the *FT* and taking into account the properties of the Gaussian integrals, the *wf* will be

$$\psi(x, t) = \mathrm{e}^{\mathrm{i}\omega t} \sum_{k=0}^{\infty} \frac{(\mathrm{i}\omega t)^k}{k!} \sqrt[4]{8\pi b^2} \frac{\mathrm{e}^{-x^2/2(ka^2 + 2b^2 + 2\mathrm{i}Dt)}}{\sqrt{2\pi(ka^2 + 2b^2 + 2\mathrm{i}Dt)}}, \tag{24}$$

namely a time-dependent, complex, Poisson superposition of centered Gaussian *wf*s. The same is true for the second example with propagator $\mathfrak{N}(2\mathrm{i}Dt) * \mathfrak{P}(\mathrm{i}\omega t, \mathfrak{D}_a)$ in (C.16) with an initial Gaussian *wf* (B.2): the *wf FT* in fact now is

$$\hat{\psi}(u, t) = \mathrm{e}^{\mathrm{i}\omega t(\cos au - 1)} \sqrt[4]{\frac{2b^2}{\pi}} \mathrm{e}^{-(b^2 + \mathrm{i}Dt)u^2}$$
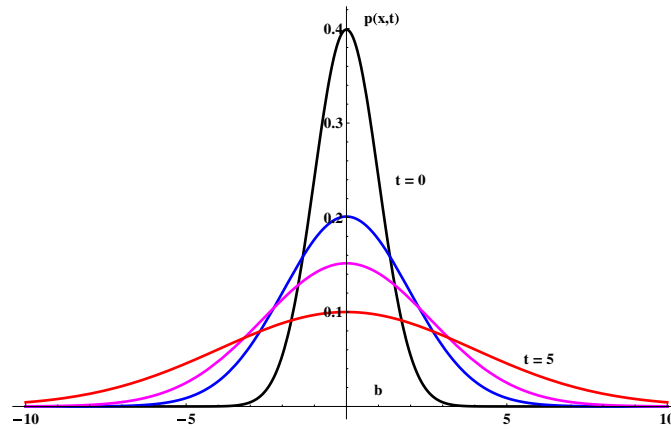
Figure 9. The *pdf* (23) of a normal-Poisson process $\mathfrak{N}(2Dt) * \mathfrak{P}(\omega t, \mathfrak{D}_a)$ with a Gaussian initial law.
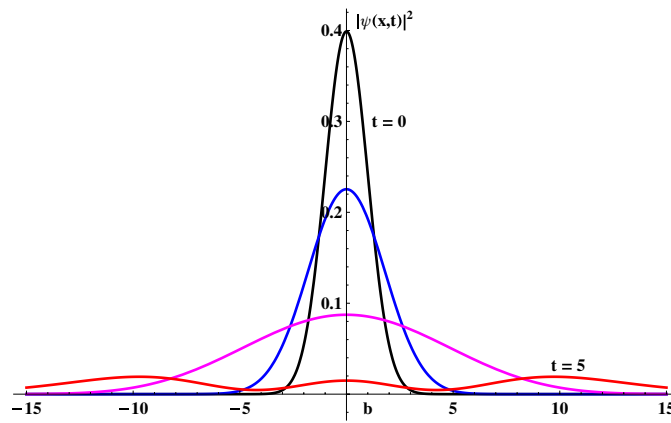

Figure 10. The square modulus of the normal-Poisson–Schrödinger *wf* (25) with a Gaussian initial *wf*.

so that the *wf* itself will be

$$\psi(x,t) = e^{i\omega t} \sum_{k=0}^{\infty} \frac{(i\omega t)^k}{k!} \frac{\sqrt[4]{8\pi b^2}}{2^k} \sum_{j=0}^{k} \binom{k}{j} \frac{e^{-[x-(k-2j)a]^2/4(b^2+iDt)}}{\sqrt{4\pi(b^2+iDt)}}. \tag{25}$$

In conclusion, while the plots of $p(x,t)$ in (22) and (23) simply display the too familiar story of a diffusing, unimodal, bell-shaped curve—and the same is of course true for $|\psi(x,t)|^2$ in (24)—for $|\psi(x,t)|^2$ in (25) we instead have again a separation of the wave packet in two symmetrical sub-packets drifting away from the center where in any case a decreasing, rump maximum is left behind (see figure 10): we then have here a trimodal evolution.
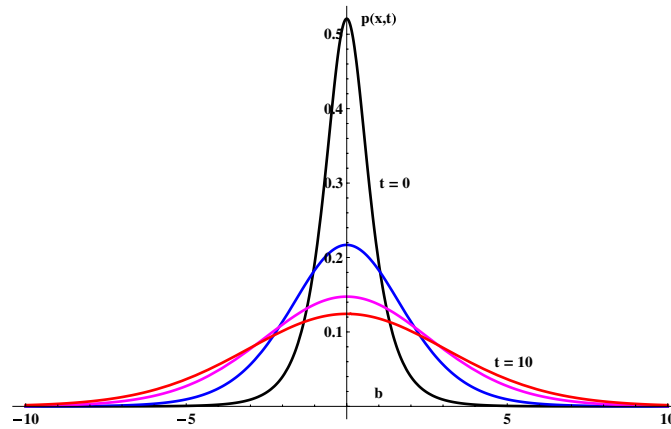
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Figure 11. The *pdf* (27) of a Lévy process with a relativistic *qm* background noise and an initial law of the same family.

4.5. relativistic qm

In a way similar to that of the variance-Gamma, for a relativistic *qm* Lévy process with transition law (C.17) and initial distribution (B.11), but with $a = b$, we immediately have

$$\phi(u, t) = \chi(u, t)\varphi_0(u) = e^{(\nu+\omega t)(1-\sqrt{1+a^2u^2})} \tag{26}$$

$$p(x, t) = \frac{(\nu + \omega t)e^{\nu+\omega t}}{\pi a} \frac{K_1(\sqrt{(\nu + \omega t)^2 + x^2/a^2})}{\sqrt{(\nu + \omega t)^2 + x^2/a^2}} \tag{27}$$

and hence the process law simply is $\Re(\nu + \omega t)$; namely, it will stay always in the same relativistic *qm* family but with a time-dependent parameter. The *pdf* $p(x, t)$ is shown in figure 11 and has the usual bell-like, unimodal, diffusing form. For the corresponding LS evolution on the other hand we have from (B.13) and (C.19) that the normalized *wf*s are

$$\hat{\psi}(u, t) = \gamma(u, t)\hat{\psi}_0(u) = \sqrt{\frac{a}{2e^{2\nu}K_1(2\nu)}} \, e^{(\nu+i\omega t)(1-\sqrt{1+a^2u^2})} \tag{28}$$

$$\psi(x, t) = \frac{(\nu + i\omega t)\, e^{i\omega t}}{\sqrt{a\pi K_1(2\nu)}} \frac{K_1(\sqrt{(\nu + i\omega t)^2 + x^2/a^2})}{\sqrt{(\nu + i\omega t)^2 + x^2/a^2}}. \tag{29}$$

We show in figure 12 how this $|\psi(x, t)|^2$ behaves, and in particular, at variance with the previous Lévy *pdf* (27), we find here again a case of bimodality: the *wf* square modulus shows two symmetric maxima drifting away from the center of the distribution.

5. Unimodality versus multimodality

The distributions of a process can undergo temporal changes of a qualitative nature, and the changes in modality are among the most interesting ones. It is difficult at present to give a complete analysis of this problem since many questions still remain unanswered, but an exhaustive review of the most important results about the Lévy processes can be found in [21]. Essentially a Lévy process is called *unimodal* (independently from its initial distribution) when the laws φ^t are unimodal for every $t > 0$. We learn first then that the *sd* processes always
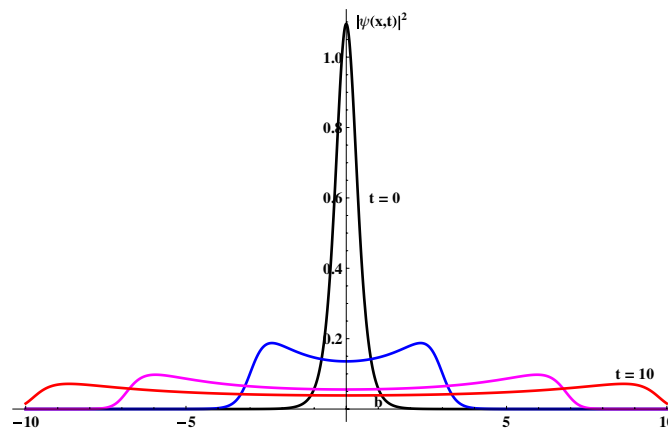
Figure 12. The square modulus of the relativistic *qm wf* (29) with an initial *wf* of the same family.

are unimodal all along their evolution: this is perfectly coherent with what we have found in our inquiry because all our processes (with the exception of the compound Poisson processes) are *sd*, and then they stay correctly unimodal for every $t > 0$. The case of the compound Poisson process $\mathfrak{N}_\sigma * \mathfrak{P}(\lambda, \mathfrak{N}_a)$, on the other hand, is recovered according to another result: a symmetric Lévy process (as all our processes are) is unimodal if and only if its Lévy measure is unimodal in $x = 0$. As a consequence it is not surprising that all our examples consistently were unimodal: what is new, in contrast, is that the evolutions associated with these Lévy processes by the LS equation (14) display a multimodality (in particular a bimodality) which manifests itself in time even when the initial distribution is unimodal.

As we have already mentioned, for (non-Schrödinger) Lévy processes, bimodality has been shown to arise in the presence of confining potentials [12]. In fact, the stationary states of nonlinear oscillators driven by particular Lévy noises display an apparent bimodality in given ranges of the potential parameters. In these cases, the authors contend that 'qualitatively the occurrence of the bimodal structure can be understood as a trade-off between the relatively high probability for large amplitude of the Lévy noise, and the sharp increase in the slope $\propto |x|^4$ of the quartic potential relative to the harmonic case'. In other words, this qualitative behavior is made contingent on the interplay between the jump length and the confining potential strength. At first sight, this interpretation does not seem to be immediately extendable to our LS examples because they are all instances of *free evolutions*, in the sense that we did not add external potentials to our LS equations. However, it is also well known that, in the usual Gaussian case, the Nelson stochastic mechanics [2, 5] achieves its breakthrough of getting the Schrödinger equation only by introducing a dynamics either by means of modified Newton equations or by means of stochastic variational principles: a dynamics which is otherwise sometimes modeled by means of a so-called quantum potential. In the same sense then we could conjecture that here, in our model of LS equations, we have introduced some kind of hidden dynamics which—in association with the Lévy jumps of the background noise—can account for the multimodality arising in the evolutions. Of course at present this is just an analogy because we do not have an explicit dynamical theory of the Lévy processes in the same sense in which the stochastic mechanics is a dynamical theory of Brownian motion. But this apparently points to a path of research that we will tread in the future.

Let us finally remark that in the previous papers about bimodality and Lévy flights [12], the discussion was typically confined only to the stable (in particular Cauchy) processes, the results were mostly produced from numerical simulations and they pertained essentially to stationary

states of confining potentials. In this context, these authors managed to find also the critical values (both for the parameters and for the time evolution) beyond which the bimodality arises. In this paper, we have instead found mostly analytical results about non-stationary states both of the general (not only stable) Lévy processes and of their LS counterparts: this substantially broadens the scope of the previous research in the field. Moreover, from the explicit expressions of the LS *wf*s listed in section 4 we can find the values of the *critical bifurcation time* t_0 at which the bimodality shows up from the initial unimodality. It is apparent indeed that—because of the symmetry of the system—t_0 would be the solution of the equation

$$\left[\partial_x^2 |\psi(x,t)|^2\right]_{x=0} = 0. \tag{30}$$

Take for example the two Cauchy–Schrödinger *wf*s (18) and (19): it is easy then to see that in these cases the solutions of (30) are respectively

$$t_0 = \frac{b}{c}, \qquad t_0 = 1.081\,42\,\frac{b}{c},$$

where b/c is a constant with the dimensions of a time which depends on the characteristics of both the background Lévy noise and the initial distribution, while the value of the coefficient in the second formula comes from the numerical solution of a simple, transcendent equation.

6. Conclusions

We presented in the previous sections several examples of free wave packets that are solutions of the LS equation without potentials (14). We started by generalizing the relation between Brownian motion and Schrödinger equation, and by associating the kinetic energy of a physical system with the generator of a symmetric Lévy process, namely to a pseudo-differential operator whose symbol is the *lch* η of an *id* law. This amounts to supposing that the LS equation is based on an underlying Lévy process that can have both Gaussian (continuous) and non-Gaussian (jumping) components. The use of all the *id*, even non-stable, processes on the other hand is important and physically meaningful because there are significant cases that are in the domain of our LS picture, without being in that of the stable (fractional) Schrödinger equation. In particular, as discussed in [1, 6], the simplest form of a relativistic, free Schrödinger equation can be associated with a particular type of *sd*, non-stable process acting as background noise. Moreover, in many instances of the LS equation, the new energy–momentum relations can be seen as corrections to the classical relations for small values of certain parameters [1]. It must also be remembered that—at variance with the stable, fractional case—our model is not tied to the use of background noises with infinite variances: these can be finite even in purely non-Gaussian models—as in the case of the relativistic, free Schrödinger equation—and can then be used as a legitimate measure of the dispersion. Finally, let us recall that a typical non-stable, Student Lévy noise seems to be suitable for applications in the models of halo formation in intense beam of charged particles in accelerators [8, 18, 22].

It was then important to explore the general behavior of the diffusing LS *wf*s: we systematically approached this problem by defining a procedure allowing us to analyze several combinations of initial *wf*s and background Lévy noises, and by comparing Lévy processes and free LS wave packets. We have then remarked that virtually in all our examples we witnessed a similar qualitative behavior: first of all the LS wave packets diffuse, in the sense that they broaden in a very regular way. As it is known the variance of a Lévy process—when it exists—grows linearly with time, exactly as in the usual diffusions. Of course for stable, non-Gaussian noises there is no variance, and we get instead an anomalous sub- and

super-diffusive behaviors. The corresponding LS wave packets show a similar qualitative behavior, but it is not always easy to calculate their variances.

A second feature is represented by the multimodality of the LS *wf*s even when the initial *wf* is unimodal. In fact, we found that in virtually all our examples the wave packet splits into sub-packets symmetrically and smoothly drifting away from the center: a behavior which—in similar conditions—is present neither in the free Lévy processes considered nor in the (Gaussian) free Schrödinger *wf*s. It is interesting to remark, then, that the unique instance with a similar bimodal behavior that has been found earlier [12] deals with *confined* Lévy flights. In our opinion, the bimodality found in our examples could then be connected to the combined effect of Nelson dynamics, and Lévy jumps in the background noise, and it would be interesting to explore if this behavior shows up again in the form of rings and shells respectively for the two- and three-dimensional LS equation.

It would be important now to explicitly give in full detail the formal association between LS *wf*s and the underlying Lévy processes, namely a true generalized stochastic mechanics. In particular, we would need to show that with every *wf* solution of the LS equation we can associate a well defined Lévy process: the techniques of the stochastic calculus applied to Lévy processes are today in full development [13, 14, 23], and to the best of our knowledge there is no apparent, fundamental impediment along this road. Finally, it would be relevant to explore this Lévy–Nelson stochastic mechanics by adding suitable potentials to our LS equation, and by studying the corresponding possible stationary and coherent states: all that will be the subject of future papers.

Appendix A. Compound Poisson laws

Among the *id*, non-*sd* laws the Poisson case stands as the most important example, but the simple Poisson law $\mathfrak{P}(\lambda)$ is neither symmetric nor *ac* with probability concentrated on the integer numbers according to the usual Poisson distribution:

$$f(x) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \delta_k(x), \qquad \varphi(u) = e^{\lambda(e^{iu}-1)}, \qquad \ell(x) = \lambda \delta_1(x).$$

Since $\mathfrak{P}(\lambda)$ is neither centered nor symmetric the generator of the corresponding Lévy process will not be self-adjoint. Take then a symmetric (not necessarily *ac* or *id*) law $\mathfrak{H}$ with *chf* $\vartheta(u) = e^{\zeta(u)}$ and build the compound Poisson law $\mathfrak{P}(\lambda, \mathfrak{H})$ with *chf*

$$\varphi(u) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \vartheta^k(u) = e^{\lambda[\vartheta(u)-1]}, \qquad \eta(u) = \lambda[\vartheta(u) - 1].$$

When $\mathfrak{H}$ is also *ac* with *pdf* $h(x)$ the law of $\mathfrak{P}(\lambda, \mathfrak{H})$ is

$$f(x) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} h^{*k}(x), \qquad h^{*k} = \begin{cases} \overbrace{h * \cdots * h}^{k \text{ times}}, & k = 1, 2, \ldots \\ \delta_0, & k = 0. \end{cases}$$

This compound Poisson law $\mathfrak{P}(\lambda, \mathfrak{H})$ has as Lévy triplet $\mathcal{L} = (0, 0, \lambda h)$, but it is still not *ac* even if $\mathfrak{H}$ has a density: in fact for $k = 0$ we always have a degenerate law δ_0. The laws of the increments of the corresponding compound Poisson process $\mathfrak{P}(\omega t, \mathfrak{H})$ with $\omega = \lambda/\tau$ are then the time-dependent mixtures

$$p(x, t) = e^{-\omega t} \sum_{k=0}^{\infty} \frac{(\omega t)^k}{k!} h^{*k}(x),$$

while its self-adjoint generator (without singularities at $x = 0$) is

$$[Av](x) = \lambda \int_{-\infty}^{+\infty} [v(x + y) - v(x)]h(y)\,\mathrm{d}y.$$

Its sample trajectories are now up and down staircase functions, with steps at Poisson random times, and random jump heights distributed according to the symmetric law $\mathfrak{H}$. Since however for $k = 0$ the law is degenerate in $x = 0$, these sample trajectories stick at $x = 0$ for a finite time (with probability 1), and the marginal distribution of the process is not *ac*.

In order to overcome this problem, take another independent, symmetric, *ac*, *id* law $\mathfrak{H}_0$ with *pdf* $h_0(x)$, *chf* $\vartheta_0(u) = \mathrm{e}^{\zeta_0(u)}$ and Lévy triplet $\mathcal{L}_0 = (0, \beta_0, \ell_0)$ and consider the law $\mathfrak{H}_0 * \mathfrak{P}(\lambda, \mathfrak{H})$ obtained by addition (convolution) so that

$$\varphi(u) = \vartheta_0(u)\,\mathrm{e}^{\lambda(\vartheta(u)-1)}, \qquad \eta(u) = \zeta_0(u) + \lambda(\vartheta(u) - 1),$$

while the *pdf* is

$$f(x) = \mathrm{e}^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} (h_0 * h^{*k})(x),$$

namely a mixture of *ac* laws. The law $\mathfrak{H}_0 * \mathfrak{P}(\lambda, \mathfrak{H})$ will have the Lévy triplet $\mathcal{L} = (0, \beta_0, \lambda h + \ell_0)$ and will also be symmetric if both h and h_0 are symmetric. The laws of the increments of the corresponding Lévy process will then be

$$\varphi(t) = \vartheta_0^{t/\tau}\,\mathrm{e}^{\lambda t(\vartheta-1)/\tau}, \qquad \eta(u, t) = \frac{t}{\tau}\zeta_0(u) + \omega t[\vartheta(u) - 1]$$

so that the process will be the superposition of two independent processes: an $\mathfrak{H}_0$–Lévy process plus a $\mathfrak{P}(\omega t, \mathfrak{H})$ compound Poisson process. Its trajectories are now the paths of the $\mathfrak{H}_0$–Lévy process, interspersed with Poisson random jumps with size law $\mathfrak{H}$. If then $h_0(x, t)$ is the *pdf* of $\vartheta_0^{t/\tau}(u)$, the t-increment *pdf*s of our process will be

$$p(x, t) = \mathrm{e}^{-\omega t} \sum_{k=0}^{\infty} \frac{(\omega t)^k}{k!} [h_0(t) * h^{*k}](x)$$

and the self-adjoint process generator

$$[Av](x) = \frac{\beta_0^2}{2}\partial_x^2 v(x) + \int_{y \neq 0} [v(x + y) - v(x)][\lambda h(y) + \ell_0(y)]\,\mathrm{d}y.$$

The relevant particular case of a Gaussian $\mathfrak{H}_0$ is discussed in section 2.4.

Appendix B. Initial states

We define here a list of possible initial *pdf*s and *wf*s. To simplify our calculations we will choose the initial *pdf*s to be centered and symmetric, and whenever convenient we will take pairs f_0, ψ_0 satisfying the relation $f_0 = |\psi_0|^2$. To evidence the meaning of the involved quantities the space a, b and time τ scaling parameters will be explicitly taken into account.

Initial laws and *wf*s with $f_0 = |\psi_0|^2$ are in the Gauss $\mathfrak{N}_b$ case

$$f_0(x) = \frac{\mathrm{e}^{-x^2/2b^2}}{\sqrt{2\pi b^2}}, \qquad \varphi_0(u) = \mathrm{e}^{-b^2 u^2/2}, \tag{B.1}$$

$$\psi_0(x) = \frac{\mathrm{e}^{-x^2/4b^2}}{\sqrt[4]{2\pi b^2}}, \qquad \hat{\psi}_0(u) = \sqrt[4]{\frac{2b^2}{\pi}}\,\mathrm{e}^{-b^2 u^2}. \tag{B.2}$$

We remark that, while ψ_0 is just the square root of f_0, $\hat{\psi}_0$ is the *FT* of ψ_0 and its relation to φ_0 is given by equation (12). The two *wf*s, moreover, are both normalized in L^2.

In the Cauchy $\mathfrak{C}_b = \mathfrak{T}_b(1)$ case initial laws and *wf*s are

$$f_0(x) = \frac{1}{b\pi} \frac{b^2}{b^2 + x^2}, \qquad \varphi_0(u) = \mathrm{e}^{-b|u|}, \tag{B.3}$$

$$\psi_0(x) = \frac{1}{\sqrt{b\pi}} \sqrt{\frac{b^2}{b^2 + x^2}}, \qquad \hat{\psi}_0(u) = \frac{\sqrt{2b}}{\pi} K_0(b|u|), \tag{B.4}$$

where K_0 is the modified Bessel function of order 0.

For the 3-Student $\mathfrak{T}_b(3)$ case the initial laws and *wf*s are

$$f_0(x) = \frac{2}{b\pi} \left(\frac{b^2}{b^2 + x^2} \right)^2, \qquad \varphi_0(u) = \mathrm{e}^{-b|u|}(1 + b|u|) \tag{B.5}$$

$$\psi_0(x) = \sqrt{\frac{2}{b\pi}} \frac{b^2}{b^2 + x^2}, \qquad \hat{\psi}_0(u) = \sqrt{b}\, \mathrm{e}^{-b|u|}. \tag{B.6}$$

It is also very easy to show that $\hat{\psi}_0$ is the right *FT* of ψ_0 and that $\varphi_0 = \hat{\psi}_0 * \hat{\psi}_0$.

In the general variance-Gamma case $\mathfrak{VG}_b(\nu)$, to make calculations possible, we will not always choose pairs of initial *pdf*s and *wf*s satisfying $\psi_0 = \sqrt{f_0}$. A possible example then is

$$f_0(x) = \frac{2}{2^\nu \Gamma(\nu)\sqrt{2\pi}\, b} \left(\frac{|x|}{b} \right)^{\nu - \frac{1}{2}} K_{\nu - \frac{1}{2}} \left(\frac{|x|}{b} \right), \tag{B.7}$$

$$\varphi_0(u) = \left(\frac{1}{1 + b^2 u^2} \right)^\nu, \tag{B.8}$$

$$\psi_0(x) = \sqrt{\frac{2\Gamma\left(\nu + \frac{1}{2}\right)}{b\pi\, \Gamma(\nu)\Gamma\left(2\nu - \frac{1}{2}\right)}} \left(\frac{|x|}{b} \right)^{\nu - \frac{1}{2}} K_{\nu - \frac{1}{2}} \left(\frac{|x|}{b} \right), \tag{B.9}$$

$$\hat{\psi}_0(u) = \sqrt{\frac{b\,\Gamma(2\nu)}{\sqrt{\pi}\,\Gamma\left(2\nu - \frac{1}{2}\right)}} \left(\frac{1}{1 + b^2 u^2} \right)^\nu, \tag{B.10}$$

where the functions are chosen in order to have an evolution easy to calculate. As a matter of fact, the usual relation $f_0 = |\psi_0|^2$ could be easily restored in the particular case of $\nu = 1$, namely for an initial Laplace law $\mathfrak{L}_b = \mathfrak{VG}_b(1)$. This particular case, however, is not really easier than the general case of the variance-Gamma process. In fact, as we will see in (C.7), the parameter affected by the time evolution is exactly ν, so that it is of no help to start with $\nu = 1$ if it immediately becomes $\nu \neq 1$.

In the relativistic *qm* $\mathfrak{R}_b(\nu)$ case we will choose as initial *chf* and *wf FT* respectively

$$f_0(x) = \frac{\nu \mathrm{e}^\nu K_1(\sqrt{\nu^2 + x^2/b^2})}{b\pi \sqrt{\nu^2 + x^2/b^2}}, \qquad \varphi_0(u) = \mathrm{e}^{\nu(1 - \sqrt{1 + b^2 u^2})}, \tag{B.11}$$

$$\psi_0(x) = \frac{\nu K_1(\sqrt{\nu^2 + x^2/b^2})}{\sqrt{\pi b K_1(2\nu)(\nu^2 + x^2/b^2)}}, \tag{B.12}$$

$$\hat{\psi}_0(u) = \sqrt{\frac{b}{2K_1(2\nu)}}\, \mathrm{e}^{-\nu\sqrt{1 + b^2 u^2}}, \tag{B.13}$$

which are in a relation similar to that of (B.7)–(B.10).

Appendix C. Transition laws and propagators

We will list here a few examples of background Lévy noises by paying attention to pick up processes with a known transition *pdf* associated with the evolution equation (9) and a known propagator associated with the free LS equation (14).

Take first a $\mathfrak{N}_a$ law with Lévy triplet $\mathcal{L} = (0, a, 0)$

$$f(x) = \frac{e^{-x^2/2a^2}}{\sqrt{2\pi a^2}}, \qquad \varphi(u) = e^{-a^2 u^2/2}.$$

The transition law of the corresponding Lévy (Wiener) process is then $\mathfrak{N}(2Dt)$ with $D = a^2/2\tau$, namely

$$q(x, t) = \frac{e^{-x^2/4Dt}}{\sqrt{4\pi Dt}}, \qquad \chi(u, t) = e^{-Dtu^2} \tag{C.1}$$

and the *pdf* evolution equation (9) is the usual Fokker–Planck equation

$$\partial_t p(x, t) = D\partial_x^2 p(x, t).$$

The corresponding LS propagator $\mathfrak{N}(2\mathrm{i}Dt)$ is again formally normal albeit with an imaginary variance:

$$g(x, t) = \frac{e^{-x^2/4\mathrm{i}Dt}}{\sqrt{4\pi \mathrm{i}Dt}}, \qquad \gamma(u, t) = e^{-\mathrm{i}Dtu^2} \tag{C.2}$$

and hence the LS equation (14) is the usual free Schrödinger equation

$$\mathrm{i}\partial_t \psi(x, t) = -D\partial_x^2 \psi(x, t).$$

From the Cauchy law $\mathfrak{C}_a$, a typical stable, non-Gaussian law with Lévy triplet $\mathcal{L} = (0, 0, a/\pi x^2)$ and with

$$f(x) = \frac{1}{a\pi} \frac{a^2}{a^2 + x^2}, \qquad \varphi(u) = e^{-a|u|},$$

we get the transition law $\mathfrak{C}_{ct}$ of the Cauchy process with $c = a/\tau$:

$$q(x, t) = \frac{1}{\pi ct} \frac{c^2 t^2}{c^2 t^2 + x^2}, \qquad \chi(u, t) = e^{-ct|u|} \tag{C.3}$$

and the corresponding process equation (9):

$$\partial_t p(x, t) = \int_{y \neq 0} [p(x + y, t) - p(x, t)] \frac{c}{\pi y^2} \, dy. \tag{C.4}$$

On the other hand the LS propagator $\mathfrak{C}_{\mathrm{i}ct}$ is

$$g(x, t) = \frac{1}{\mathrm{i}\pi} \frac{ct}{c^2 t^2 - x^2}, \qquad \gamma(u, t) = e^{-\mathrm{i}ct|u|} \tag{C.5}$$

and the LS equation (14)

$$\mathrm{i}\partial_t \psi(x, t) = -\int_{y \neq 0} [\psi(x + y, t) - \psi(x, t)] \frac{c}{\pi y^2} \, dy. \tag{C.6}$$

Remark that, at variance with the transition *pdf* (C.3), the Cauchy–Schrödinger propagator (C.5) has two simple poles in $x = \pm ct$ drifting away from the center $x = 0$ with velocity c.

Take now a *sd*, non-stable variance-Gamma law $\mathfrak{VG}_a(\lambda)$ with symmetric Lévy triplet $\mathcal{L} = (0, 0, \lambda e^{-|x|/a}/|x|)$ and with

$$f(x) = \frac{2}{2^\lambda \Gamma(\lambda)\sqrt{2\pi} a} \left(\frac{|x|}{a}\right)^{\lambda - \frac{1}{2}} K_{\lambda - \frac{1}{2}}\left(\frac{|x|}{a}\right), \qquad \varphi(u) = \left(\frac{1}{1 + a^2 u^2}\right)^\lambda.$$

The transition law will then be $\mathfrak{V}\mathfrak{G}_a(\omega t)$ with $\omega = \lambda/\tau$:

$$q(x,t) = \frac{2}{2^{\omega t}\Gamma(\omega t)\sqrt{2\pi}\,a}\left(\frac{|x|}{a}\right)^{\omega t - \frac{1}{2}} K_{\omega t - \frac{1}{2}}\left(\frac{|x|}{a}\right) \tag{C.7}$$

$$\chi(u,t) = \left(\frac{1}{1+a^2 u^2}\right)^{\omega t} \tag{C.8}$$

and the corresponding process equation (9):

$$\partial_t p(x,t) = \omega \int_{y\neq 0}[p(x+y,t) - p(x,t)]\frac{\mathrm{e}^{-|y|/a}}{|y|}\mathrm{d}y$$

so that the evolution will only affect the parameter λ, while a will always be the same. Then for the LS propagator $\mathfrak{V}\mathfrak{G}_a(\mathrm{i}\omega t)$ we have

$$g(x,t) = \frac{2}{2^{\mathrm{i}\omega t}\Gamma(\mathrm{i}\omega t)\sqrt{2\pi}\,a}\left(\frac{|x|}{a}\right)^{\mathrm{i}\omega t - \frac{1}{2}} K_{\mathrm{i}\omega t - \frac{1}{2}}\left(\frac{|x|}{a}\right) \tag{C.9}$$

$$\gamma(u,t) = \left(\frac{1}{1+a^2 u^2}\right)^{\mathrm{i}\omega t}, \tag{C.10}$$

while the LS equation (14) becomes

$$\mathrm{i}\partial_t \psi(x,t) = -\omega \int_{y\neq 0}[\psi(x+y,t) - \psi(x,t)]\frac{\mathrm{e}^{-|y|/a}}{|y|}\mathrm{d}y.$$

We will consider then two examples of *id*, non-*sd* background noise: for notations and details see section 2.4 and appendix A. Take first the law $\mathfrak{N}_\sigma * \mathfrak{P}(\lambda,\mathfrak{N}_a)$ discussed in section 2.4. From its *chf* we see that, with $\omega = \lambda/\tau$ and $D = \sigma^2/2\tau$, the transition law $\mathfrak{N}(2Dt) * \mathfrak{P}(\omega t, \mathfrak{N}_a)$ is

$$q(x,t) = \mathrm{e}^{-\omega t}\sum_{k=0}^{\infty}\frac{(\omega t)^k}{k!}\frac{\mathrm{e}^{-x^2/2(ka^2+2Dt)}}{\sqrt{2\pi(ka^2+2Dt)}} \tag{C.11}$$

$$\chi(u,t) = \mathrm{e}^{\omega t(\mathrm{e}^{-a^2 u^2/2}-1)}\,\mathrm{e}^{-Dtu^2}. \tag{C.12}$$

The corresponding Wiener–Poisson process *pdf*s have then an elementary form as time-dependent Poisson mixtures of time-dependent normal laws and the corresponding process equation (9) will become

$$\partial_t p(x,t) = D\partial_x^2 p(x,t) + \omega \int_{-\infty}^{+\infty}[p(x+y,t) - p(x,t)]\frac{\mathrm{e}^{-y^2/2a^2}}{\sqrt{2\pi a^2}}\,\mathrm{d}y.$$

The LS propagator $\mathfrak{N}(2\mathrm{i}Dt) * \mathfrak{P}(\mathrm{i}\omega t, \mathfrak{N}_a)$ is now

$$g(x,t) = \mathrm{e}^{-\mathrm{i}\omega t}\sum_{k=0}^{\infty}\frac{(\mathrm{i}\omega t)^k}{k!}\frac{\mathrm{e}^{-x^2/2(ka^2+2\mathrm{i}Dt)}}{\sqrt{2\pi(ka^2+2\mathrm{i}Dt)}} \tag{C.13}$$

$$\gamma(u,t) = \mathrm{e}^{\mathrm{i}\omega t(\mathrm{e}^{-a^2 u^2/2}-1)}\,\mathrm{e}^{-\mathrm{i}Dtu^2} \tag{C.14}$$

and the LS equation (14) becomes

$$\mathrm{i}\partial_t \psi(x,t) = -D\partial_x^2 \psi(x,t) - \omega \int_{-\infty}^{+\infty}[\psi(x+y,t) - \psi(x,t)]\frac{\mathrm{e}^{-y^2/2a^2}}{\sqrt{2\pi a^2}}\,\mathrm{d}y.$$

Take then the law $\mathfrak{N}_\sigma * \mathfrak{P}(\lambda, \mathfrak{D}_a)$ discussed in section 2.4: from its *lch* $\eta(u, t) = \omega t(\cos au - 1) - Dtu^2$ we see that the law of the corresponding Lévy process is $\mathfrak{N}(2Dt) * \mathfrak{P}(\omega t, \mathfrak{D}_a)$ and hence

$$q(x, t) = e^{-\omega t} \sum_{k=0}^{\infty} \frac{(\omega t)^k}{k!} \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} \frac{e^{-[x-(k-2j)\,a]^2/4Dt}}{\sqrt{4\pi Dt}} \tag{C.15}$$

$$\chi(u, t) = e^{\omega t(\cos au - 1) - Dtu^2},$$

while the process equation (9) is

$$\partial_t p(x, t) = D\partial_x^2 p(x, t) + \omega \frac{p(x+a, t) - 2p(x, t) + p(x-a)}{2}.$$

The LS propagator $\mathfrak{N}(2\mathrm{i}Dt) * \mathfrak{P}(\mathrm{i}\omega t, \mathfrak{D}_a)$ instead is

$$g(x, t) = e^{-\mathrm{i}\omega t} \sum_{k=0}^{\infty} \frac{(\mathrm{i}\omega t)^k}{k!} \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j} \frac{e^{-[x-(k-2j)\,a]^2/4\mathrm{i}Dt}}{\sqrt{4\pi \mathrm{i}Dt}} \tag{C.16}$$

$$\gamma(u, t) = e^{\mathrm{i}\omega t(\cos au - 1) - \mathrm{i}Dtu^2}$$

and the LS equation (14) is

$$\mathrm{i}\partial_t \psi(x, t) = -D\partial_x^2 \psi(x, t) - \omega \frac{\psi(x+a, t) - 2\psi(x, t) + \psi(x-a)}{2}.$$

Finally from the relativistic *qm chf* of $\mathfrak{R}_a(\lambda)$ we see that the corresponding Lévy process $\mathfrak{R}_a(\omega t)$ will have as transition law

$$q(x, t) = \frac{\omega t\, e^{\omega t} K_1(\sqrt{\omega^2 t^2 + x^2/a^2})}{\pi a \sqrt{\omega^2 t^2 + x^2/a^2}}, \tag{C.17}$$

$$\chi(u, t) = e^{\omega t(1 - \sqrt{1 + a^2 u^2})}, \tag{C.18}$$

with $\omega = \lambda/\tau$ as usual. We can also explicitly write the process equation (9) as

$$\partial_t p(x, t) = \omega \int_{y \neq 0} [p(x+y, t) - p(x, t)] \frac{K_1(|y|/a)}{\pi |y|}\, \mathrm{d}y.$$

On the other hand the LS propagator $\mathfrak{R}_a(\mathrm{i}\omega t)$ will be given by

$$g(x, t) = \frac{\mathrm{i}\omega t\, e^{\mathrm{i}\omega t} K_1(\sqrt{-\omega^2 t^2 + x^2/a^2})}{\pi a \sqrt{-\omega^2 t^2 + x^2/a^2}}, \tag{C.19}$$

$$\gamma(u, t) = e^{\mathrm{i}\omega t(1 - \sqrt{1 + a^2 u^2})} \tag{C.20}$$

with singularities in $x = \pm a\omega t$ and corresponds to the LS equation

$$\mathrm{i}\partial_t \psi(x, t) = -\omega \int_{y \neq 0} [\psi(x+y, t) - \psi(x, t)] \frac{K_1(|y|/a)}{\pi |y|}\, \mathrm{d}y.$$

We remember here, as remarked in section 2.5, that—after reabsorbing an irrelevant constant term mc^2 in a phase factor of the *wf*—this is essentially the integro-differential form of the relativistic, free Schrödinger equation (15) with $\omega = \lambda/\tau = mc^2/\hbar$, $a = \hbar/mc$.

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