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Cointegrating Jumps: an Application to Energy Facilities*

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Abstract

Based on the concept of self-decomposable random variables we discuss the application of a model for a pair of dependent Poisson processes to energy facilities. Due to the resulting structure of the jump events we can see the self-decomposability as a form of cointegration among jumps. In the context of energy facilities, the application of our approach to model power or gas dynamics and to evaluate transportation assets seen as spread options is straightforward. We study the applicability of our methodology first assuming a Merton market model with two underlying assets; in a second step we consider price dynamics driven by an exponential mean-reverting Geometric Ornstein-Uhlenbeck plus compound Poisson that are commonly used in the energy field. In this specific case we propose a price spot dynamics for each underlying that has the advantage of being treatable to find non-arbitrage conditions. In particular we can find close-form formulas for vanilla options so that the price and the Greeks of spread options can be calculated in close form using the Margrabe formula [5] (if the strike is zero) or some well known approximations as in Deng et al. [8].

*The technique here discussed does not reflect EGC view.

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1 Introduction and Motivation

Several research studies have shown that the spot dynamics of commodity markets is subjected to mean reversion, seasonality and jumps (see for example Cartea and Figueroa [1]). In addition, some methodologies have also been proposed to take dependency into account based on correlation and co-integration. However, these approaches can become mathematically complex or non-treatable when leaving the Gaussian-Itō world.

In this paper we address the problem of dependency in the 2-dimensional case and start considering 2-dimensional jump diffusion processes with a 2-dimensional compound Poisson component. We then introduce an intuitive approach to model the dependency of 2-dimensional Poisson processes based on the self-decomposability (see Cufaro Petroni [6], Cufaro Petroni and Sabino [7], Sato [10]) of the exponential random variables used for its construction. We will see indeed in the subsequent sections that given two independent exponential *rv*'s $Y, Z \sim \mathfrak{E}(\lambda)$, and a 0-1 Bernoulli *rv* $B(1) \sim \mathfrak{B}(1, 1-a)$ with $a = \mathbf{P}\{B(1) = 0\}$, then also the *rv* defined as

$$X = aY + Z_a \qquad\qquad Z_a = B(1)Z \tag{1}$$

is an exponential $\mathfrak{E}(\lambda)$ resulting in a weighed sum of Y and Z where $0 < a < 1$ is the *deterministic* weight of Y, while $B(1)$ is the *random* weight of Z. In other words X is nothing else than the exponential Y down a-rescaled, plus another independent, but *intermittent* with frequency $1 - a$, exponential Z. It is apparent on the other hand that, by construction, X and Y are not independent and it is possible to show that a also represents precisely their correlation coefficient. This result is a direct consequence of the self-decomposability of the exponential laws.

As a matter of fact, moreover, we could also produce pairs of a-correlated exponentials $X \sim \mathfrak{E}(\lambda)$ and $Y' \sim \mathfrak{E}(\mu)$ with different parameters by reformulating the previous relation as

$$X = \gamma Y' + Z_a \tag{2}$$

where $\gamma = \mu a/\lambda$, so that $0 < \gamma < \frac{\mu}{\lambda}$. Considering then X and Y' as two random times with a positive random delay Z_a, the mathematical concept of self-decomposability can help describing their co-movement and can answer some common questions arising in the financial context:

- Once a financial institution defaults how long should one wait for a dependent institution to default too?
- A market receives a news interpreted as a shock: how long should one wait to see the propagation of that shock onto a dependent market?
- If different companies are interlinked, what is the impact on insurance risk?

Questions like the ones above are covered by the special case $\gamma > 1$. Our model is then rich enough to describe cases where the second random time event does not only occur after the first one. Similar results based on linear structure of exponential *rv*can be found in Iyer et al. [13] whose purpose was to model a multi-component reliability system.

It is worthwhile to notice that this bivariate exponential model implies a copula function (see Cufaro Petroni and Sabino [7]) that is neither chosen upfront nor whose parameters are estimated from market data: here the copula function does not define the model, but rather the opposite.

Based on the self-decomposability of the exponentional *rv*'s we are able to construct 2-dimensional Poisson processes with dependent marginals (see Cufaro Petroni and Sabino [7]). Because of the relationship among random times, the two Poisson processes can be seen linked with a form of cointegration between their jumps.

In the context of energy facilities, the application of our approach to model price dynamics and to evaluate transportation assets seen as spread options is straightforward. To this purpose, we consider the TTF and NCG gas markets and assume that each spot price dynamics is driven by an exponential mean-reverting Geometric Ornstein-Uhlenbeck (GOU) plus compound Poisson. In this specific case we propose a stochastic dynamics of the spot prices that is slightly different form the one in Cartea and Figueroa [1] with the advantage of being more treatable to find non-arbitrage conditions. In particular we can find close-form formulas for vanilla options hence the price and the Greeks of spread options can be calculated in close form using the Margrabe formula [5] (if the strike is zero) or some well known approximations as in Deng et al. [8]. In any case our approach implies an explicit algorithm for the simulation of the dependent Poisson processes and can be used in Monte Carlo simulations.

Finally we compare the results obtained by our approach to the ones obtained assuming that the two compound Poisson processes are independent or contain a common Poisson component.

The extension to the multi-dimensional case will be the goal of future studies as well as the extension to different dynamics other than Poisson. However, under the assumption that only two underlyings have jump component, the price and the Greeks of spread options can be obtained by the moment-matching methodology proposed in Pellegrino and Sabino [14].

The paper is organized as follows. Section 2 summarizes the results for 2-dimensional Poisson process that we presented in Cufaro Petroni and Sabino [7]. In section 3 we consider 2-dimensional jump diffusion processes having a Geometric Brownian Motion (GBM) and GOU diffusive component. We also apply our methodology to the 2-factor Schwartz-Smith model [11] with jump diffusion where we find analytical solutions for vanilla options as well. Section 4 presents the risk neutral formulas for plain vanilla and spread options given the price dynamics introduced in Section 3 and given the different types of 2-dimensional Poisson components. Section 5 illustrates our approach with practical examples: we first assume a pure GBM plus jump model (Merton model) and compare the results obtained by our approach to the ones obtained assuming that the two compound Poisson processes are independent or contain a common Poisson component. In a second step we calibrate the parameters of a 2-dimensional GOU plus jumps dynamics to model the TTF and NCG day-head prices. Finally we compare the price of a transportation between these two hubs assuming the three types of Poisson configuration mentioned above. Section 6 concludes the paper with an overview of future studies and possible further applications.

2 Dependent Poisson processes

A law with density (*pdf*) $f(x)$ and characteristic function (*chf*) $\varphi(u)$ is said to be *self-decomposable* (*sd*) (see Sato [10] and Cufaro Petroni [6]) when for every $0 < a < 1$ we can find another law with *pdf* $g_a(x)$ and *chf* $\chi_a(u)$ such that

$$\varphi(u) = \varphi(au)\chi_a(u)$$

This definition selects an important family of laws with many relevant properties. Remark however that, while a *sd* $\chi_a(u)$ can be explicitly expressed in terms of $\varphi(u)$, its corresponding *pdf* $g_a(x)$ can not be given in a general, elementary form from $f(x)$. We will also say that a random variable (*rv*) X is *sd* when its law is *sd*: looking at the definition this means that for every $0 < a < 1$ we can always find two *independent* *rv*'s Y (with the same law of X), and Z_a with *pdf* $g_a(x)$ and *chf* $\chi_a(u)$ such that *in distribution*

$$X \overset{d}{=} aY + Z_a$$

We can look at this, however, also from a different point of view: to the extent that for $0 < a < 1$ the law of Z_a is known, we can define the *rv*

$$X = aY + Z_a$$

which by self-decomposability will now have the same law of Y. It would be easy to show that a also plays the role of the correlation coefficient between X and Y, namely

$$r_{XY} = a$$

It is well known, in particular, that the exponential laws $\mathfrak{E}_1(\lambda)$ with *pdf* and *chf*

$$f_1(x) = \lambda e^{-\lambda x}\mathbb{1}_{x\geq 0} \qquad\qquad \varphi_1(u) = \frac{\lambda}{\lambda - iu}$$

are a typical example of *sd* laws (see Sato [10]). Remark that if $Y' \sim \mathfrak{E}_1(\mu)$, then $\alpha Y' \sim \mathfrak{E}_1\left(\frac{\mu}{\alpha}\right)$ for every $\alpha > 0$, and hence in particular

$$Y = \frac{\mu}{\lambda}Y' \sim \mathfrak{E}_1(\lambda)$$

As a consequence we could also state the self-decomposability by means of *exponential rv's with different parameters* $X \sim \mathfrak{E}_1(\lambda)$ and $Y' \sim \mathfrak{E}_1(\mu)$ because of course we have

$$X = aY + Z_a = \frac{a\mu}{\lambda} Y' + Z_a$$

provided that $0 < a < 1$. In this paper, however, we will stick to the original formulation with $\mu = \lambda$. It is possible to show now (Cufaro Petroni and Sabino [7]) that the law of Z_a is a mixture of a law $\boldsymbol{\delta}_0$ degenerate in 0, and an exponential $\mathfrak{E}_1(\lambda)$, namely

$$Z_a \sim a\boldsymbol{\delta}_0 + (1-a)\mathfrak{E}_1(\lambda)$$

and this entails that Z_a can be taken as the product of two *id rv*'s: $Z \sim \mathfrak{E}_1(\lambda)$, and $B(1) \sim \mathfrak{B}(1, 1-a)$ (a Bernoulli with $a = \mathbb{P}\{B(1) = 0\}$), which are also independent from Y, namely

$$Z_a = B(1)\, Z$$

By summarizing, given two exponential *rv*'s $Y \sim \mathfrak{E}_1(\lambda)$ and $Z \sim \mathfrak{E}_1(\lambda)$, and a Bernoulli $B(1) \sim \mathfrak{B}(1, 1-a)$ (all mutually independent) the *rv*

$$X = aY + B(1)Z$$

is again an exponential $\mathfrak{E}_1(\lambda)$ defined as the weighed sum of Y and Z: while a is the *deterministic* weight of Y, the weight of Z is *random* and is represented by another (independent from both Y and Z) 0-1 Bernoulli *rv* $B(1) \sim \mathfrak{B}(1, 1-a)$. In other words X is nothing else than the exponential Y *down a-rescaled*, plus another independent, but *intermittent* with frequency $1-a$, exponential Z. The self-decomposability of the exponential laws ensures then that, if both the parameters of Y and Z are λ, also X marginally is an $\mathfrak{E}_1(\lambda)$ for every $0 < a < 1$. It is apparent on the other hand that, by construction, X and Y are not independent and it is easy to show that a represents their correlation coefficient

As initially suggested in Iyer et al. [13], we take now a sequence of *iid rv*'s

$$X_k = aY_k + B_k(1)Z_k \qquad k = 1, 2, \dots$$

in such a way that for every k: X_k, Y_k, Z_k are $\mathfrak{E}_1(\lambda)$, $B_k(1)$ is $\mathfrak{B}(1, 1-a)$, and $Y_k, Z_k, B_k(1)$ are mutually independent. Add moreover $X_0 = Y_0 = Z_0 = 0$, $\mathbb{P}$-*a.s.* to the list, and then define the point processes

$$\begin{aligned} T_n &= \sum_{k=0}^{n} X_k \sim \mathfrak{E}_n(\lambda) \qquad n = 0, 1, 2, \dots \\ S_n &= \frac{\lambda}{\mu} \sum_{k=0}^{n} Y_k \sim \mathfrak{E}_n(\mu) \qquad n = 0, 1, 2, \dots \end{aligned}$$

where $\mathfrak{E}_n(\lambda)$ are Erlang (gamma) laws with *pdf*'s and *chf*'s

$$f_n(x) = \lambda \frac{(\lambda x)^{n-1}}{(n-1)!} e^{-\lambda x} \mathbb{1}_{x \geq 0} \qquad \varphi_k(u) = \left(\frac{\lambda}{\lambda - iu}\right)^n \qquad n = 0, 1, 2, \dots$$

where it is understood that $\mathfrak{E}_0 = \boldsymbol{\delta}_0$. We will finally denote with $N(t) \sim \mathfrak{P}(\lambda t)$ and $M(t) \sim \mathfrak{P}(\mu t)$ the *dependent* Poisson processes associated respectively to T_n and S_n, and for our purposes we are interested in finding an explicit form of

$$p_{m,n}(t) = \mathbb{P}\{M(t) = m,\ N(t) = n\} \qquad n, m = 0, 1, 2, \dots \qquad t \geq 0$$

To this end we first introduce the parameter

$$\gamma = \frac{a\mu}{\lambda}$$

and the shorthand notations

$$\begin{aligned} \pi_k(\alpha) &= e^{-\alpha \frac{\alpha^k}{k!}} & k = 0, 1, \dots \\ \beta_\ell(n) &= \binom{n}{\ell} a^{n-\ell}(1-a)^\ell & \ell \leq n = 0, 1, \dots \end{aligned}$$

respectively for the distributions of a Poisson $\mathfrak{P}(\alpha)$, a binomial $\mathfrak{B}(n, 1-a)$ (it is understood that $\beta_0(0) = 1$) and a binomial mixture of shifted Erlang laws $\mathfrak{E}_\ell(\lambda)$, and then we prove (see Cufaro Petroni and Sabino [7]) the following result

Proposition 2.1. *When $\gamma \geq 1$, namely $a\mu \geq \lambda$, we have*

$$\begin{aligned} p_{m,n}(t) &= \begin{cases} 0 & n > m \geq 0 \\ Q_{n,n}(t) & m = n \geq 0 \\ Q_{m,n}(t) - Q_{m,n+1}(t) & m > n \geq 0 \end{cases} \\ Q_{m,n}(s,t) &= \sum_{k=n}^{m}(-1)^k \sum_{j=k}^{m} \binom{j}{k} \frac{\pi_{m-j}(\mu t)}{(-a)^j} \sum_{\ell=0}^{n} \beta_\ell(n)\pi_{j+\ell}(\lambda t)\Phi(j+1; j+\ell+1; \lambda t) \end{aligned}$$

When $\gamma \leq 1$, namely $a\mu \leq \lambda$, we have

$$p_{m,n}(t) = \begin{cases} A_{m,n}(t) - A_{m,n+1}(t) + B_{m,n}(t) - B_{m,n-1}(t) & n > m \geq 0 \\ A_{n,n}(t) - A_{n,n+1}(t) + B_{n,n}(t) + C_{n,n}(t) & m = n \geq 0 \\ A_{m,n}(t) - A_{m,n+1}(t) + C_{m,n}(t) - C_{m,n+1}(t) & m > n \geq 0 \end{cases}$$

where we define for every $n, m \geq 0$

$$A_{m,n}(t) = \pi_m(\mu t) \sum_{k=0}^{n} \beta_k(n) \left[1 + \pi_k(\lambda t - a\mu t) - \sum_{j=0}^{k} \pi_j(\lambda t - a\mu t)\right]$$

while for $n \geq m \geq 0$, and $\lambda t - a\mu t = w$ for short, it is

$$B_{m,n}(t) = \pi_m(\mu t) \sum_{k=0}^{n-m} \pi_k\left(\frac{w}{a}\right) \sum_{\ell=0}^{n+1} \beta_\ell(n+1) \frac{w^\ell k!}{(k+\ell)!}\, \Phi\left(\ell, k+\ell+1, \frac{1-a}{a} w\right)$$

and for $m \geq n \geq 1$ *it is (for* $n = 0$ *we have* $C_{m,0}(t) = 0$*)*

$$C_{m,n}(t) = \frac{e^{-(1-a)\mu t}}{a^m} \sum_{\ell=1}^{n} \beta_\ell(n) \sum_{k=n}^{m} \sum_{j=0}^{\ell-1} \binom{k+\ell-j-1}{k} (-1)^{\ell-1-j} \pi_j(\lambda t) \pi_{m+\ell-j}(a\mu t) \Phi(k+\ell-j, m+\ell-j+1, a\mu t)$$

and $\Phi(j+1; j+\ell+1; \lambda t)$ *for* $0 \leq \ell \leq n \leq j \leq m$ *are the confluent hypergeometric functions that are in fact elementary functions as proved Cufaro Petroni and Sabino [7].*

Proof: See Cufaro Petroni and Sabino [7] including the more general case for $p_{m,n}(s,t) = \mathbb{P}\{M(s) = m,\, N(t) = n\}\, n, m = 0, 1, 2, \ldots \quad t \geq 0$. Remark that in the boundary case $\boldsymbol{\gamma = 1}$ the previous two expressions coherently return the same result. □

3 The Market Models

In this section we adapt the model described in Section 2 to the financial context. We consider an usual Black-Scholes (BS) market and a market with geometric Ornstein Uhlenbeck (GOU) processes with jumps similar to the one adopted by Cartea and Figueroa [1]. Finally we focus on the Schwartz-Smith model with double jumps that can be seen as a complete cointegrated model with jumps.

Hereafter, compared to Section 2, $N_1(t)$ and $N_2(t)$ replace $M(t)$ and $N(t)$ and λ_1 and λ_2 replace μ and λ, respectively.

3.1 The GBM plus Jumps Case

Consider a BS market with two risky underlying assets whose dynamics are driven by SDEs with the following solution (Merton model):

$$S_i(T) = \exp\left[\log S_i(0) + \left(\mu_i - \frac{1}{2}\sigma_i^2\right) T + \sigma_i W_i(T) + \sum_{n_i=1}^{N_i(T)} \log J_i^{n_i}\right], \quad i = 1, 2, \tag{3}$$

with $dW_1(t)dW_2(t) = \rho^{(W)} dt$ and log-normal jumps:

$$J_i = M_i \exp\left(-\frac{\nu_i^2}{2} + \nu_i Z_i\right), \quad i = 1, 2. \tag{4}$$

where $Z_i \sim N(0,1)$ and $Corr(Z_1 Z_2) = \rho^{(D)}$. We assume that the compound Poisson processes and BM are independent.

We now concentrate on the logarithm:

$$\begin{aligned}\log S_i(T) &\stackrel{d}{=} \log S_i(0) + \left(\mu_i - \frac{1}{2}\sigma_i^2\right) T + \sigma_i W_i(T) + N_i(T)\log M_i \\ &\quad -\frac{\nu_i^2}{2} N_i(T) + \nu_i \sum_{n_i=1}^{N_i(T)} Z_i^{n_i}, \quad i = 1, 2. \end{aligned} \tag{5}$$

The equations above can be rewritten as:

$$\begin{aligned}\log S_i(T) &\stackrel{d}{=} \log S_i(0) + \left(\mu_i - \frac{1}{2}\sigma_i^2\right) T + N_i(T)\log M_i \\ &\quad -\frac{\nu_i^2}{2} N_i(T) + \sqrt{\sigma_i^2 T + N_i(T)\nu_i^2}\, H_i, \quad i = 1, 2, \end{aligned} \tag{6}$$

where given n and m, $(H_1, H_2) \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1, & \rho^{(J),n,m} \\ \rho^{(J),n,m}, & 1 \end{pmatrix}\right)$ The calculation of $\rho^{(J),n,m}$ can be found in the Appendix A.

For simplicity we denote

$$v_i^{(J,n)}(T) = \left(\sigma_i^{(J,n)}\right)^2 = \sigma_i^2 T + n\nu_i^2 = v_i^{(C)}(T) + v_i^{(D,n)}, \tag{7}$$

where $v_i^{(C)}$ and $v_i^{(D)}$ denote the terminal variances of the continuous and discontinuous parts. In case the continuous part of the SDE has a time-dependent volatility function, it is easy to see that the formulas still hold by replacing $v_i^{(C)}$ by $\int_0^T \sigma_i^2(s)ds$.

No-arbitrage conditions imply (see Joshi [3] pag 344):

$$\mu_i - r = -\lambda_i \mathbb{E}[J_i - 1] \quad i = 1, 2. \tag{8}$$

3.2 The Ornstein-Uhlenbeck plus Jumps Case

Energy markets often display mean-reversion and jumps. We here consider a one-factor model plus jumps similar to the one introduced in Cartea and Figueroa [1]. Consider a market driven by a stochastic process whose solution is:

$$S_i(t) = F_i(0, t)\exp\{U_i(t) + h(t)\}, \quad i = 1, 2, \tag{9}$$

where h(t) is a pure deterministic function and $U_i(t)$ is

$$U_i(t) = U_i(0)e^{-k_i t} + \sigma_i \int_0^t e^{-k_i(t-s)} dW_i(s) + e^{-k_i t} \sum_{n_i=1}^{N_i(t)} Y_i^{n_i} = U_i^C(t) + U_i^D(t) \tag{10}$$

whose SDE is:

$$dU_i(t) = -k_i U_i(t)dt + \sigma_i dW_i(t) + e^{-k_i t} Y_i dN_i(t). \tag{11}$$

$Y_i^{n_i}$ are copies of $Y_i \sim N(M_i, \nu_i^2)$ and $Corr(Y_1, Y_2) = \rho^{(D)}$. Remark that compared to the GBM case the *rv*'s $Y_i^{n_i}$ are not in terms of logarithms. The spot SDE is slightly different from the one adopted in Cartea and Figueroa [1], indeed the exponential term that multiplies the jump component is chosen such that the solution has no random jumps with time-dependent jump size. Should we have considered as Cartea and Figueroa [1],

$$dU_i(t) = -k_i U_i(t)dt + \sigma_i dW_i(t) + Y_i dN_i(t),$$

the solution would have been

$$U_i(t) = U_i(0)e^{-k_i t} + \sigma_i \int_0^t e^{-k_i(t-s)} dW_i(s) + e^{-k_i t} \sum_{n_i=1}^{N_i(t)} Y_i^{n_i} e^{k_i T_i^{n_i}}$$

where $T_i^{n_i}$ are the jump times. This setting leads to less tractable option formulas as it will be shown here below.

In order to get no-arbitrage conditions, we impose $\mathbb{E}\left[S(T)|\mathcal{F}_t\right] = F(t,T)$ and for simplicity we look at $\mathbb{E}[S(T)] = F(0,T)$ to adjust our parameters and functions. We then need to compute $\mathbb{E}\left[e^{U_i^C(t)+U_i^D(t)}\right] = \mathbb{E}\left[e^{U_i^C(t)}\right]\mathbb{E}\left[e^{U_i^D(t)}\right]$.

It is well known that:

$$\mathbb{E}\left[e^{U_i^C(t)}\right] = \exp\left(\mathbb{E}\left[U_i^C(t)\right] - \frac{1}{2}\mathbb{V}ar\left[U_i^C(t)\right]\right) = e^{a_i(t)}. \tag{12}$$

with:

$$\begin{aligned} \mathbb{E}\left[U_i^C(t)\right] &= U(0)e^{-k_i t}, \\ \mathbb{V}ar\left[U_i^C(t)\right] &= \frac{\sigma_i^2}{2k_i}\left(1 - e^{-2k_i t}\right). \end{aligned} \tag{13}$$

Hereafter we will assume that $U_i(0) = 0, i = 1, 2$ that does not change the applicability of the model. Finally we need to calculate:

$$\mathbb{E}\left[e^{U_i^D(t)}\right] = \mathbb{E}\left[\exp\left(e^{-k_i t}\sum_{n_i=1}^{N_i(t)} Y_i^{n_i}\right)\right] = e^{b_i(t)}. \tag{14}$$

Knowing the moment-generating function of the compound Poisson process:

$$\phi(u) = \mathbb{E}\left[\exp\left\{u\sum_{n=1}^{N_i(t)} Y_i^{n_i}\right\}\right] = \exp\left\{\lambda_i t\left(\phi_{Y_i}(u) - 1\right)\right\} \tag{15}$$

where

$$\phi_{Y_i}(u) = \exp\left\{M_i u + \frac{1}{2}\nu_i^2 u^2\right\} \tag{16}$$

we easily obtain the required expected value.

$$\mathbb{E}\left[e^{U_i^D(t)}\right] = \phi\left(e^{-k_i t}\right). \tag{17}$$

$$b_i(t) = \lambda_i t\left(e^{e^{-k_i t}\left(M_i + \frac{1}{2}e^{-k_i t}\nu_i^2\right)} - 1\right) \tag{18}$$

Based on the results above, non-arbitrage is given by $h_i(t) = -a_i(t) - b_i(t)$. The equations for the spot dynamics above can be rewritten as:

$$\begin{aligned}\log S_i(t) \quad &\stackrel{d}{=} \quad \log F_i(0,t) - b_i(t) + N_i(t) M_i e^{-k_i t} + \frac{1}{2}\nu_i^2 e^{-2k_i t} N_i(t) \\ &-\frac{1}{2}\left(\mathbb{V}ar\left[U_i^{(C)}(t)\right] + \nu_i^2 e^{-2k_i t} N_i(t)\right) + \\ &\sqrt{\mathbb{V}ar\left[U_i^{(C)}(t)\right] + \nu_i^2 e^{-2k_i t} N_i(t)} H_i. \end{aligned} \tag{19}$$

Where $H_i, i = 1, 2$ have been defined in the previous section.

3.3 The Schwartz-Smith plus Jumps Case

Consider the two factor Schwartz-Smith model (see Schwartz Smith [11]):

$$\begin{aligned} U_1(t) &= U_1(0)e^{-kt} + \sigma_1 \int_0^t e^{-k(t-s)} dW_1(s) + e^{-kt} \sum_{n_1=1}^{N_1(t)} Y_1^{n_1} \\ U_2(t) &= U_2(0) + \mu t + \sigma_2 W_2(t) + \sum_{n_2=1}^{N_2(t)} Y_2^{n_2} \\ U(t) &= U_1(t) + U_2(t). \end{aligned} \tag{20}$$

where $S(t) = F(0,t)e^{h(t)+U(t)}$ and we assume that the jumps of both process share the same distribution $Y_1, Y_2 \sim N(M, \nu)$. Simply taking the differential and some algebra:

$$dU(t) \quad = \quad -k\left(\mu + U_2(t) - U(t)\right) dt + \sigma dW + Y\left(e^{-kt} dN_1 + dN_2\right) \tag{21}$$

Should we consider the OU plus compound Poisson as in Cartea and Figueroa [1] the process $U(t)$ would be:

$$\begin{aligned} dU(t) &= -k\left(\mu + U_2(t) - U(t)\right) dt + \sigma_1 dW_1 + \sigma_2 dW_2 + Y\left(dN_1 + dN_2\right) \\ &= -k\left(\mu + U_2(t) - U(t)\right) dt + \sigma dW + Y dN(t). \end{aligned} \tag{22}$$

where $\sigma^2 = \sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2\rho^{(W)}$. With the latter equation above (22), the log of the spot process can be expressed in terms of one BM and a compound Poisson-like process.

With the same procedure outlined in the previous subsection, no arbitrage conditions can be obtained by taking the (conditional) expectation of the spot process:

$$
\begin{aligned}
\mathbb{E}\left[e^{U^C(t)}\right] &= \mathbb{E}\left[e^{U_1(0)e^{-kt}+\sigma_i\int_0^t e^{-k(t-s)}dW_1(s)+U_2(0)+\mu t+\sigma_2 W_2(t)}\right] \\
&= \exp\left(\mathbb{E}\left[U^C(t)\right]-\frac{1}{2}\mathbb{V}ar\left[U^C(t)\right]\right) = e^{a(t)}.
\end{aligned}
\tag{23}
$$

Assuming once more $U_1(0)=0$ and $U_2(0)=0$ we have:

$$
\begin{aligned}
\mathbb{E}\left[U^C(t)\right] &= \mu t, \\
\mathbb{V}ar\left[U^C(t)\right] &= \frac{\sigma_1^2}{2k}\left(1-e^{-2kt}\right)+\sigma_2^2 t+\frac{2\rho\sigma_1\sigma_2}{k}\left(1-e^{-kt}\right).
\end{aligned}
\tag{24}
$$

For the discontinuous component we have:

$$
\begin{aligned}
e^{b(t)} &= \mathbb{E}\left[e^{U^D(t)}\right] = \mathbb{E}\left[\exp\left(e^{-kt}\sum_{n_1=1}^{N_1(t)}Y_1^{n_1}+\sum_{n_1=1}^{N_2(t)}Y_2^{n_2}\right)\right] = \\
&= \sum_{m_1,m_2=0}^{+\infty} p_{m_1,m_2}\left[\exp\left(e^{-kt}\sum_{n_1=1}^{m_1}Y_1^{n_1}+\sum_{n_1=1}^{m_2}Y_2^{n_2}\right)\right] = \\
&= \sum_{m_1,m_2=0}^{+\infty} p_{m_1,m_2}\phi_Y\left(e^{-kt}\right)^{m_1}\times\phi_Y\left(1\right)^{m_2}
\end{aligned}
\tag{25}
$$

As done in Section 5.2, non-arbitrage is given by $h(t)=-a(t)-b(t)$ where $b(t)$ can be computed numerically.

In contrast, in the case of Cartea and Figueroa [1], we have:

$$
e^{b(t)} = \mathbb{E}\left[e^{U^D(t)}\right] = \mathbb{E}\left[\exp\left(e^{-kt}\sum_{n_1=1}^{N_1(t)}Y_1^{n_1}e^{kT^{n_1}}+\sum_{n_1=1}^{N_2(t)}Y_2^{n_2}\right)\right]
\tag{26}
$$

that is more complex to treat

After some algebra, the log-spot dynamics above can be rewritten as:

$$
\begin{aligned}
\log S_i(t) &\stackrel{d}{=} \log F(0,t)-b(t)+\mu t+N_1(t)e^{-kt}\left(M+e^{-kt}\frac{\nu^2}{2}\right)+N_2(t)\left(M+\frac{\nu^2}{2}\right)- \\
&- \frac{1}{2}\left\{\mathbb{V}ar\left[U^{(C)}(t)\right]+\left(e^{-2k_it}N_1(t)+N_2(t)\right)\right\}\nu^2+ \\
&+ \sqrt{\mathbb{V}ar\left[U^C(t)\right]+\left(e^{-2k_it}N_1(t)+N_2(t)\right)\nu^2}\epsilon.
\end{aligned}
\tag{27}
$$

4 Risk-neutral Pricing Formulas

4.1 The simple European Plain Vanilla Options Case

In order to simplify the calculation and the notation, we represent the price of a call option at time zero $c(0)$ in terms of an abstract BS formula

$$c(0) = BS\left(P_0, K, r, T, v, q\right). \tag{28}$$

Where P_0, K, r, T, v, q denote the arguments: initial price, strike, risk-free rate, maturity, terminal variance and dividend yield, respectively, for the Black-Scholes formula.

- **GBM Case**. We need to rearrange Equation (6) given the market of Equation (3) such that we can apply the abstract BS formula for the GBM-plus-jumps case:

$$\log S_i(T) = \log S_i(0) + N_i(T)\log M_i + \lambda_i(1-M_i)T - \frac{v_i^{(J,N_i(T))}}{2} + \sqrt{v_i^{(J,N_i(T))}}H_i \tag{29}$$

  The price of a call (put) option on underlying asset $i=1$ given the GBM-plus-jumps market of Equation (3) is then:

$$c(0) = \sum_{n=0}^{\infty} \pi_{n_1}(\lambda_1 T) BS(S_1^{(n)}(0), K, r, T, v_1^{(J,n)}(T), 0). \tag{30}$$

  where

$$S_1^{(n)}(0) = S_1(0) M_1^n \exp\left[\lambda_1 T(1-M_1)\right]. \tag{31}$$

  and $v_1^{(J,n)}(T)$ is defined in Equation (7).

- **GOU Case**. In contrast for the OU-plus-jumps market of Equation (9), the starting point argument for the abstract BS formula is:

$$S_1^{(n)}(0) = F_1(0,T)e^{p_i^n(T)}, \tag{32}$$

  where $v_1^{(J,n)}(T) = \mathbb{V}ar\left[U_1^{(C)}(T)\right] + ne^{-2k_1T}\nu_1^2$ and

$$p_1^n(t) = -b_1(t) + ne^{-k_1 t}\left(\frac{1}{2}e^{-k_1 t}\nu_1^2 + M_1\right). \tag{33}$$

- **Schwartz-Smith Case**. Assuming Equation (20) a semi-closed form formula can be found following the procedure outlined in the GBM and GOU cases.

$$c(0) = \sum_{n_1,n_2=0}^{\infty} \mathbb{P}\left(N_1(T)=n_1, N_2(T)=n_2\right) BS(S^{(n_1,n_2)}(0), K, r, T, v^{(J,n_1,n_2)}(T), 0). \tag{34}$$

where

$$S^{(n_1,n_2)}(0) = F(0,t)e^{-b(t)+\mu t+n_1(t)e^{-kt}\left(M+e^{-kt}\frac{\nu^2}{2}\right)+n_2(t)\left(M+\frac{\nu^2}{2}\right)}, \tag{35}$$

and

$$v^{(J,n_1,n_2)}(T) = \mathbb{V}ar\left[U^C(T)\right] + \left(e^{-2k_iT}n_1 + n_2\right)\nu^2 \tag{36}$$

Moreover, for both settings in Equations (20) and (22) given the non-arbitrage conditions, the simple algorithm to generate the dependent Poisson processes of Section 2 offers an easy Monte Carlo implementation to compute the price of vanilla options.

Having obtained the risk-neutral conditions on each underlying asset, it is straightforward to obtain formulas for spread options.

4.2 Spread Options Case

The application to spread options is the native framework to compare our approach with cointegrated jumps compared to other jump-diffusion cases. We start considering a spread option with zero-strike, based on the results of Margrabe [5], with the same conditioning approach applied in Subsection (4.2) we get that the price of a spread option with zero strike given the market of Equation (3) or Equation (9) is:

$$s(0) = \sum_{n_1,n_2=0}^{\infty} \mathbb{P}\left(N_1(T)=n_1; N_2(T)=n_2\right) BS(S_1^{(n_1)}(0), S_2^{(n_2)}(0), 0, T, v^{(M,n_1,n_2)}(T), 0)^1, \tag{37}$$

where $v^{(M,n_1,n_2)}(T) = v_1^{(J,n_1)}(T) + v_2^{(J,n_2)}(T) - 2\rho^{(J,n_1,n_2)}\sqrt{v_1^{(J,n_1)}v_2^{(J,n_2)}}$ is the spread terminal variance. The definition of S_2^m and $v_2^{(J,m)}$ follows from the subsection above.

In the literature different analytical approximations are available when the strike is not zero (see for instance Deng and Lee [8] or Kirk [4]), the extension to the jump diffusion case is just a matter of adapting the parameters of the approximation. Employing Monte Carlo methods is not complicated because the simulation of the 2-dimensional path is not a complex task and as well as is the 2-dimensional Poisson generation.

We will deserve future studies to a market of more than two assets that have a jump component. The current framework cannot cope with multi-asset spread options unless one considers that the third asset has no jump term. Pricing multi-assets spread options then can be tackle via simulation, analytical approximations as done in Deng and Lee [9] and Pellegrino and Sabino [15] or by applying moment matching and using one of the solutions available for two legs as explained in Pellegrino and Sabino [14].

In the following, we compare three different Poisson models:

[1] when the Margabe option is written on the spot it can be seen that the price is independent of r, in contrast to the situation of a spread option on the forward.

- **Independent Jumps**. $N_1(t)$ and $N_2(t)$ are independent Poisson processes. The spread option formula is:

$$s(0) = \sum_{n_1,n_2=0}^{\infty} \pi_{n_1}(\lambda_1 T)\pi_{n_2}(\lambda_2 T) BS(S_1^{(n_1)}(0), S_2^{(n_2)}(0), 0, T, v^{(M,n_1,n_2)}(T), 0). \tag{38}$$

- **One Common Jump**. $N_i(t) = N(t) + N_i^X$, $i = 1, 2$, where $N(t)$ and N_i^X are all mutually independent Poisson processes. The spread option formula is:

$$\begin{aligned} s(0) = & \sum_{n=0, n_1, n_2 \geq n}^{\infty} \pi_{n_1-n}(\lambda_1^X T)\pi_{n_2-n}(\lambda_2^X T)\pi_n(\lambda T) \times \\ & BS(S_1^{(n_1-n)}(0), S_2^{(n_2-n)}(0), 0, T, v^{(M,n_1-n,n_2-n)}, 0) \end{aligned} \tag{39}$$

- **Cointegrated Jumps**. $N_i(t)$, $i = 1, 2$ described in Subsection 4.2. The spread option formula is:

$$s(0) = \sum_{n_1,n_2=0}^{\infty} \mathbb{P}\left(N_1(T) = n_1; N_2(T) = n_2\right) BS(S_1^{(n_1)}(0), S_2^{(n_2)}(0), 0, T, v^{(M,n_1,n_2)}(T), 0) \tag{40}$$

  where $p_{n_1,n_2} = \mathbb{P}\left(N_1(T) = n_1; N_2(T) = n_2\right)$ are defined in Proposition 2.1.

The payoff of the spread options above considers the values of the two underlying at the same time T. Other types of spread options instead look at the two underlying at different times, e.g. the payoff may be $(S_1(T_1) - S_2(T_2))^+$, $T_2 < T_1$. In this case one needs to readapt the formulas and consider the probabilities $p_{n_1 n_2} = \mathbb{P}\left(N_1(T_1) = n_1; N_2(T_2) = n_2\right)$ and they can be found in Cufaro Petroni and Sabino [7].

5 Numerical Experiments

In this section we presents the numerical experiments assuming the GBM and GOU dynamics plus jumps explained in the previous sections. The case with GBM considers realistic parameters and is meant to study the spread option values with different types of bivariate Poisson processes; in contrast the GOU case is based on real data of TTF and NCG day-ahead prices.

5.1 GBM. Application to Spread Options

We compare the spread option value obtained using Equations (38)-(40) changing the correlation between the two Poisson processes. In particular, assuming $N_i = N(t) + N_i^X(t)$ we have $\mathbb{C}ov[N_1(t), N_2(t)] = \mathbb{V}ar[N(t)] = \lambda t$, then the instantaneous

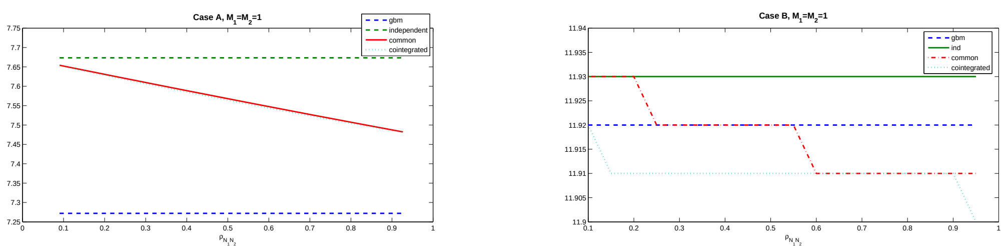
Table 1: Parameters of the GBM and Compound Poisson processes

(a) Continuous Part.

	No Jump		With Jumps
	Case A	Case B	
$S_1(0)$	100	100	100
$S_2(0)$	100	100	100
σ_1	0.49	0.37	0.2
σ_2	0.35	0.23	0.15
$\rho^{(W)}(\%)$	96	60	80

(b) Discontinuous Part

	Case A	Case B
$\rho^{(D)}(\%)$	99	50
λ_1	20	40
λ_2	20	20
ν_1	0.10	0.05
ν_2	0.07	0.04
M_1	1.1	1.05
M_2	1.1	1.05



(a) Case A

(b) Case B

Figure 1: Spread Option Values in the Cases A and B when $M_1 = M_2 = 1$

correlation is $\rho_{N_1N_2} = \frac{\lambda}{\sqrt{\lambda_1\lambda_2}}$; in case of cointegrated jumps this can be obtained numerically. Naturally, in case $\lambda_1 \neq \lambda_2$ the perfect correlation cannot be obtained.

We consider two cases:

Case A . $\lambda_1 = \lambda_2 = 20$, where for cointegrated jumps $\gamma = a < 1$.

Case B . $\lambda_1 = 40$, $\lambda_2 = 20$ where for cointegrated jumps $\gamma < 1$ can assume values lower or higher than 1, when $a < 0.5$ and $a > 0.5$, respectively.

The parameters are shown in Table (5.1). In both cases we consider an at-the-money spread option with zero strike, $K = 0$ and maturity $T = 1$ such that we can use the exact Margrabe formula. One can use some approximation for the spread option value for non-zero strikes without changing the validity of our tests because.

We also compute the spread value with a pure GBM with no jumps with the parameters of the first column of the first table in (5.1) that are chosen such that they match the average spread terminal variance $v^{(M,\lfloor\lambda_1 T\rfloor,\lfloor\lambda_2 T\rfloor)}$, where $\lfloor\cdot\rfloor$ denotes the integer part.

We first observe that the behavior of Equations (38)-(40) depends on the values of the probabilities p_{n_1,n_2} and the values of the BS formulas separately. The former quantities do not depend on the distribution of the jumps while the latter ones are independent on the structure of dependence between the Poisson processes.

Table 2: Spread Option Values without jumps and independent Compound Poisson processes.

	No Jump		Independent Jump	
	Case A	Case B	Case A	Case B
Option Value	7.27	11.92	25.23	19.27

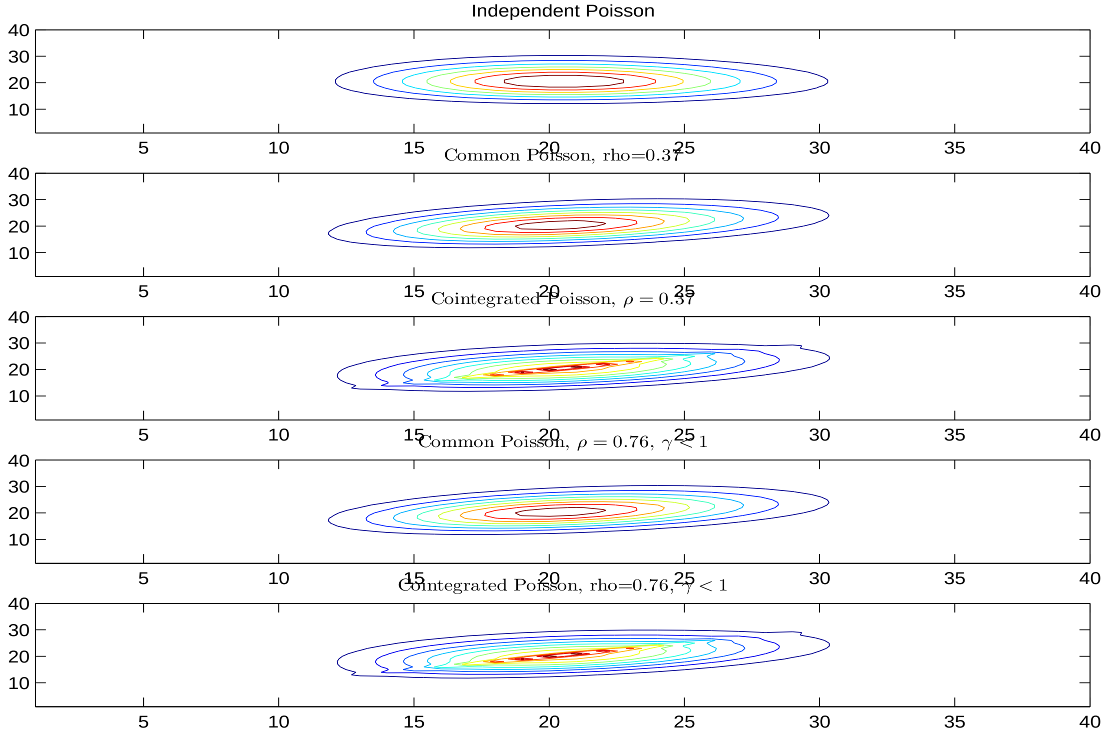
Table 3: Spread Option Values with Common and Cointegrated Compound Poisson.

	Case A		Option Value Case A		Case B		Option Value Case B	
a	$\rho_{N_1N_2}(\%)$	λ	Common	Cointegrated	$\rho_{N_1N_2}(\%)$	λ	Common	Cointegrated
0.1	9	1.80	24.30	24.22	7	2.09	18.87	18.87
0.15	14	2.71	23.76	23.64	11	3.13	18.66	18.67
0.2	18	3.63	23.20	23.05	15	4.16	18.45	18.46
0.25	23	4.55	22.63	22.44	18	5.19	18.25	18.26
0.3	27	5.47	22.04	21.81	22	6.21	18.04	18.05
0.35	32	6.40	21.42	21.16	26	7.23	17.83	17.83
0.4	37	7.34	20.78	20.48	29	8.24	17.61	17.62
0.45	41	8.29	20.11	19.78	33	9.25	17.40	17.40
0.5	46	9.24	19.41	19.05	36	10.25	17.18	17.18
0.55	51	10.20	18.68	18.29	40	11.25	16.97	16.96
0.6	56	11.17	17.90	17.49	43	12.24	16.75	16.74
0.65	61	12.16	17.08	16.64	47	13.23	16.53	16.51
0.7	66	13.15	16.20	15.74	50	14.21	16.30	16.29
0.75	71	14.17	15.25	14.78	54	15.19	16.08	16.06
0.8	76	15.20	14.21	13.75	57	16.16	15.85	15.83
0.85	81	16.26	13.06	12.61	61	17.13	15.62	15.60
0.9	87	17.36	11.74	11.33	64	18.09	15.38	15.37
0.95	93	18.53	10.14	9.82	67	19.05	15.15	15.14

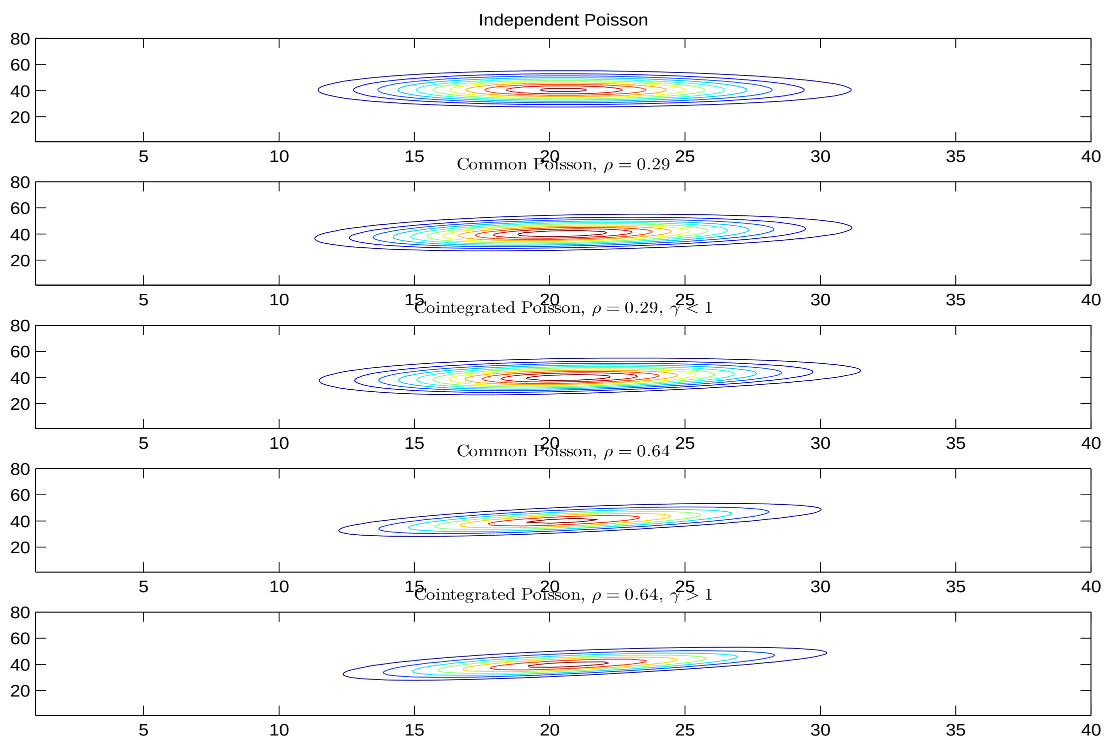
Figure (5.1) clearly shows that the expected jump size has a relevant impact in the option value because under the assumption $M_1 = M_2 = 1$ the price in almost independent on the choice of the Poisson model.

Figures (5.1) show the difference among the joint probabilities of the Poisson processes. The isolines of the contour plot of $p_{n_1n_2}$ resemble to a sort of ellipse whose axis are parallel to the X-Y axis that is expected for independent Poisson both in case A and B. The positive correlation of the Poisson processes in the other configuration is reflected by the fact that the axis are now rotated counterclockwise. In addition for cointegrated Poisson the higher value of the probabilities is more concentrated around the expected value.

This is more evident in the figure with $\lambda_1 = \lambda_2$ and it is worth noticing once more that for $\gamma > 1$ the matrix $p_{n_1n_2}$ is lower triangular for the cointegrated Poisson while it is full for the structure with one common Poisson process. Finally, Table 5.1 displays the results with expected jump size different from zero. λ where chosen such



(a) $\lambda_1 = \lambda_2 = 20$



(b) $\lambda_1 = 40$, $\lambda_2 = 20$

that the correlations of the 'cointegrated' and 'common' Poisson processes coincides.

The effect of the correlation between the Poisson processes is noticeable. The value of the spread option is decreasing when $\rho_{N_1N_2}$ is increasing that is in line with the intuition because the spread terminal variance decreases.

In the case A, the jump sizes are perfectly correlated and the spread option values using the common Poisson setting is always higher than the values obtained with our methodology. This is somehow reflected by the concentration of the isoline of the probabilities. Using a common Poisson reduces the spectrum of jump events, for instance in an extreme setting where $\lambda = \lambda_1 = \lambda_2$, $N_1(t)$ cannot jump more that $N_2(t)$, while this is not the case for the cointegrated Poisson process. In contrast, the choice of the Poisson model has no remarkable effect on the price of the spread option in the configuration B. This in our opinion does not diminish the value of our methodology, as shown for the results of the configuration A, because in any case the probabilities $p_{n_1n_2}$ differ between the two different Poisson examples. With the same $\rho_{N_1N_2}$ the price of the spread option seems to highly depend on the number of the jumps of both processes rather than when they occurred and that explains the small differences. Furthermore, assuming for instance $\lambda_2^X = 0$ implies $\lambda_2 = \lambda$ and $N_2(t)$ cannot have more jumps than $N_2(t)$ that coincides with the properties of our model only if $\gamma > 1$, that means that our model gives a richer set of combinations.

5.2 GOU. Application to Gas Transports

In this section we apply our methodology in order to price a gas transport between TTF and NCG hubs modeled as a spread option (NCG minus TTF). We assume that each dynamics behaves as a GOU plus a compound Poisson and calibrate the parameters. As done for the GBM example we do not consider transport costs (no strike) and adopt the Margrabe formula in Equations (38)-(40).

In particular, the calculation date is end of December 2013 with a historical time window of 2 years for the estimation period. We concentrate then on the transportation value for the first and second quarters, Q1, Q2 [2].

The estimation procedure can be split into two steps. As a first step, after filtering out the time-dependent components of each process, one can estimate the parameters of the one-dimensional processes, $\theta_i = (k_i, \sigma_i, \lambda_i, M_i, \nu_i)$. As a second step then, one can estimate the remaining joint parameters defined by the two-dimensional model[3].

Consider an equally spaced time grid $t_0, t_1, \ldots, t_T$ with $t_{i+1} - t_i = \Delta t$ and the Euler scheme of each SDE in Equation (41)

$$U_i(t+1) = (1 - k_i\Delta t)\, U_i(t) + \sigma_i\sqrt{\Delta t}\epsilon_{i,t+1} + e^{-k_it}\mathbb{1}_i(t+1)Y_i. \tag{41}$$

[2]The technique here discussed does not reflect EGC view.

[3]A slightly different example on how to derive the parameters of GOU process can be found at http://de.mathworks.com/help/fininst/simulating-electricity-prices-with-mean-reversion-and-jump-diffusion.html

where

$$\mathbb{1}_i(t+1) = \begin{cases} 1, & \text{with probability } \lambda_i \Delta t \\ 0, & \text{with probability } 1-\lambda_i \Delta t. \end{cases} \tag{42}$$

Hence the transition density is a combination of Gaussian densities:

$$p_i\left(U_i(t+1), t+1|U_i(t), t\right) = \left(1-\lambda_i \Delta t\right) \mathcal{N}\left(\mu_i^C(t), \sigma^C(t)\right) + \lambda_i \Delta t \mathcal{N}\left(\mu_i^J(t), \sigma^J(t)\right). \tag{43}$$

$\mathcal{N}(x)$ denotes the density function of a Gaussian random variable and $\mu_i^C(t) = \left(1-k_i\Delta t\right) U_i(t)$, $\mu_i^J(t) = \left(1-k_i\Delta t\right) U_i(t) + M_i e^{-k_i t}$ and $\sigma_i^C(t) = \sigma_i \sqrt{\Delta t}$, $(\sigma_i^J(t))^2 = \sigma_i^2 \Delta t + e^{-2k_i t}\nu_i^2$. The parameters $\theta_i = (k_i, \sigma_i, \lambda_i, M_i, \nu_i)$ can be calibrated by minimizing the log-likelihood function with the usual constrains on the parameters:

$$\theta_i = argmin \sum_{t=0}^{T-1} \log\left(p_i\left(U_i(t+1), t+1|U_i(t), t\right)\right). \tag{44}$$

The calibration of the parameters for the two-dimensional process depends on the model specification written in Section 2.

- **Independent Jumps**. In case of independent Poisson processes the joint probability are simply:

$$\begin{aligned} p_{0,0} &= \left(1-\lambda_1\Delta t\right)\left(1-\lambda_2\Delta t\right), & p_{1,0} &= \lambda_1\Delta t\left(1-\lambda_2\Delta t\right) \\ p_{0,1} &= \left(1-\lambda_1\Delta t\right)\lambda_2\Delta t, & p_{1,1} &= 1-p_{0,1}-p_{1,0}-p_{0,0}. \end{aligned} \tag{45}$$

  The only two remaining parameters to estimate are $\rho^{(W)}$ and $\rho^{(J)}$ and can be obtained by minimizing the log-likelihood of the two dimensional process. The transition density is

$$\begin{aligned} & p\left(U(t+1), t+1|U(t), t\right) = \\ & \mathcal{N}\left(\mu^{CC}(t), \Sigma^{CC}(t)\right) p_{0,0} + \mathcal{N}\left(\mu^{CJ}(t), \Sigma^{CJ}(t)\right) p_{0,1} + \\ & \mathcal{N}\left(\mu^{JC}(t), \Sigma^{JC}(t)\right) p_{1,0} + \mathcal{N}\left(\mu^{JJ}(t), \Sigma^{JJ}(t)\right) p_{1,1}. \end{aligned} \tag{46}$$

  where

$$\mu^{CC}(t) = \left(\mu_1^C(t), \mu_2^C(t)\right) \quad , \quad \Sigma^{CC}(t) = \begin{pmatrix} (\sigma_1^C)^2 & \rho^{(W)}\sigma_1^C\sigma_2^C \\ \rho^{(W)}\sigma_1^C\sigma_2^C & (\sigma_2^C)^2 \end{pmatrix},$$

$$\mu^{CJ}(t) = \left(\mu_1^C(t), \mu_2^J(t)\right) \quad , \quad \Sigma^{CJ}(t) = \begin{pmatrix} (\sigma_1^C)^2 & \rho^{(W)}\sigma_1^C\sigma_2^C \\ \rho^{(W)}\sigma_1^C\sigma_2^C & (\sigma_2^J)^2 \end{pmatrix},$$

$$\mu^{JC}(t) = \left(\mu_1^J(t), \mu_2^C(t)\right) \quad , \quad \Sigma^{JC}(t) = \begin{pmatrix} (\sigma_1^J)^2 & \rho^{(W)}\sigma_1^C\sigma_2^C \\ \rho^{(W)}\sigma_1^C\sigma_2^C & (\sigma_2^C)^2 \end{pmatrix},$$

$$\mu^{JJ}(t)\left(\mu_1^J(t), \mu_2^J(t)\right) \quad , \quad \Sigma^{JJ}(t) = \begin{pmatrix} (\sigma_1^J)^2 & \rho^{(J)}\sigma_1^J\sigma_1^J \\ \rho^{(J)}\sigma_1^J\sigma_2^J & (\sigma_2^J)^2 \end{pmatrix}.$$

  We do not neglect the $o\left(\Delta t^2\right)$ terms that are necessary to estimate $\rho^{(D)}$.

- **Common Jumps**. One cannot detect the presence of the common Poisson process only looking at each log process independently.

  After some algebra, the pair $(\mathbb{1}_i(t+1))$ is bi-dimensional Bernoulli *rv*with:

$$p_{0,0} = 1 - \left(\lambda_1^X + \lambda_2^X + \lambda\right)\Delta t, \qquad p_{1,0} = \lambda_1^X \Delta t$$
$$p_{0,1} = \lambda_2^X \Delta t, \qquad p_{1,1} = \lambda\Delta t = 1 - p_{0,1} - p_{1,0} - p_{0,0}. \tag{47}$$

  In contrast to the case above, we neglect $o\left(\Delta t^2\right)$ terms that implies that we are neglecting the possibility that the Poisson processes $N_1^X(t)$ and $N_2^X(t)$ jump simultaneously in the unit of time Δt. The functional form of the transition density is the one of Equation (46), with different probability weights.

- **Cointegrated Jumps**

  - Case 1. $\gamma > 1$. Based on the results of Proposition (2.1) up to $O(\Delta^2)$ terms we have:

$$p_{0,0} = 1 - \lambda_1\Delta t, \qquad p_{1,0} = (\lambda_1 - \lambda_2)\Delta t - \lambda_1\left(\frac{\lambda_1}{\gamma} - \lambda_2\right)\Delta t^2$$
$$p_{0,1} = 0, \qquad p_{1,1} = \lambda_2\Delta t. \tag{48}$$

  - Case 2. $0 < \gamma \leq 1$.

$$p_{0,0} = 1 - (\lambda_1 + \lambda_2(1-\gamma))\Delta t, \qquad p_{0,1} = \lambda_2(1-\gamma)\Delta t$$
$$p_{1,0} = (\lambda_1 - \gamma\lambda_2)\Delta t, \qquad p_{1,1} = \gamma\lambda_2\Delta t. \tag{49}$$

    Here above we neglect $o(\Delta t^2)$ terms (see Appendix B for the proof of this last case)

The results of the calibration are shown in Table 5.2. The expected jump sizes and their correlation are very small and negligible. Comparing the values of λ and a or γ the correlation between the two Poisson processes is also small. Based on the study in Section 5.1 we can expect that the selection of a specific Poisson model will not bring a remarkable difference.

Table 5.2 shows the values of the transportation in Q1 and Q2 with different dynamics. The prices obtained with the different configurations meet the expectations after having a look at the estimated parameters. Remark that in this case the prices with cointegrated jumps are higher than those with common jumps; this is explained by the fact that the correlation parameters are different, the latter configuration has higher values both for ρ^W and $\rho_{N_1N_2}$.

Once more, although our methodology is parameterized by γ and a that is the correlation between the exponential *rv*'s that construct the Poisson process, it implies a structure that goes beyond the linear correlation.

Table 4: Market parameters for NCG and TTF

(a) Parameters of the Single Underlyings.

Market	k	σ_i	μ_i	ν_i	λ_i
NCG	9.75	0.09	0.001	0.07	35.37
TTF	26.38	0.15	0.003	0.02	26.70

(b) Common Parameters

Method	$\rho^{(W)}(\%)$	$\rho^{(D)}(\%)$	λ	a
Independent	43	-1.0	NA	NA
Common	40	-1.0	8.93	NA
Cointegrated	35	-1.0	NA	0.27

Table 5: TTF-NCG Gas Transport Prices

	Transportation Value		
	Independent	Common	Cointegrated
Q1	49.14	49.35	49.79
Q2	34.99	35.48	36.27

6 Conclusion and Future Studies

Based on the concept of self-decomposability we have studied the use of the 2-dimensional co-dependent Poisson processes proposed in Cufaro Petroni and Sabino [7] to model energy derivatives and in general to price spread options. Due to the particular relationships among inter arrival times, we can see this dependence as a form of coitegration among jumps.

To put into the context of modeling energy market and facilities, we have shown how combine 2-dimensional compound Poisson processes with Geometric Browian Motions and Geometric Ornstein and Uhlenbeck dynamics. In the latter case, we have also proposed a dynamics for day-ahead prices that allows (semi-)closed formulas for plain vanilla options with an easy derivation of risk-neutral conditions.

Focusing on the pricing of spread options, we have compared the option prices using our methodology and different types of Poisson processes. We have shown that our methodology can cope with a wide range of possibilities that go beyond the pure correlation between marginal Poisson and can answer several questions that arise in the financial context.

In our study we have considered transportation assets between two gas hubs but the applicability can be extended other financial situations. Straightforward applications are in credit and insurance risk where our approach can answer questions regarding the time of contagion or time of propagation of certain information.

In addition, the self-decomposability and subordination technique can be promising tools to study dependency beyond the Gaussian-Itō world. For instance, in Cufaro Petroni and Sabino [7] we have detailed how to obtain dependent Erlang

(Gamma) *rv*'s that can be used to create and simulate dependent variance gamma processes. Furthermore in recent papers, Sexton and Hanzon [12] and Hanzon et al. [2] have studied the use of two sided Exponential-Polynomial-Trigonometric (ETP) density functions to option pricing where EPT are distributions with a strictly proper rational characteristic function. Due to the fact that the Erlang and exponential distributions belong to this class, it will be worthwhile to investigate the use of self-decomposability to create dependence for this larger class of distributions.

A Calculation of $\rho^{(J)}$

- The GBM plus Jumps Case.

$$
\begin{aligned}
\rho^{(J),n,m} &= Corr\left[\sigma_1 W_1(T) + \sqrt{n}\nu_1 Z_1, \sigma_2 W_2(T) + \sqrt{m}\nu_2 Z_2\right] = \\
&= = \frac{\rho^{(W)}\sigma_1\sigma_2 T + \rho^{(D)}\sqrt{nm}\nu_1\nu_2}{\sqrt{v_1^{(J,n)}(T)v_2^{(J,m)}(T)}}.
\end{aligned}
\tag{50}
$$

- The Ornstein-Uhlenbeck plus Jumps Case.

$$
\begin{aligned}
\rho^{(J),n,m} &= Corr\left(L_1, L_2\right) = \\
&= \frac{\frac{\rho^{(W)}\sigma_1\sigma_2}{2\sqrt{k_1 k_2}}\sqrt{1-e^{-2k_1 t}}\sqrt{1-e^{-2k_2 t}} + \rho^{(D)}\sqrt{nm}\nu_1\nu_2 e^{-(k_1+k_2)t}}{\sqrt{v_1^{(J,n)}(T)v_2^{(J,m)}(T)}}
\end{aligned}
\tag{51}
$$

where $L_1 = \sigma_1 \int_0^t e^{-k_1(t-s)}dW_1(s) + \sqrt{n}\nu_1 e^{-k_1 t} Z_1$ and $L_2 = \sigma_2 \int_0^t e^{-k_2(t-s)}dW_2(s) + \sqrt{m}\nu_2 e^{-k_2 t} Z_2$.

B Calculation of $p_{0,0}$, $p_{0,1}$, $p_{1,0}$ and $p_{1,1}$

Based on the results in Cufaro Petroni and Sabino [7] the joint *cdf* of $X_1 \sim \mathfrak{E}_1(\lambda_1)$ and $X_2 \sim \mathfrak{E}_1(\lambda_2)$ is

$$
H(x_1, x_2) = \mathbb{1}_{x_1 \wedge \frac{x_2}{\gamma} \geq 0}\left[\left(1 - e^{-\lambda_1\left(x_1 \wedge \frac{x_2}{\gamma}\right)}\right) - e^{-\lambda_2 x_2}\left(1 - e^{-(\lambda_1 - \gamma\lambda_2)\left(x_1 \wedge \frac{x_2}{\gamma}\right)}\right)\right]
\tag{52}
$$

For Δt small we can assume that no more that one jump can occur hence:

$$
\begin{aligned}
p_{1,1} &= \mathbb{P}(X_1 \leq \Delta t, X_2 \leq \Delta t) = H(\Delta t, \Delta t) \\
p_{1,0} &= \mathbb{P}(X_1 \leq \Delta t, X_2 \geq \Delta t) = F_1(\Delta t) - H(\Delta t, \Delta t) \\
p_{0,1} &= \mathbb{P}(X_1 \geq \Delta t, X_2 \leq \Delta t) = F_2(\Delta t) - H(\Delta t, \Delta t) \\
p_{0,0} &= \mathbb{P}(X_1 \geq \Delta t, X_2 \geq \Delta t) = 1 - p_{1,1} - p_{0,1} - p_{0,1}
\end{aligned}
$$

Finally, the results in section 5.2 are obtained considering only $O(\Delta t)$ terms and splitting between $\gamma > 1$ and $0 \leq \gamma \leq 1$.

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