Methodol Comput Appl Probab (2013) 15:147–163
DOI 10.1007/s11009-011-9228-9

Pricing and Hedging Asian Basket Options with Quasi-Monte Carlo Simulations

Nicola Cufaro Petroni · Piergiacomo Sabino

Received: 11 September 2009 / Revised: 27 February 2011 /
Accepted: 2 May 2011 / Published online: 26 May 2011

Abstract In this article we consider the problem of pricing and hedging high-dimensional Asian basket options by Quasi-Monte Carlo simulations. We assume a Black–Scholes market with time-dependent volatilities, and we compute the deltas by means of the Malliavin Calculus as an extension of the procedures employed by Kohatsu-Higa and Montero (Physica A 320:548–570, 2003). Efficient path-generation algorithms, such as Linear Transformation and Principal Component Analysis, exhibit a high computational cost in a market with time-dependent volatilities. To face this challenge we then introduce a new and faster Cholesky algorithm for block matrices that makes the Linear Transformation more convenient. We also propose a new-path generation technique based on a Kronecker Product Approximation. Our procedure shows the same accuracy as the Linear Transformation used for the computation of deltas and prices in the case of correlated asset returns, while requiring a shorter computational time. All these techniques can be easily employed for stochastic volatility models based on the mixture of multi-dimensional dynamics introduced by Brigo et al. (2004a, Risk 17(5):97–101, b).

Keywords Computational finance · Quasi-Monte Carlo algorithms · Malliavin Calculus · Pricing and hedging options · Asian basket options

AMS 2000 Subject Classification 91G60

N. Cufaro Petroni
Dipartimento di Matematica and TIRES, Università di Bari,
INFN Sezione di Bari, via E. Orabona 4, 70125 Bari, Italy
e-mail: cufaro@ba.infn.it

P. Sabino (✉)
Dipartimento di Matematica, Università di Bari, via E. Orabona 4, 70125 Bari, Italy
e-mail: sabino@dm.uniba.it

P. Sabino
E.ON Energy Trading SE, Holzstrasse 6, 40221 Düsseldorf, Germany
e-mail: piergiacomo.sabino@eon.com

1 Introduction and Motivation

In a few recent papers Dahl and Benth (2002) and Wang (2009) have investigated the efficiency and the computational cost of the Principal Component Analysis (PCA) used in the Quasi-Monte Carlo (QMC) simulations for the pricing of high-dimensional Asian basket options in a multi-dimensional Black–Scholes (BS) model with constant volatilities. In particular they have shown the essential role of the Kronecker product both for a fast implementation, and for identifying the effective dimension in the analysis of variance (ANOVA, see below). Since the convergence rate of the QMC method is $O\left(N^{-1}\log^d N\right)$—here N is the number of simulation trials, and d the nominal dimension of the problem—the theoretically higher asymptotic convergence rate of QMC could not be practically achieved in high dimensions. On the other hand, particular applications in finance (see Paskov and Traub 1995) have shown that the QMC provides an accuracy higher than the standard Monte Carlo (MC) even for high dimensions.

To explain the success of the QMC in high dimensions Caflisch et al. (1997) have introduced two notions of effective dimensions based on the ANOVA of the integrand function. Consider an integrand function f for a MC problem with nominal dimension d, and let $\mathscr{A} = \{1, \ldots, d\}$ denote the labels of the input variables of the function f: the effective dimension of f in the *superposition sense* is the smallest integer d_S such that $\sum_{|u|\le d_S} \sigma^2(f_u) \ge p\sigma^2(f)$, where f_u is a function with variables in the set $u \subseteq \mathscr{A}$, $\sigma^2(\cdot)$ denotes the variance of the given function, $|u|$ is the cardinality of the set and $0 \le p \le 1$ (for instance $p = 0.99$). On the other hand the effective dimension of f in the *truncation sense* is the smallest integer d_T such that $\sum_{u\subseteq\{1,2,\ldots,d_T\}} \sigma^2(f_u) = p\sigma^2(f)$. In other words the truncation dimension indicates the number of variables essential to capture the given function f, while the superposition dimension takes into account that, for some f's, the inputs might influence the outcome through their joint action within smaller groups.

Different techniques have been proposed for a dimension reduction: the PCA decomposition and the Brownian bridge (BB) however achieve this result independently from the particular payoff of a European option. Imai and Tan (2006) have instead proposed a general dimension reduction construction, the Linear Transformation (LT), that depends on the payoff function, and that minimizes the effective dimension in the truncation sense. Several studies have investigated the efficiency of the dimension reduction produced by these approaches. Wang (2009), for example, has shown that the accuracy of the QMC simulations depends on both the dimension reduction technique, and the quasi-random points. He also proved that the PCA decomposition is always outperforming the BB as a result of the different grouping strategies developed (see the cited article for more details). Moreover, Papageorgiou (2002) has demonstrated that the accuracy of the QMC method used for the pricing of certain specific derivative contracts is not substantially improved by a BB construction. Finally Imai and Tan (2006) have shown that the LT approach is more accurate than the standard PCA and BB, but has a higher computational cost. In a previous paper one of the authors (Sabino 2011) has described how to efficiently implement this technique and, even with a slower computer, has obtained computational times that are about 30 times shorter than those originally presented by Imai and Tan (2006).

In the present paper we address the problem of the time-dependent volatilities, and since we can no longer rely on the properties of the Kronecker product, our task

will be computationally harder. In order to reduce this computational complexity, we first introduce a fast Cholesky (CH) decomposition algorithm tailored for block matrices: this will already severely lower the computational cost. Then we present a new path-generation technique based on the Kronecker Product Approximation (KPA) of the correlation matrix of a multi-dimensional Brownian path: this returns a suboptimal ANOVA decomposition with a remarkable computational advance. In the case of time-dependent volatilities, the BB procedure also requires a slightly different algorithm (see Sabino 2009 for details), but on the basis of the previous observations we have decided not to include it in this study.

Our numerical simulations consist first in calculating the Randomized QMC (RQMC) estimation of both the prices and the deltas of high-dimensional Asian basket options in a BS market with time-dependent volatilities. In order to compute the deltas we extend to a dependent multi-assets model the procedure employed by Kohatsu-Higa and Montero (2003) in a single-asset setting. To do that we take advantage of the Malliavin Calculus, and we allow a certain flexibility to enhance the localization techniques introduced by Fournié et al. (1999). As far as the computation of the Asian options prices is concerned, the KPA and LT approaches are checked both in terms of accuracy and computational cost. We show that the LT procedure is more efficient than the PCA—even from a computational point of view—provided that we adopt our CH algorithm and the approach described in Sabino (2011). The KPA and the PCA constructions perform equally well in terms of accuracy, but with the former requiring a considerably shorter computational time. In the same vein, the KPA and the LT display comparable accuracies in the computation of the deltas. Finally we contrast our simulations—also using the standard CH and the PCA decomposition methods—with both pseudo-random and Latin Hypercube Sampling (LHS) generators.

We finally remark that all the methods described here can accommodate a market with stochastic volatility where the evolution of the risky securities is modeled by a mixture of multi-dimensional dynamics as in the papers by Brigo et al. (2004a, b). It is noteworthy to say instead that none of these procedures can be applied to the Heston-like multi-dimensional stochastic volatility models. In principle we might still use the LT for the Euler discretization of the Heston model, but this could be no longer applicable within more realistic frameworks involving discrete random variables as proposed for instance by Alfonsi (2005).

The paper is organized as follows: the Section 2 describes Asian options, while the Section 3 discusses some path-generation techniques and in particular, presents the fast CH algorithm and the KPA construction. Section 4 shows then the numerical simulations for the Asian option pricing, and Section 5 explains how to represent the deltas of Asian basket options as expected values with the aid of Malliavin Calculus, and shows their estimated values by RQMC. Section 6 finally summarizes the most important results and concludes the paper.

2 Asian Basket Options

Assume a multi-dimensional BS market with M risky securities and one risk-free asset. Denote $\mathbf{B}(t) = (B_1(t), \ldots, B_M(t))$ an M-dimensional Brownian motion (BM) with correlated components and $(\mathcal{F}_t)_{t\geq 0}$ the filtration generated by this BM.

Moreover, denote ρ_{ik} the constant instantaneous correlation between $B_i(t)$ and $B_k(t)$, $S_i(t)$ the i-th asset price at time t, $\sigma_i(t)$ the instantaneous time-dependent volatility of the i-th asset return and r the continuously compounded risk-free rate. In the risk-neutral probability, we assume that the dynamics of the risky assets are

$$dS_i(t) = rS_i(t)\,dt + \sigma_i(t)\,S_i(t)\,dB_i(t), \qquad i = 1, \ldots, M. \tag{1}$$

The solution of Eq. 1 is

$$S_i(t) = S_i(0)\exp\left[\int_0^t \left(r - \frac{\sigma_i^2(s)}{2}\right)ds + \int_0^t \sigma_i(s)\,dB_i(s)\right], \quad i = 1, ..., M. \tag{2}$$

Discretely monitored Asian basket options are derivative contracts that depend on the arithmetic mean of the prices assumed by a linear combination of the underlying securities at precise times $t_1 < t_2 \cdots < t_N = T$, where T is the maturity of the contract. By the risk-neutral pricing formula (see for instance Lamberton and Lapeyre 1996) the fair price of the contract at time t is

$$a(t) = e^{r(T-t)}\mathbb{E}\left[\left(\sum_{i=1}^{M}\sum_{j=1}^{N} w_{ij}\,S_i\left(t_j\right) - K\right)^{+}\Big|\mathscr{F}_t\right], \tag{3}$$

with the assumption that $\sum_{i,j} w_{ij} = 1$.

Pricing Asian options by simulation hence requires the discrete averaging of the solution (2) at a finite set of times $\{t_1, \ldots, t_N\}$. This sampling procedure yields

$$S_i(t_j) = S_i(0)\exp\left[\int_0^{t_j}\left(r - \frac{\sigma_i^2(t)}{2}\right)dt + Z_i(t_j)\right] \quad i = 1, \ldots, M,\ j = 1, \ldots, N, \tag{4}$$

where the components of the vector

$$(Z_1(t_1), \ldots, Z_1(t_N);\ Z_2(t_1), \ldots, Z_2(t_N);\ \ldots;\ Z_M(t_1), \ldots, Z_M(t_N))^T$$

are $M \times N$ normal random variables with zero mean and the following covariance matrix

$$\Sigma_{MN} = \begin{pmatrix} \Sigma(t_1) & \Sigma(t_1) & \ldots & \Sigma(t_1) \\ \Sigma(t_1) & \Sigma(t_2) & \ldots & \Sigma(t_2) \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma(t_1) & \Sigma(t_2) & \ldots & \Sigma(t_N) \end{pmatrix}, \tag{5}$$

where the elements of the $M \times M$ submatrices $\Sigma(t_n)$ are $(\Sigma(t_n))_{ik} = \int_0^{t_n} \rho_{ik}\sigma_i(s)\sigma_k(s)ds$ with $i, k = 1, \ldots, M$; $n = 1, \ldots, N$. This setting is also suitable for time-dependent correlations. In the case of constant volatilities the covariance matrix is

$$\Sigma_{MN} = \begin{pmatrix} t_1\Sigma & t_1\Sigma & \ldots & t_1\Sigma \\ t_1\Sigma & t_2\Sigma & \ldots & t_2\Sigma \\ \vdots & \vdots & \ddots & \vdots \\ t_1\Sigma & t_2\Sigma & \ldots & t_N\Sigma \end{pmatrix}, \tag{6}$$

where now Σ denotes the $M \times M$ covariance matrix of the logarithmic returns of the assets. It follows from the last equation that the covariance matrix Σ_{MN} can be

represented as $R \otimes \Sigma$, where $\otimes$ denotes the Kronecker product and R is the auto-covariance matrix of a single BM. This simplification is not possible in the case of time-dependent volatilities. We recall that the elements of R are

$$R_{ln} = t_l \wedge t_n, \quad l, n = 1, \ldots N, \tag{7}$$

and that R is invariant under reflections about the diagonal.

Definition 1 (Boomerang Matrix) The square matrix $B \in \mathbb{R}^{n_B \times n_B}$ is a boomerang matrix if it exists a vector $\mathbf{b} = (b_1, \ldots, b_{n_B}) \in \mathbb{R}^{n_B}$ such that

$$B_{hp} = b_{h \wedge p}, \quad h, p = 1, \ldots, n_B. \tag{8}$$

In this case $\mathbf{b}$ takes the name of elementary vector associated to B.

As a consequence R is boomerang, and in general the auto-covariance matrix of every Gaussian process is boomerang. This definition can also be extended to block matrices as follows.

Definition 2 (Block Boomerang Matrix) Partition the rows and the columns of a square matrix $B \in \mathbb{R}^{n_B \times n_B}$ to obtain:

$$B = \begin{pmatrix} B_{11} & \ldots & B_{1P} \\ \vdots & \ddots & \vdots \\ B_{P1} & \ldots & B_{PP} \end{pmatrix}, \tag{9}$$

where for $h, p = 1, \ldots, P$, $B_{hp} \in \mathbb{R}^{D \times D}$ designates the (h, p) square submatrix and $n_B = P \times D$; then B is a boomerang block matrix if we can find P matrices $B_1, \ldots, B_P$ with $B_h \in \mathbb{R}^{D \times D}$, $h = 1, \ldots, P$ such that

$$B_{hp} = B_{h \wedge p}, \quad h, p = 1, \ldots, n_B. \tag{10}$$

The vector $\mathbf{b} = (B_1, \ldots, B_P)^T$ takes the name of elementary block vector associated to B.

From these definitions we find that Σ_{MN} is block boomerang.

The payoff at maturity of the Asian basket option now is $a(T) = (g(\mathbf{Z}) - K)^+$ with

$$g(\mathbf{Z}) = \sum_{k=1}^{M \times N} \exp(\mu_k + Z_k) \tag{11}$$

where $\mathbf{Z} \sim \mathcal{N}(0, \Sigma_{MN})$ and

$$\mu_k = \ln(w_{k_1 k_2} S_{k_1}(0)) + r t_{k_2} - \int_0^{t_{k_2}} \frac{\sigma_{k_1}^2(t)}{2}\, dt \tag{12}$$

with $k_1 = (k-1) \bmod M$; $k_2 = \lfloor (k-1)/M \rfloor + 1$; $k = 1, \ldots, M$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

3 Path-Generation Techniques

From the previous discussion it comes out that the pricing of Asian basket options by simulation requires an averaging on the sample trajectories of an M-dimensional BM. In general, if $\mathbf{Y} \sim \mathcal{N}(0, \Sigma_Y)$ and $\mathbf{X} \sim \mathcal{N}(0, I)$ are two N-dimensional Gaussian random vectors, we will always be able to write $\mathbf{Y} = C\mathbf{X}$, where C is a matrix such that:

$$\Sigma_Y = CC^T. \tag{13}$$

and the core problem consists in finding the matrix C. In our case Σ_Y coincides with Σ_{MN} of Eq. 5. The accuracy of the standard MC method does not depend on the choice of the matrix C because the order of the random variables is not important. However, a choice of C that reduces the nominal dimension would improve the efficiency of the (R)QMC method, and in the following we discuss a few possible cases.

3.1 Cholesky Construction

The CH decomposition simply finds the matrix C among all the *lower triangular matrices*. In the case of constant volatilities the matrix Σ_{MN} is the Kronecker product of R and Σ, and the Kronecker product is compatibile with a CH decomposition (see Pitsianis and Van Loan 1993). In fact, denoting by $C_{\Sigma_{MN}}$, C_R and C_Σ the CH matrices associated to Σ_{MN}, R and Σ respectively, we have

$$C_{\Sigma_{MN}} = C_R \otimes C_\Sigma. \tag{14}$$

This now entails a remarkable reduction of the computational cost: it turns out indeed that a $O\left((M \times N)^3\right)$ computation is reduced to a $O\left(M^3\right) + O\left(N^3\right)$ one.

When time-dependent volatilities are considered, however, we can no longer use these properties of the Kronecker product: in their stead, since Σ_{MN} is a block boomerang matrix, we can take advantage of the following result:

Proposition 1 *Let $B \in \mathbb{R}^{n_B \times n_B}$ be a block boomerang matrix and let $(B_1, \ldots, B_P)^T$, where $B_h \in \mathbb{R}^{D \times D}$, $h = 1, \ldots, P$ with $n_B = P \times D$, be its associated elementary block vector. Then the CH matrix C_B associated to B is*

$$C_B = \begin{pmatrix} C_1 & 0 & \ldots & 0 \\ \vdots & C_2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_1 & C_2 & \ldots & C_P \end{pmatrix} \tag{15}$$

where the $D \times D$ blocks C_h, $h = 1, \ldots, P$ are

$$C_h = \mathrm{Chol}\,(B_h - B_{h-1}) \tag{16}$$

with Chol *denoting the CH factorization. We also assume $B_0 = 0$.*

Proof Consider the h^{th} row of C_B and the m^{th} row of its transposed matrix; we then have

$$(C_1, \ldots, C_h, 0, \ldots, 0)^T \cdot \left(C_1^T, \ldots, C_m^T, 0, \ldots, 0\right)^T = \sum_{l=1}^{h\wedge m} C_l C_l^T$$
$$= \sum_{l=1}^{h\wedge m} (B_l - B_{l-1}) = B_{h\wedge m}$$

and this concludes the proof. □

3.2 Principal Component Analysis

Acworth et al. (1998) have proposed a path generation technique based on the PCA. Following this approach we consider the spectral decomposition of Σ_{MN}

$$\Sigma_{MN} = E\Lambda E^T = \left(E\Lambda^{1/2}\right)\left(E\Lambda^{1/2}\right)^T, \tag{17}$$

where Λ is the diagonal matrix of all the positive eigenvalues of Σ_{MN} sorted in decreasing order and E is the orthogonal matrix ($EE^T = I$) of all the associated eigenvectors. The matrix C solving Eq. 13 then is $E\Lambda^{1/2}$. The amount of variance explained by the first k principal components is the ratio: $\frac{\sum_{i=1}^{k}\lambda_i}{\sum_{i=1}^{d}\lambda_i}$ where d is the rank of Σ_{MN}. The PCA construction permits the statistical ranking of the normal factors, while this is not possible by the CH decomposition. For a market with constant volatilities, the Kronecker product reduces this calculation to the computation of eigenvalues and eigenvectors of the two smaller matrices R and Σ. All these simplifications, on the other hand, are no longer valid for the time-dependent volatilities. Nevertheless we can still reduce the computational cost of a PCA decomposition in a different way.

Let M_1, M_2, M_3 and M_4 be respectively $p \times p$, $p \times q$, $q \times p$ and $q \times q$ matrices, and suppose that M_1 and M_4 are invertible. Assume then

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix}$$

and define $S_1 = M_4 - M_3 M_1^{-1} M_2$ and $S_4 = M_1 - M_2 M_4^{-1} M_3$, namely the Schur complements of M_1 and M_4 respectively. Then by Schur's lemma the inverse M^{-1} is:

$$M^{-1} = \begin{pmatrix} S_4 & -M_1^{-1} M_2 S_1^{-1} \\ -M_4^{-1} S_4^{-1} & S_1^{-1} \end{pmatrix}. \tag{18}$$

Taking into account the previous result it is possible to prove the following proposition

Proposition 2 *Let $B \in \mathbb{R}^{n_B \times n_B}$ be a block boomerang matrix, and $(B_1, \ldots, B_P)^T$—where $B_h \in \mathbb{R}^{D\times D}, h = 1, \ldots, P$ with $n_B = P \times D$—its associated elementary block vector: then the inverse of B is symmetric block tri-diagonal. The blocks on the lower (and upper) diagonal are $T_l = -\left(B_{l+1} - B_l\right)^{-1}$, $l = 1, \ldots, P-1$ while those on the*

diagonal are $D_m = (B_m - B_{m-1})^{-1}(B_{m+1} - B_{m-1})(B_{m+1} - B_m)^{-1}$, $m = 1, \ldots, P$, *with the assumption that* $B_0 = B_{N+1} = 0$*:*

$$B^{-1} = \begin{pmatrix} D_1 & T_1 & 0 & \ldots & 0 \\ T_1 & D_2 & T_2 & \ddots & \vdots \\ 0 & T_2 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & T_{P-1} \\ 0 & 0 & 0 & T_{P-1} & D_P \end{pmatrix} \tag{19}$$

This property can be used to reduce the computational cost of evaluating the PCA decomposition in the case of time-dependent volatilities and in general for multi-dimensional Gaussian processes. Indeed, if B is a non-singular square matrix then the eigenvalues of B^{-1} are the reciprocal of the eigenvalues of B and the eigenvectors coincide.

3.3 Linear Transformation

Imai and Tan (2006) have considered the following class of LT as a solution of Eq. 13:

$$C^{\mathrm{LT}} = C^{\mathrm{Ch}} A \tag{20}$$

where C^{Ch} is the CH matrix associated to the covariance matrix of the normal random vector to be generated, and A is an orthogonal matrix, i.e. $AA^T = I$. The matrix A is introduced with the main purpose of minimizing the effective dimension of a simulation problem in the truncation sense. Imai and Tan (2006) have proposed to approximate an arbitrary function g, such that $(g - K)^+$ is the payoff function of a European derivative contract, with its first order Taylor expansion around $\hat{\varepsilon}$

$$g(\boldsymbol{\varepsilon}) = g(\hat{\boldsymbol{\varepsilon}}) + \sum_{l=1}^{n} \frac{\partial g}{\partial \varepsilon_l}\Big|_{\boldsymbol{\varepsilon}=\hat{\boldsymbol{\varepsilon}}} \Delta\varepsilon_l. \tag{21}$$

The approximated function is linear in the standard normal random vector $\Delta\varepsilon$. Considering an arbitrary point— such as $\hat{\boldsymbol{\varepsilon}} = \mathbf{0}$—as expansion starting point, we can derive the first column of the optimal orthogonal matrix A^*. It is possible to find the complete matrix by expanding g about different points and then compute the optimization algorithm. Imai and Tan (2006) have set: $\hat{\varepsilon}_1 = \mathbf{0} = (0, 0, \ldots, 0)$, $\hat{\varepsilon}_2 = (1, 0, \ldots, 0), \ldots, \hat{\varepsilon}_n = (1, \ldots, 1, 0)$, where the k-th point has $k - 1$ non-zero components. The optimization can then be formulated as follows:

$$\max_{\mathbf{A_{\cdot k}} \in \mathbf{R^n}} \left(\frac{\partial g}{\partial \varepsilon_k}\Big|_{\boldsymbol{\varepsilon}=\hat{\boldsymbol{\varepsilon}}_{\mathbf{k}}} \right)^2, \quad k = 1, \ldots, n, \tag{22}$$

subject to $\|\mathbf{A_{\cdot k}}\| = 1$ and $\mathbf{A^*_{\cdot j}} \cdot \mathbf{A_{\cdot k}} = 0$; $j = 1, \ldots, k-1$; $k \leq n$. In the case of Asian basket options we have

$$g(\boldsymbol{\varepsilon}) = g(\hat{\boldsymbol{\varepsilon}}) + \sum_{l=1}^{NM} \left[\sum_{i=1}^{NM} \exp\left(\mu_i + \sum_{k=1}^{NM} C_{ik} \hat{\varepsilon}_k \right) C_{il} \right] \Delta\varepsilon_l. \tag{23}$$

Imai and Tan (2006) have then proved the following result:

Proposition 3 *Consider an Asian basket options in a BS model, and define*

$$\mathbf{d}^{(\mathbf{p})} = \left(e^{\left(\mu_1 + \sum_{k=1}^{p-1} C_{1k}^*\right)}, \ldots, e^{\left(\mu_{MN} + \sum_{k=1}^{p-1} C_{MN,k}^*\right)} \right)^T \tag{24}$$

$$\mathbf{B}^{(\mathbf{p})} = \left(C^{\mathrm{Ch}}\right)^T (\mathbf{d}^{(\mathbf{p})}), \quad p = 1, \ldots, MN. \tag{25}$$

Then the p-th column of the optimal matrix A^* is

$$\mathbf{A}_{\cdot\mathbf{p}}^* = \pm \frac{\mathbf{B}^{(\mathbf{p})}}{\|\mathbf{B}^{(\mathbf{p})}\|} \quad p = 1, \ldots, MN. \tag{26}$$

The matrices $C_{ik}^*, k < p$ have already been found in the previous $p-1$ steps, while $\mathbf{A}_{\cdot\mathbf{p}}$ must be orthogonal to all the other columns. This condition can be easily met by an incremental QR decomposition, as described in Sabino (2011).

3.4 Kronecker Product Approximation

In a time-dependent volatility market the covariance matrix Σ_{MN} has time-dependent blocks. The multi-dimensional BM is the unique source of risk in the BS market and the generation of the trajectories of the 1-dimensional BM does depend on the volatilities. As a consequence we propose to find a constant covariance matrix of the assets H in order to approximate, in an appropriate sense, the matrix Σ_{MN} as a Kronecker product of R and H. In the following we illustrate the proposed procedure called Kronecker Product Approximation (KPA). Pitsianis and Van Loan (1993) have proved the following proposition:

Proposition 4 *Suppose $G \in \mathbb{R}^{m\times n}$ and $G_1 \in \mathbb{R}^{m_1\times n_1}$ with $m = m_1 m_2$ and $n = n_1 n_2$. Consider the problem of finding $G_2^* \in \mathbb{R}^{m_1\times n_1}$ that realizes the minimum*

$$\min_{G_2 \in \mathbb{R}^{m_1\times n_1}} \| G - G_1 \otimes G_2 \|_F^2, \tag{27}$$

where $\|\cdot\|_F^2$ denotes the Frobenius norm. For fixed $h = 1, \ldots, m_2$ and $l = 1, \ldots, n_2$ denote $\mathscr{R}(G)_{hl}$ the $m_1 \times n_1$ matrix defined by the rows $h, h+m_2, h+2m_2, \ldots, h+(m_1-1)m_2$ and the columns $l, l+n_2, l+2n_2, \ldots, l+(n_1-1)n_2$ of the original matrix G. The elements of G_2^* then are

$$(G_2^*)_{hl} = \frac{\mathrm{Tr}\left(\mathscr{R}(G)_{hl}^T G_1\right)}{\mathrm{Tr}\left(G_1 G_1^T\right)} \quad h = 1, \ldots, m_2,\ l = 1, \ldots, n_2, \tag{28}$$

where Tr *denotes the trace of a matrix.*

In our setting we have $G = \Sigma_{MN}$, $G_1 = R$ and $G_2 = H$, and we remark that for any $i, j = 1, \ldots, N$, $\mathscr{R}(\Sigma_{MN}))_{ij}$ is a $N \times N$ boomerang matrix. Moreover, given two general $N \times N$ boomerang matrices A and B, we can prove by direct computation that

$$\mathrm{Tr}(A^T B) = \mathrm{Tr}(AB) = \sum_{j=1}^{N} (2(N-j)+1)\, a_{jj} b_{jj}. \tag{29}$$

We then perform the PCA decomposition of $R \otimes H$ by relying on the properties of the Kronecker product, but, if we use the PCA decomposition of the matrix $F = R \otimes H$, we do not get the required path. In order to produce the required trajectory we then take

$$\mathbf{Z} = C^{\mathrm{KPA}}\boldsymbol{\varepsilon} = C_{\Sigma_{MN}}(C_F)^{-1}E_H\Lambda_H^{1/2}\boldsymbol{\varepsilon} \tag{30}$$

where $C_{\Sigma_{MN}}$ and C_F are the CH matrices associated to Σ_{MN} and F, respectively, and $E_H\Lambda_H^{1/2}$ is the PCA decomposition of F: the matrix C^{KPA} turns out to be the correct covariance matrix because, denoting $P = E_H\Lambda_H^{1/2}$, we have

$$C^{\mathrm{KPA}}\left(C^{\mathrm{KPA}}\right)^T = C_{\Sigma_{MN}}(C_F)^{-1}PP^T\left[(C_F)^{-1}\right]^T C_{\Sigma_{MN}}^T = C_{\Sigma_{MN}}C_{\Sigma_{MN}}^T = \Sigma_{MN}$$

since $PP^T = C_F C_F^T = F$. Our fundamental assumption is here that the principal components of $\mathbf{Z}$ are not too different from those of the normal random vector $\mathbf{Z}'$ whose covariance matrix is F. We expect that the KPA decomposition would produce an effective dimension higher than that obtained by the PCA decomposition, but with a substantial boon from the computational standpoint. Due to properties of the Kronecker product, indeed, the Eq. 30 becomes

$$\mathbf{Z} = C_{\Sigma_{MN}}\left(C_R^{-1} \otimes C_H^{-1}\right)E_H\Lambda_H^{1/2}\boldsymbol{\varepsilon}, \tag{31}$$

where C_R and C_H are the CH matrices of R and H, respectively, and this matrix multiplication can be carried out quickly by block-matrices multiplication and taking advantage of the fact that (due to the Propositions 1 and 2) C_R^{-1} is a sparse bi-diagonal matrix.

4 Computing the Option Price

We will now estimate the fair price of an Asian option on a basket of $M = 10$ underlying assets with $N = 250$ sampled points in the BS model, with time-dependent volatilities having the following expression

$$\sigma_i(t) = \hat{\sigma}_i(0)\,\exp\left(-t/\tau_i\right) + \sigma_i(+\infty), \quad i = 1, \ldots M. \tag{32}$$

The parameters chosen for the simulation are listed in Table 1, and of course we have $\hat{\sigma}_i(0) = \sigma_i(0) - \sigma_i(+\infty)$. We implement this numerical investigation in two steps: first we test the effectiveness of our path-generation procedures on the dimension reduction and compute their computational times; then we compare the accuracy of the simulations.

Table 1 Inputs parameters

$S_i(0) = 100, \quad \forall i = 1 \ldots, N$
$K = 100$
$r = 4\%$
$T = 1$
$\sigma_i(0) = 10\% + \frac{i-1}{9}40\% \quad i = 1 \ldots, N$
$\sigma_i(+\infty) = 9\% \quad \forall i = 1 \ldots, N$
$\tau_i = 1.5 \quad \forall i = 1 \ldots, N$
$\rho_{ij} \subset \{0, 40\} \quad i, j = 1 \ldots, N$

Table 2 Effective dimensions

Ch	PCA	LT	KPA
Time-dependent volatilities			
$\rho = 0\%$			
$d_T > 1900$	$d_T = 14$	$d_T = 10$	$d_T = 19$
$\rho = 40\%$			
$d_T > 1900$	$d_T = 9$	$d_T = 8$	$d_T = 11$

Table 2 shows the effective dimensions obtained by the different the path-generation methods considered ($p = 0.99$). The LT construction provides the lowest effective dimension, while the PCA decomposition performs almost as well as the LT approach only for the correlation case, and the KPA returns a slightly higher effective dimension. The CH decomposition collects 98.58% and 98.70% of the total variance for $d_T \approx 2000$, respectively in the uncorrelated and in the correlated cases. To have a more detailed comparison look at the Table 3 which displays the elapsed times measured in Sabino (2011) by using an *ad hoc* incremental QR algorithm for the LT, and assuming constant volatilities equal to the $\sigma_i(0)$ of Table 1. The computation was implemented in MATLAB running on a laptop with an Intel Pentium M, processor 1.60 GHz and 1 GB of RAM. We computed 50 optimal columns for the LT technique. The CH algorithm for block boomerang matrices has almost the same cost as the one relying on the properties of the Kronecker product. As a consequence the LT also requires almost the same computational cost, while the PCA needs a time almost 20 times longer because now we can not rely on the properties of the Kronecker product. In contrast, the KPA has almost the same computational time as the PCA in the constant volatility case, and is the best performing path-generation procedure from a computational time point of view. We have applied Proposition 2 to implement the PCA, and we have computed the eigenvalues and eigenvectors of Σ_{MN} relying only on the *sparse* function of MATLAB. The development of algorithms tailored for the computation of the eigenvalues and eigenvectors of tri-diagonal symmetric block matrices is still in progress: hopefully they could further reduce the computational time and their performance will be presented in future papers.

In the second part of our investigation we launched a simulation to estimate the Asian option price using 10 replications, each consisting in 8192 random points, following the strategy in Imai and Tan (2006). We used again different random generators: standard MC, LHS and RQMC generators. Concerning the computational times of the price estimation, the CPU ratio between LHS and RQMC is almost 1

Table 3 Computational times in seconds

Ch	PCA	LT	KPA
Constant volatilities			
$\rho = 0\%$			
0.60	25.77	71.14	
$\rho = 40\%$			
0.59	25.55	71.02	
Time-dependent volatilities			
$\rho = 0\%$			
0.62	565.77	71.65	28.25
$\rho = 40\%$			
0.62	568.55	71.20	28.33

while standard MC is 1.33 faster. As suggested in Glasserman (2004), we considered both the product of the root mean square error (RMSE), and the square root of the total computational time of the simulation as a measure of total accuracy of the simulation. We denote this latter quantity as *Err*.

We used a Matoušek affine plus random digital shift scrambled version (see Matoušek 1998) of the Sobol sequence satisfying Sobol's property A (see Sobol 1976), and we also shunned generating a 2,500-dimensional Sobol' sequence by using the Latin Supercube Sampling (LSS) method (Owen 1998). Briefly speaking this sampling procedure is a scheme for producing high-dimensional sequences out from sets of lower-dimensional sequences. For instance, a 2,500-dimensional low discrepancy sequence can be concatenated from 100 sets of 25-dimensional low discrepancy sequences by suitably randomizing the run order of the points. For a theoretical justification of the LSS method, see Owen (1998).

LHS can also be seen as an intermediate solution between pseudo- and quasi-random points in terms of accuracy enhancement by stratification. It can be proved indeed that LHS gives good variance reductions when the target function is the sum of one-dimensional functions (see Stein 1987). On the other hand, the LT method is designed to capture the lower effective dimension in the truncation sense for linear combinations. As a consequence we should have a high accuracy when just running the simulation using a combination of LHS and LT. We expect that the KPA technique produces a suboptimal decomposition in the sense of ANOVA, with the advantage of a lower computational effort. Our setting is organized to check how large the improvement given by every factorization is. Tables 4 and 5 present the results of this investigation. The prices in Table 4 are all in statistical agreement. Those obtained with the CH decomposition are almost independent from the random number generator: KPA, PCA and LT all provide good improvements in both the LHS and RQMC implementations for all the strike prices. The LT, however, has an apparent advantage with respect to PCA and KPA, and this is still more conspicuous in the uncorrelated case. In contrast, we observe that the KPA- and PCA-based simulations give almost the same accuracy, assuming both correlated and uncorrelated asset returns. Considering the total computational cost and the accuracy we remark that the KPA performs better than the standard PCA. Moreover, all these

Table 4 Estimated at-the money prices and errors

		$\rho = 0\%$		$\rho = 40\%$	
		Price	Err	Price	Err
MC	Ch	3.180	0.700	5.190	1.400
	KPA	3.120	0.700	5.190	1.410
	PCA	3.110	0.750	5.200	1.490
	LT	3.110	0.710	5.210	1.420
LHS	Ch	3.120	0.420	5.200	0.680
	KPA	3.120	0.310	5.201	0.190
	PCA	3.120	0.330	5.201	0.200
	LT	3.120	0.170	5.201	0.069
RQMC	Ch	3.112	0.410	5.195	0.500
	KPA	3.122	0.047	5.201	0.033
	PCA	3.121	0.053	5.201	0.040
	LT	3.122	0.018	5.201	0.019

Table 5 At-the-money estimated Δ's (10^{-2}) and errors (10^{-4}) with RQMC

LT		KPA		PCA		CH	
Δ	Err	Δ	Err	Δ	Err	Δ	Err
$\rho = 0\%$							
6.18	0.43	6.18	0.60	6.18	0.51	6.21	0.82
6.20	0.40	6.21	0.49	6.20	0.51	6.23	0.60
6.23	0.46	6.23	0.47	6.23	0.58	6.25	0.65
6.27	0.40	6.27	0.44	6.27	0.54	6.28	0.76
6.31	0.32	6.31	0.50	6.31	0.57	6.33	0.65
6.36	0.30	6.36	0.52	6.36	0.49	6.38	0.60
6.41	0.27	6.41	0.50	6.41	0.46	6.43	0.65
6.47	0.27	6.47	0.49	6.47	0.49	6.49	0.76
6.53	0.27	6.54	0.50	6.53	0.55	6.55	0.71
6.60	0.35	6.61	0.42	6.60	0.57	6.61	0.60
$\rho = 40\%$							
5.478	0.030	5.484	0.060	5.481	0.077	5.468	0.599
5.535	0.033	5.541	0.060	5.537	0.082	5.525	0.654
5.594	0.029	5.600	0.060	5.597	0.077	5.587	0.654
5.654	0.033	5.661	0.065	5.657	0.077	5.640	0.545
5.717	0.040	5.723	0.070	5.718	0.082	5.710	0.599
5.781	0.044	5.789	0.065	5.784	0.071	5.770	0.708
5.848	0.041	5.853	0.052	5.851	0.071	5.832	0.654
5.916	0.044	5.921	0.060	5.918	0.077	5.900	0.599
5.985	0.028	5.991	0.050	5.987	0.071	5.971	0.545
6.055	0.032	6.061	0.060	6.057	0.077	6.046	0.654

constructions can be employed in stochastic and local volatility models based on the mixture of multi-dimensional dynamics for basket options, as done in Brigo et al. (2004a).

5 Computing the Sensitivities

In the financial jargon a *Greek* is the derivative of an option price with respect to a parameter, and hence represents a measure of the price sensitivity with respect to that parameter. The deltas (Δ's) in particular are the components of the gradient of the discounted expected outcome of the option with respect to the initial values of the assets. The problem of computing the Greeks in finance has been studied by several authors: here we extend the methodology employed by Kohatsu-Higa and Montero (2003), based on the use of Malliavin Calculus, to the multi-assets case. The main difficulties of this extension lie in the fact that the assets are now correlated, so that the formulas in Kohatsu-Higa and Montero (2003) can not be directly extended to the multi-dimensional case. The localization technique introduced by Fournié et al. (1999) should moreover generally control all the components of the multi-dimensional BM to improve the accuracy of the estimation. We write the dynamics (Eq. 1) with respect to an M-dimensional BM $\mathbf{W}(t)$ with uncorrelated components

$$dS_i(t) = rS_i(t)dt + S_i(t)\sigma_i(t)\sum_{m=1}^{M}\alpha_{im}(t)dW_m(t) \quad i = 1, \ldots, M, \tag{33}$$

where $\sum_{m=1}^{M}\alpha_{im}\alpha_{km} = \rho_{ik}$. We also take $\sigma_{im}(t) = \sigma_i(t)\sum_{m=1}^{M}\alpha_{im}$.

The Malliavin calculus is a theory of variational stochastic calculus which provides the tools to compute derivatives and integrals by parts of random variables (see Nualart 2006 for more details on Malliavin Calculus). Let us denote by $D_s^1, \ldots, D_s^M$ the Malliavin derivatives with respect to the components of $\mathbf{W}(t)$, while $\delta^{\mathrm{Sk}} = \sum_{m=1}^{M} \delta_m^{\mathrm{Sk}}$ stands for the Skorohod integral with δ_m^{Sk} representing the Skorohod integral on a single $W_m(t)$. The domains of both the Malliavin derivatives, and the Skorohod integral will be denoted denoted by $\mathbb{D}^{1,2}$ and $\mathrm{dom}(\delta^{\mathrm{Sk}})$ respectively, while δ^{Kr} is the Kronecker delta. We prove then the following proposition:

Proposition 5 *With* $\mathbf{x} = \mathbf{S}(0)$ *and* $m(T) = \sum_{i=1}^{M} \sum_{j=1}^{N} w_{ij} S_i(t_j)$ *take*

$$G_k = \frac{\partial m(T)}{\partial x_k} = \frac{\sum_{j=1}^{N} w_{kj} S_k(t_j)}{x_k}, \quad k = 1, \ldots, M \tag{34}$$

Then, since $a(T) \in \mathbb{D}^{1,2}$, *the k-th delta (the k-th component of the gradient) is*

$$\Delta_k = \frac{\partial a(0)}{\partial x_k} = e^{-rT} \mathbb{E}\left[a'(T) G_k\right] = e^{-rT} \mathbb{E}\left[a(T) \sum_{m=1}^{M} \delta_m^{\mathrm{Sk}}(G_k u_m)\right] \tag{35}$$

where $\mathbf{u} = (u_1, \ldots, u_M) \in \mathrm{dom}(\delta^{\mathrm{Sk}})$, $\mathbf{z} = (z_1, \ldots, z_m) \in \mathrm{dom}(\delta^{\mathrm{Sk}})$, $G_k \mathbf{u} \in \mathrm{dom}(\delta^{\mathrm{Sk}})$ *and*

$$\frac{z_m(s)}{\sum_{h=1}^{M} \int_0^T z_h(s) D_s^h m(T) ds} = u_m(s)$$

$$\sum_{h=1}^{M} \int_0^T z_h(s) D_s^h m(T) ds \neq 0, \quad \text{a.s.}$$

Proof Compute

$$D_s^h a(T) = a'(T) D_s^h m(T) \quad h = 1, \ldots, M \tag{36}$$

multiply the above equation by G_k and by $z_h(t)$—so that $\mathbf{z} \in \mathrm{dom}(\delta^{\mathrm{Sk}})$—and finally sum for all $h = 1, \ldots, M$ and integrate:

$$\sum_{h=1}^{M} \int_0^T G_k z_h(s) D_s^h a(T) ds = \sum_{h=1}^{M} \int_0^T G_k z_h(s) a'(T) D_s^h m(T) ds. \tag{37}$$

Due to the definition of $\mathbf{u}$, and to the fact that $a'(T) G_k$ does not depend on s, we can write

$$a'(T) G_k = \sum_{m=1}^{M} \int_0^T u_m(s) G_k D_s^m a(T)) ds. \quad k = 1, \ldots, M \tag{38}$$

Compute now the expected value of both sides in Eq. 38

$$\mathbb{E}\left[a'(T) G_k\right] = \mathbb{E}\left[\sum_{m=1}^{M} \int_0^T u_m(s) G_k D_s^m a(T) ds\right] \tag{39}$$

and remark that by duality

$$\Delta_k = \mathbb{E}\left[a(T))\delta^{\mathrm{Sk}}(G_k\mathbf{u})\right] \quad k = 1, \ldots, M. \tag{40}$$

This concludes the proof. □

Proposition 5 allows a certain flexibility in choosing either the process $\mathbf{u}$, or better $\mathbf{z}$. We take then $z_h = \alpha_k \delta^{\mathrm{Kr}}_{hk}$; $h, k = 1, \ldots, M$, $\alpha_k = 1, \forall k$, and this implies that to compute the k-th delta we can consider only the k-th term of the Skorohod integral, thus reducing the computational cost. In particular, this choice is motivated by the fact that in this way the localization technique needs to control only $\delta^{\mathrm{Sk}}_k(\cdot)$, and hence only the k-th component of $\mathbf{W}(t)$. We now define L_k and calculate for $k = 1, \ldots, M$

$$L_k = \int_0^T D^k_s m(T) ds = \sum_{i=1}^{M}\sum_{j=1}^{N} w_{ij} S_i(t_j) \int_0^{t_j} \sigma_{ik}(s) ds, \tag{41}$$

$$\int_0^T D^k_s G_k ds = \sum_{j=1}^{N} \frac{w_{jk} S_k(t_j)}{x_k} \int_0^{t_j} \sigma_{kk}(s) ds = \sum_{j=1}^{N} \frac{w_{jk} S_k(t_j)}{x_k} \int_0^{t_j} \sigma_k(s) ds, \tag{42}$$

$$\int_0^T D^k_s L_k ds = \sum_{j=1}^{N} w_{ij} S_i(t_j) \left(\int_0^{t_j} \sigma_{ik} ds\right)^2, \tag{43}$$

so that

$$\Delta_k = \mathbb{E}\left[a(T)\delta^{\mathrm{Sk}}_k\left(\frac{G_k}{L_K}\right)\right], \quad k = 1, \ldots, M. \tag{44}$$

Due to the properties of the Skorohod integral we then have for $k = 1, \ldots, M$

$$\delta_k\left(\frac{G_k}{L_K}\right) = \frac{G_k}{L_K} W_k(T) - \frac{1}{L_k^2}\left(L_k \int_0^T D^k_s G_k ds - G_k \int_0^T D^k_s L_k ds\right), \tag{45}$$

Remark that with a different choice of $\mathbf{z}$ (for instance $z_h = \alpha_h$) Δ_k would linearly depend on the whole M-dimensional BM, eventually making the localization technique less efficient.

We finally investigate the applicability of the RQMC approach to estimate the expected value in Eq. 44 for $k = 1, \ldots, M$. Take the same input parameters as in Section 4, and generate the trajectories (the values $S_i(t_j)$, $i = 1, \ldots, M$, $j = 1, \ldots, N$) in Eqs. 41–43 as done in that section. Consider then the α_{im} as the elements of the CH matrix associated to ρ_{im}, $i, m = 1, \ldots, M$: the Table 5 compares the deltas obtained only with RQMC, with the same number of scenarios as in Section 4. We adopted the same LT procedure used to estimate the option price, and not that for the integrand function in Eq. 44: at first sight this does not seem to be an optimal choice; but, would we have applied the LT for the integrand function in Eq. 44, $M = 10$ decomposition matrices (one for each delta) should have been considered. This would have increased the CPU time by at least 1/3 of the total time (or even more, due to the larger number of terms to compute) thus making the calculation less convenient. Table 5 shows that the PCA, LT and KPA approaches perform almost equally well in terms of total accuracy, with the LT giving better results only for correlated assets. In terms of total accuracy (Err), the KPA performs better than

the PCA, the CH construction displaying Err's that are even 10 times higher. As previously stated, g can be considered a good approximation for the payoff function in Eq. 44, but in the Malliavin expression $a(T)$ is multiplied by a random weight that depends on the Gaussian vector $\mathbf{Z}$. The PCA and the KPA instead concentrate most of the variation in the first dimensions of $\mathbf{Z}$: this can be considered as an explanation for the almost equal accuracy of the LT, PCA and KPA procedurs.

6 Conclusions

We have considered the problem of computing both the fair price, and the deltas of high-dimensional Asian basket options in a BS market with time-dependent volatilities by QMC simulations. In order to extend the QMC superior performances to higher dimensions we need to employ path-generation techniques chiefly tailored to reduce the nominal dimension. The LT and the PCA constructions try to accomplish this task by exploiting the concept of ANOVA. For time-dependent volatilities in a BS economy, however, the computational cost of the LT and the PCA cannot be reduced by making use of the Kronecker product properties, so that the computation is more difficult. To face this challenge, we have first produced a new and faster CH algorithm for block matrices that remarkably cuts down the computational burden and hence makes the LT procedure even more convenient than the PCA. Then, we have presented a new path-generation technique, the KPA, that in the usual applications is as accurate as the PCA, but is even more convenient with respect to the computational costs. In addition, we proved that the KPA improves the RQMC performances for both the estimation of the fair price, and the calculation of the deltas of Asian basket options in a BS model with time-dependent volatilities. In this setting the KPA provides the same accuracy of the LT in the case of correlated asset returns, and in the estimation of the deltas. We have also extended to the multi-assets case the procedures—based on the Malliavin Calculus—adopted by Kohatsu-Higa and Montero (2003) for the computation of the sensitivities. Finally, all these results can be easily applied to local volatility models that are based on the mixture of multi-dimensional dynamics for basket options, as done in Brigo et al. (2004a).

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