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Pricing Exchange Options with Correlated Jump-diffusion Processes (Manuscript ID RQUF-2017-0008)

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We study the applicability to energy facilities of a model for correlated Poisson processes generated by self-decomposable jumps. In this context the implementation of our approach, both to shape power or gas dynamics, and to evaluate transportation assets seen as spread or exchange options, is rather natural. In particular we first enhance the Merton market with two underlying assets making jumps at times ruled by correlated Poisson processes. Here however - at variance with the existing literature - the correlation is no longer provided only by a systemic common source of synchronous macroeconomic shocks, but also by a delayed synaptic propagation of the shocks themselves between the assets. In a second step we consider a price dynamics driven by an exponential mean-reverting geometric Ornstein-Uhlenbeck plus compound Poisson: a combination which is well suited for the energy markets. In our specific instance, for each underlying we adopt a jumping price spot dynamics that has the advantage of being exactly treatable to find no-arbitrage conditions. As a result we are able to find closed formulas for vanilla options, so that the price of the spread options can subsequently be calculated (again in closed form) using the Margrabe formula if the strike is zero (exchange options), or with some other suitable procedures available in the literature. The exchange option values obtained in our numerical examples show that, compared to the other Poisson models we analyzed, the dependence introduced by the self-decomposition gives more relevance to the timing of the jumps and not only to their frequency.

1. Introduction and motivations

Several research studies have shown that the spot dynamics of commodity prices is subjected to mean reversion, seasonality and jumps (see for example (5)). In addition, some methodologies have also been proposed to take into account a dependence based on the concepts of correlation and cointegration. These approaches, however, can quickly become non-treatable outside the Gaussian-Itō world. In this paper we then address the problem of dependence in the 2-dimensional case by considering 2-dimensional jump diffusion processes with a 2-dimensional compound Poisson component. In particular, to model the dependence in a 2-dimensional Poisson process, we take advantage of the self-decomposability (see (7, 8, 20)) of the exponential random variables (rv) used for its construction. It is possible to show (see (8)) indeed that given two independent exponential

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The technique here discussed does not reflect UGC view

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rv's $Y, Z \sim \mathfrak{E}(\lambda)$, and a Bernoulli *rv* $B(1) \sim \mathfrak{B}(1, 1-a)$ with $a = \mathbf{P}\{B(1) = 0\}$, then also the *rv* $X = aY + B(1)Z$ is an exponential $\mathfrak{E}(\lambda)$ resulting in a weighed sum of Y and Z: a is the *deterministic* weight of Y, while $B(1)$ is the *random* weight of Z. This fact is a direct consequence of the self-decomposability of the exponential laws. It is apparent on the other hand that, by construction, X and Y are not independent, and it is possible to show that a also represents precisely their correlation coefficient. By the way it is also possible produce pairs of a-correlated exponentials $X \sim \mathfrak{E}(\lambda)$ and $W \sim \mathfrak{E}(\mu)$ with different parameters just by reformulating the previous relation as $X = a\mu W/\lambda + B(1)Z$.

Considering now our correlated, exponential *rv*'s X and Y (or W) as two random times with a positive random delay $B(1)Z$, the present model can help describing their co-movement and can answer some commonplace questions arising in a financial context:

- if a financial institution defaults, how long should one wait for a dependent institution to default?
- if a market receives some news interpreted as a shock, how long should one wait to see the propagation of that shock onto a dependent market?
- if different companies are interlinked, what is the impact on insurance risk?

It turns out that questions like these are covered by the special case $a\mu/\lambda > 1$, so that the present model is rich enough to describe cases where the second random time event occurs before the first one. Similar results based on linear structure of exponential *rv*'s can also be found elsewhere in the literature (23) in papers whose main purpose is to model a multi-component reliability system.

By taking advantage of the self-decomposability of the exponential laws we are then able to construct 2-dimensional Poisson processes with dependent marginals (8), and – with some terminological stretching – the two Poisson processes can be seen as linked by some kind of cointegration between their jumps.

Based on these observations, we propose to introduce a form of, say, synaptic risk-interactions in analogy with what happens for the diffusion of information on a network: the individual reaction in every node propagates with a delay to other nodes, inducing then a new reaction and so on. For example, several previous papers (see (6, 2)) consider a Merton model with two underlying assets where the macro-economic shocks to the system are modeled by common arrival jumps with correlated jump sizes, while independent shocks are added to represent the idiosyncratic components. At variance with (6, 2) we here assume that the macro-economic shocks can impact each underlying asset with random and correlated time delays. This relative timing of the correlated Poisson processes apparently allows for an enhanced flexibility of the model in the practical applications because we no longer have to rely only on common shocks (see (8) for further details on cross-correlation and relative timing).

In the context of the energy facilities, the application of our approach to modeling the price dynamics, and to evaluating the transportation assets seen as spread or exchange options, is rather straightforward. To this purpose, we consider the German *EEX* and French *Powernext* power markets and we assume that each spot price dynamics is driven by an exponential mean-reverting geometric Ornstein-Uhlenbeck (GOU) plus a compound Poisson. In this specific case we adopt a stochastic dynamics of the spot prices that is slightly different from the one proposed in other studies (see (5)), but with the relevant advantage of being more treatable. In particular we are able to find closed-form formulas for vanilla options so that the price of spread options can be calculated in closed form using the Margrabe formula (15) (if the strike is zero), or some other well known approximation as proposed in the existing literature (see (18, 1, 3)). In any case our approach also suggests an explicit algorithm for the simulation of the dependent Poisson processes that can be used in the Monte Carlo simulations.

Finally in the numerical investigation, in order to better highlight the impact of the jump timing of the Poisson processes, we neglect the idiosyncratic component. We then compare the effect of the timing of the Poisson processes based on the self-decomposition to the one of two independent Poisson processes, or to the one of correlated Poisson processes where their correlation is produced by a common Poisson part.

Our numerical examples show that, compared to the other Poisson models we analyzed, modeling the dependence between Poisson process via self-decomposition gives more relevance to the dependence structure between the two markets, to the timing of the jumps and not only to their frequency. In particular the exchange option prices calculated with our methodology coupled to GOU processes with jumps may seem counterintuitive or may even seem wrong if one relies on pure linear correlation assumptions.

The paper is organized as follows: the Section 2 summarizes the notations and the results for a 2-dimensional Poisson process as discussed in (8). In the Section 3 we then consider 2-dimensional jump diffusion processes having a Geometric Brownian Motion (GBM) and a GOU diffusive component. We also apply our method to the 2-factor Schwartz-Smith model in (21) with jump diffusion where we find analytical solutions for vanilla options as well. Subsequently the Section 4 presents the risk neutral formulas for plain vanilla and exchange options given the price dynamics introduced in Section 3 and for several different types of 2-dimensional Poisson components. The Section 5 illustrates then our approach with practical examples: we assume a pure GBM plus jump model and compare the results obtained by our approach to the ones obtained assuming that the two compound Poisson processes are independent, or contain a common Poisson component. In a second step we show the applicability of our approach with GOU plus jumps dynamics to price an interconnection between the *EEX* and *Powernext* day-head prices. The Section 6 finally concludes the paper with an overview of future inquiries and possible further applications.

2. Joint Poisson distributions

A law with density (*pdf*) $f(x)$ and characteristic function (*chf*) $\varphi(u)$ is said to be *self-decomposable* (*sd*) (see (20, 7)) if for every $0 < a < 1$ we can find another law with *pdf* $g_a(x)$ and *chf* $\chi_a(u)$ such that

$$\varphi(u) = \varphi(au)\chi_a(u)$$

We will also say that a *rv* X is *sd* when its law is *sd*: looking at the definition this means that for every $0 < a < 1$ we can always find two *independent* *rv*'s Y (with the same law of X), and Z_a with *pdf* $g_a(x)$ and *chf* $\chi_a(u)$ such that *in distribution*

$$X \stackrel{d}{=} aY + Z_a$$

From a reverse point of view, when for $0 < a < 1$ the law of Z_a is known, we can define the *rv* $X = aY + Z_a$ which (by self-decomposability) will now have the same law of Y. It would be easy to show that a also plays the role of the correlation coefficient between X and Y, namely

$$r_{XY} = a$$

It is well known, in particular, that the exponential laws $\mathfrak{E}(\lambda)$ with *pdf* and *chf*

$$f_1(x) = \lambda e^{-\lambda x} \qquad x \geq 0 \qquad\qquad \varphi_1(u) = \frac{\lambda}{\lambda - iu}$$

are a typical example of *sd* laws (see (20)), and it is possible to show (see (8)) that in this case Z_a turns out to take the form

$$Z_a = B(1)\, Z \sim a\,\boldsymbol{\delta}_0 + (1-a)\,\mathfrak{E}(\lambda)$$

where $Z \sim \mathfrak{E}(\lambda)$, and $B(1) \sim \mathfrak{B}(1, 1-a)$ (a Bernoulli with $a = \boldsymbol{P}\{B(1) = 0\}$) are independent from Y, so that its law is a mixture of a $\boldsymbol{\delta}_0$ degenerate in 0, and an exponential $\mathfrak{E}(\lambda)$. In conclusion, given two exponential *rv*'s $Y \sim \mathfrak{E}(\lambda)$ and $Z \sim \mathfrak{E}(\lambda)$, and a Bernoulli $B(1) \sim \mathfrak{B}(1, 1-a)$ (all mutually independent) the *rv* $X = aY + B(1)Z \sim \mathfrak{E}(\lambda)$ is again an exponential which is now a-correlated with Y.

Now, for given $\lambda_1, \lambda_2 > 0$ and $0 < a < 1$, take first a sequence of *iid rv*'s

$$X_k = aY_k + B_k(1)Z_k \qquad k = 1, 2, \dots$$

in such a way that for every k the *rv*'s X_k, Y_k, Z_k are $\mathfrak{E}(\lambda_2)$, $B_k(1)$ is $\mathfrak{B}(1, 1-a)$, and $Y_k, Z_k, B_k(1)$ are mutually independent (we understand that $X_0 = Y_0 = Z_0 = 0$, $\mathbb{P}$-*a.s.*), and then for $n = 0, 1, 2, \dots$ define the two point processes

$$S_n = \frac{\lambda_2}{\lambda_1} \sum_{k=0}^{n} Y_k \sim \mathfrak{E}_n(\lambda_1) \qquad T_n = \sum_{k=0}^{n} X_k \sim \mathfrak{E}_n(\lambda_2)$$

where the symbols $\mathfrak{E}_n(\lambda)$ denote the Erlang (gamma) laws with *pdf*'s and *chf*'s

$$f_n(x) = \lambda \frac{(\lambda x)^{n-1}}{(n-1)!} e^{-\lambda x} \qquad x \geq 0\,, \qquad \varphi_k(u) = \left(\frac{\lambda}{\lambda - iu}\right)^n \qquad n = 0, 1, 2, \dots$$

and it is understood that $\mathfrak{E}_0 = \boldsymbol{\delta}_0$, while $\mathfrak{E}_1(\lambda) = \mathfrak{E}(\lambda)$. Define finally the two *dependent* Poisson processes $N_1(t) \sim \mathfrak{P}(\lambda_1 t)$ and $N_2(t) \sim \mathfrak{P}(\lambda_2 t)$ associated respectively to S_n and T_n: for the purposes of the present paper it is instrumental now to have an explicit form of the joint probabilities

$$p_{m,n}(t) = \boldsymbol{P}\{N_1(t) = m,\, N_2(t) = n\} \qquad n, m = 0, 1, 2, \dots \qquad t \geq 0$$

To this end let us introduce first the shorthand notations

$$\pi_k(\alpha) = e^{-\alpha} \frac{\alpha^k}{k!} \qquad k = 0, 1, \dots$$
$$\beta_\ell(n) = \binom{n}{\ell} a^{n-\ell} (1-a)^\ell \qquad \ell \leq n = 0, 1, \dots$$

respectively for the distributions of a Poisson $\mathfrak{P}(\alpha)$, and of a binomial $\mathfrak{B}(n, 1-a)$ (it is understood that $\beta_0(0) = 1$): then it is possible to prove (see (8)) that when $a\lambda_1 \geq \lambda_2$, it is

$$p_{m,n}(t) = \begin{cases} 0 & n > m \geq 0 \\ Q_{n,n}(t) & m = n \geq 0 \\ Q_{m,n}(t) - Q_{m,n+1}(t) & m > n \geq 0 \end{cases}$$
$$Q_{m,n}(s,t) = \sum_{k=n}^{m} (-1)^k \sum_{j=k}^{m} \binom{j}{k} \frac{\pi_{m-j}(\lambda_1 t)}{(-a)^j} \sum_{\ell=0}^{n} \beta_\ell(n) \pi_{j+\ell}(\lambda_2 t)\, \Phi(j+1, j+\ell+1; \lambda_2 t)$$

while on the other hand when $a\lambda_1 \leq \lambda_2$, we have

$$p_{m,n}(t) = \begin{cases} A_{m,n}(t) - A_{m,n+1}(t) + B_{m,n}(t) - B_{m,n-1}(t) & n > m \geq 0 \\ A_{n,n}(t) - A_{n,n+1}(t) + B_{n,n}(t) + C_{n,n}(t) & m = n \geq 0 \\ A_{m,n}(t) - A_{m,n+1}(t) + C_{m,n}(t) - C_{m,n+1}(t) & m > n \geq 0 \end{cases}$$

where, taking for short $w = \lambda_2 t - a\lambda_1 t$, we understand that for every $n, m \geq 0$ it is

$$A_{m,n}(t) = \pi_m(\lambda_1 t) \sum_{k=0}^{n} \beta_k(n) \left[1 + \pi_k(w) - \sum_{j=0}^{k} \pi_j(w)\right]$$

for $n \geq m \geq 0$ it is

$$B_{m,n}(t) = \pi_m(\lambda_1 t) \sum_{k=0}^{n-m} \pi_k\left(\frac{w}{a}\right) \sum_{\ell=0}^{n+1} \beta_\ell(n+1) \frac{w^\ell k!}{(k+\ell)!} \Phi\left(\ell, k+\ell+1, \frac{1-a}{a} w\right)$$

and finally for $m \geq n \geq 1$ it is (for $n = 0$ we have $C_{m,0}(t) = 0$)

$$C_{m,n}(t) = \frac{e^{-(1-a)\lambda_1 t}}{a^m} \sum_{\ell=1}^{n} \beta_\ell(n) \sum_{k=n}^{m} \sum_{j=0}^{\ell-1} \binom{k+\ell-j-1}{k} \times (-1)^{\ell-1-j} \pi_j(\lambda_2 t) \pi_{m+\ell-j}(a\lambda_1 t)\, \Phi(k+\ell-j, m+\ell-j+1; a\lambda_1 t)$$

Here $\Phi(j+1, j+\ell+1; z)$ for $0 \leq \ell \leq n \leq j \leq m$ are the confluent hypergeometric functions (see (9) for their definition and properties) which in fact are here just elementary functions (see (8)).

3. The market models

We will now adapt the ideas presented in the Section 2 to a financial context, and to this end we will consider first a Black-Scholes (BS) market, then a market led by geometric Ornstein Uhlenbeck (GOU) processes with jumps similar to that adopted elsewhere in the literature (see (5)), and finally a Schwartz-Smith model with jumps.

3.1. *The GBM plus jumps case*

We consider a generalization of the Merton model (16) introduced in (6) and (2) with two underlying assets. The original setting in (6) and (2) assumes common arrival jumps related to macro-economic shocks and additionally independent shocks relative to the idiosyncratic component. We propose to introduce a third form of, say, synaptic interactions in analogy with what happens for the diffusion of information on a network: the individual reaction in every node propagates with a delay to other nodes, inducing then a new reaction and so on.

Following the notation in (12), we consider dynamics driven by:

$$S_i(T) = \exp\left[\log S_i(0) + \left(\mu_i - \frac{1}{2}\sigma_i^2\right) T + \sigma_i W_i(T) + \sum_{n_i=1}^{N_i(T)} \log J_{i,n_i} + \sum_{p_i=1}^{P_i(T)} \log Q_{i,p_i}\right], \quad i = 1, 2, \tag{1}$$

with $dW_1(t)\, dW_2(t) = \rho^{(W)} dt$, where $\rho^{(W)}$ denotes the constant correlation among the two BM's, and with log-normal jump sizes

$$J_i = M_i \exp\left(-\frac{\nu_i^2}{2} + \nu_i Z_i\right), \quad i = 1, 2; \tag{2}$$

where $Z_i \sim \mathfrak{N}(0,1)$ are standard normal with $\boldsymbol{Corr}(Z_1, Z_2) = \rho^{(D)}$. Analogously,

$$Q_i = A_i \exp\left(-\frac{\alpha_i^2}{2} + \nu_i Y_i\right), \quad i = 1,2; \tag{3}$$

where $Y_i \sim \mathfrak{N}(0,1)$ are independent standard normal also independent of Z_i. The parameters $M_i, A_i, \nu_i, \alpha_1$ are scalar constants, the Poisson processes $N_i(t)$ are defined in section 2 while $P_i(t)$ are two independent Poisson processes with intensity ψ_i, $i = 1,2$ also independent of $N_i(t)$.

We then focus our attention on the logarithm

$$\log S_i(T) \stackrel{d}{=} \log S_i(0) + \left(\mu_i - \frac{1}{2}\sigma_i^2\right) T + \sigma_i W_i(T) + N_i(T) \log M_i + P_i(T) \log A_i$$
$$- \frac{\nu_i^2}{2} N_i(T) - \frac{\alpha_i^2}{2} P_i(T) + \nu_i \sum_{n_i=1}^{N_i(T)} Z_{i,n_i} + \alpha_i \sum_{p_i=1}^{P_i(T)} Y_{i,p_i}, \qquad i = 1,2. \tag{4}$$

that can also be rewritten as

$$\log S_i(T) \stackrel{d}{=} \log S_i(0) + N_i(T) \log M_i + P_i(T) \log A_i + \left(\mu_i - \frac{1}{2}\sigma_i^2 - \frac{\nu_i^2}{2} N_i(T) - \frac{\alpha_i^2}{2} P_i(T)\right) T$$
$$+ \sqrt{\sigma_i^2 T + N_i(T)\nu_i^2 + P_i(T)\alpha_i^2}\, H_i, \qquad i = 1,2, \tag{5}$$

where for given $N_i(t) = \ell_i$, $P_i(t) = m_i$, $(H_1, H_2) \sim \mathfrak{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho^{(J)}_{\ell_1,\ell_2,m_1,m_2} \\ \rho^{(J)}_{\ell_1,\ell_2,m_1,m_2} & 1 \end{pmatrix}\right)$. An explicit calculation of $\rho^{(J)}_{\ell_1,\ell_2,m_1,m_2}$ can be found in the Appendix A.

For short we will denote as $v^{(J)}_{i,\ell_i,m_i}(T)$ or $\left(\sigma^{(J)}_{i,\ell_i,m_i}\right)^2$ the terminal variance

$$v^{(J)}_{i,\ell_i,m_i}(T) = \left(\sigma^{(J)}_{i,\ell_i,m_i}\right)^2 = \sigma_i^2 T + \ell_i \nu_i^2 + m_i \alpha_i^2 = v_i^{(C)}(T) + v^{(D)}_{i,\ell_i,m_i}, \tag{6}$$

where $v_i^{(C)}$ and $v_i^{(D)}$ are the terminal variances of the continuous and discontinuous parts. In the case of time-dependent volatility functions, it is easy to see that the formulas still hold by replacing $v_i^{(C)}(T)$ with $\int_0^T \sigma_i^2(s)\, ds$.

In (6) it is presented a general change of measure that allows to take into account all the jump risks. The main goal of our study, however, is to show the impact of assuming the inter-arrival times of the Poisson processes to be correlated via self-decomposition. Although the application of the change of measure in (6) is straightforward to our dynamics, we will hereafter adopt the original choice of Merton (16) where all the jump risks are unpriced.

The no-arbitrage conditions then imply (see (12) p. 344)

$$\mu_i - r = -\lambda_i \boldsymbol{E}[J_i - 1] - \psi_i \boldsymbol{E}[Q_i - 1] \quad i = 1,2. \tag{7}$$

3.2. *The Ornstein-Uhlenbeck plus jumps case*

Energy markets often display mean-reversion and jumps. By considering then a one-factor model plus a jump component similar to that introduced in the previous subsection, we take a market driven by the stochastic process

$$S_i(t) = F_i(0,t)\exp\left\{U_i(t) + h(t)\right\}, \quad i = 1,2, \tag{8}$$

where $h(t)$ is a pure deterministic function, $F_i(0,t)$ is the forward curve at time $t = 0$ and

$$U_i(t) = U_i(0)e^{-k_i t} + \sigma_i \int_0^t e^{-k_i(t-s)} dW_i(s) + e^{-k_i t}\left(\sum_{n_i=1}^{N_i(t)} Z_{i,n_i} + \sum_{p_i=1}^{P_i(t)} Y_{i,p_i}\right) = U_i^C(t) + U_i^{D_1}(t) + U_i^{D_2}(t). \tag{9}$$

Moreover the process $U_i(t)$ turns out to be driven by an OU process with jumps according to the *SDE*

$$dU_i(t) = -k_i U_i(t)dt + \sigma_i dW_i(t) + e^{-k_i t}\left(Z_i dN_i(t) + Y_i dP_i(t)\right), \tag{10}$$

where k_i represents the mean reversion rate and σ_i the diffusion coefficient. In contrast to the GBM case of the previous section, we have defined the jump sizes as Z_{i,n_i} being independent copies of $Z_i \sim \mathfrak{N}(M_i, \nu_i^2)$ and $\boldsymbol{Corr}(Z_1, Z_2) = \rho^{(D)}$. Analogously, Y_{i,p_i} are copies of $Y_i \sim \mathfrak{N}(A_i, \alpha_i^2)$ with $\boldsymbol{Corr}(Y_1, Y_2) = 0$ also independent of Z_{i,n_i}.

Our spot *SDE* however is slightly different from the one adopted in (5), because the exponential coefficient in front of the jump component is chosen in such a way that the solution has no random jumps with time-dependent jump size. This implies that our model has time-decreasing jump sizes driven by the mean reversion rate k_i. Had we considered instead, as in (5), the *SDE*

$$dU_i(t) = -k_i U_i(t)dt + \sigma_i dW_i(t) + Z_i dN_i(t) + Y_i dP_i(t),$$

the solution would have been

$$U_i(t) = U_i(0)e^{-k_i t} + \sigma_i \int_0^t e^{-k_i(t-s)} dW_i(s) + e^{-k_i t}\left(\sum_{n_i=1}^{N_i(t)} Z_{i,n_i} e^{k_i T_{i,n_i}} + \sum_{p_i=1}^{P_i(t)} Y_{i,p_i} e^{k_i G_{i,p_i}}\right),$$

where T_{i,p_i} and G_{i,p_i} are the jump times of $N_i(t)$ and $P_i(t)$, respectively. Our setting however turns out to be advantageous because it leads to more tractable option formulas as it will be shown here in the following.

The power market of the model presented is obviously incomplete and hence the discounted spot price process in the risk-neutral measure is not necessarily a martingale. From now on we assume that the model is specified in the risk-neutral measure $\mathbb{Q}$: this will be for simplicity hereafter implicitly understood in the expectations. We use the same approach of the Lemma 3.1 in (10) to get the deterministic function $h(t)$ consistent with forward curve.

In order to get no-arbitrage conditions, we then impose $\boldsymbol{E}\left[S(T)|\mathcal{F}_t\right] = F(t,T)$ and for simplicity we look at $\boldsymbol{E}[S(T)] = F(0,T)$. We then need to compute the three terms in $\boldsymbol{E}\left[e^{U_i^C(t)+U_i^{D_1}(t)+U_i^{D_2}(t)}\right] = \boldsymbol{E}\left[e^{U_i^C(t)}\right]\boldsymbol{E}\left[e^{U_i^{D_1}(t)}\right]\boldsymbol{E}\left[e^{U_i^{D_2}(t)}\right]$. It is well known that:

$$\boldsymbol{E}\left[e^{U_i^C(t)}\right] = \exp\left(\boldsymbol{E}\left[U_i^C(t)\right] - \frac{1}{2}\boldsymbol{V}\left[U_i^C(t)\right]\right) = e^{a_i(t)}. \tag{11}$$

with:

$$\begin{aligned} \boldsymbol{E}\left[U_i^C(t)\right] &= U(0)e^{-k_i t}, \\ \boldsymbol{V}\left[U_i^C(t)\right] &= \frac{\sigma_i^2}{2k_i}\left(1-e^{-2k_i t}\right). \end{aligned} \tag{12}$$

Without loss of generality, hereafter we will assume that $U_i(0) = 0, i = 1, 2$. Finally, we need then to calculate:

$$\boldsymbol{E}\left[e^{U_i^{D_1}(t)}\right] = \boldsymbol{E}\left[\exp\left(e^{-k_i t}\sum_{n_i=1}^{N_i(t)} Z_{i,n_i}\right)\right] = e^{b_i(t)}. \tag{13}$$

$$\boldsymbol{E}\left[e^{U_i^{D_2}(t)}\right] = \boldsymbol{E}\left[\exp\left(e^{-k_i t}\sum_{p_i=1}^{P_i(t)} Y_{i,p_i}\right)\right] = e^{c_i(t)}. \tag{14}$$

Knowing the moment-generating function of the compound Poisson process:

$$\phi_1(u) = \boldsymbol{E}\left[\exp\left\{u\sum_{n=1}^{N_i(t)} Z_{i,n_i}\right\}\right] = \exp\left\{\lambda_i t\left(\phi_{Z_i}(u)-1\right)\right\} \tag{15}$$

where

$$\phi_{Z_i}(u) = \exp\left\{M_i u + \frac{1}{2}\nu_i^2 u^2\right\} \tag{16}$$

we easily obtain the required expected value.

$$\boldsymbol{E}\left[e^{U_i^{D_1}(t)}\right] = \phi_1\left(e^{-k_i t}\right). \tag{17}$$

$$b_i(t) = \lambda_i t\left(e^{e^{-k_i t}\left(M_i + \frac{1}{2}e^{-k_i t}\nu_i^2\right)} - 1\right) \tag{18}$$

Analogous calculations yield:

$$c_i(t) = \psi_i t\left(e^{e^{-k_i t}\left(A_i + \frac{1}{2}e^{-k_i t}\alpha_i^2\right)} - 1\right) \tag{19}$$

Based on the results above the no-arbitrage is given by $h_i(t) = -a_i(t) - b_i(t) - c_i(t)$, and the equations for the spot dynamics above can be rewritten as:

$$\begin{aligned} \log S_i(t) &\overset{d}{=} \log F_i(0,t) - b_i(t) - c_i(t) + e^{-k_i t}\left(M_i N_i(t) + A_i P_i(t)\right) + \frac{1}{2}e^{-2k_i t}\left(\nu_i^2 N_i(t) + \alpha_i^2 P_i(t)\right) \\ &-\frac{1}{2}\left(\boldsymbol{V}\left[U_i^{(C)}(t)\right] + e^{-2k_i t}\left(\nu_i^2 N_i(t) + \alpha_i^2 P_i(t)\right)\right) + \\ &\sqrt{\boldsymbol{V}\left[U_i^{(C)}(t)\right] + e^{-2k_i t}\left(\nu_i^2 N_i(t) + \alpha_i^2 P_i(t)\right)}\, H_i \end{aligned} \tag{20}$$

where as done in the previous section for fixed $N_i(T) = \ell_i$ and $P_i(t) = m_i$ we define:

$$v^{(J)}_{i,\ell_i,m_i}(T) = \boldsymbol{V}\left[U^{(C)}_i(t)\right] + e^{-2k_i t}\left(\nu_i^2\ell_i + \alpha_i^2 m_i\right) = v^{(C)}_i(T) + v^{(D)}_{i,\ell_i,m_i}, \tag{21}$$

and $H_i,\ i = 1,2$ accordingly.

3.3. *The Schwartz-Smith plus jumps case*

In this section we consider the two factor Schwartz-Smith model as in (21) plus a jump-diffusion component:

$$\begin{aligned} U_1(t) &= U_1(0)e^{-kt} + \sigma_1\int_0^t e^{-k(t-s)}dW_1(s) + e^{-kt}\sum_{n_1=1}^{N_1(t)} Z_{1,n_1} \\ U_2(t) &= U_2(0) + \mu t + \sigma_2 W_2(t) + \sum_{n_2=1}^{N_2(t)} Z_{2,n_2} + \sum_{p=1}^{P(t)} Y_p \\ U(t) &= U_1(t) + U_2(t) \end{aligned} \tag{22}$$

where $S(t) = F(0,t)\,e^{h(t)+U(t)}$, Z_{i,n_i} being independent copies of $Z_i \sim \mathfrak{N}(M_i,\nu_i^2)$ and $\boldsymbol{Corr}(Z_1,Z_2) = \rho^{(D)}$, and where finally Y_p are copies of $Y \sim \mathfrak{N}(A,\alpha)$ once more independent of Z_{i,n_i}. Simply taking the differential and after some algebra we get

$$dU(t) = k\left(\mu + U_2(t) - U(t)\right)dt + \sigma dW(t) + Z_1\left(e^{-kt}dN_1(t) + Z_2 dN_2(t) + Y dP(t)\right) \tag{23}$$

where $\sigma^2 = \sigma_1^2 + \sigma_2^2 + 2\sigma_1\sigma_2\rho^{(W)}$.

With the same procedure outlined in the previous subsection, the no-arbitrage conditions can be obtained by taking the (conditional) expectation of the spot process

$$\begin{aligned} \boldsymbol{E}\left[e^{U^C(t)}\right] &= \boldsymbol{E}\left[e^{U_1(0)e^{-kt}+\sigma_i\int_0^t e^{-k(t-s)}dW_1(s)+U_2(0)+\mu t+\sigma_2 W_2(t)}\right] \\ &= \exp\left(\boldsymbol{E}\left[U^C(t)\right] - \frac{1}{2}\boldsymbol{V}\left[U^C(t)\right]\right) = e^{a(t)}. \end{aligned} \tag{24}$$

Assuming then once more $U_1(0) = 0$ and $U_2(0) = 0$ we have

$$\begin{aligned} \boldsymbol{E}\left[U^C(t)\right] &= \mu t, \\ \boldsymbol{V}\left[U^C(t)\right] &= \frac{\sigma_1^2}{2k}\left(1 - e^{-2kt}\right) + \sigma_2^2 t + \frac{2\rho\sigma_1\sigma_2}{k}\left(1 - e^{-kt}\right). \end{aligned} \tag{25}$$

After doing the transformation $\epsilon_1 \stackrel{d}{=} Z_1, \epsilon_2 \stackrel{d}{=} \rho^{(D)}Z_1 + \sqrt{1-(\rho^{(D)})^2}$, and some algebra we can write for the discontinuous component:

$$e^{b(t)} = \boldsymbol{E}\left[e^{U^{D_1}(t)}\right] = \boldsymbol{E}\left[\exp\left(e^{-kt}\sum_{n_1=1}^{N_1(t)} Z_{1,n_1} + \sum_{n_2=1}^{N_2(t)} Z_{2,n_2}\right)\right] =$$
$$= \sum_{\ell_1,\ell_2=0}^{+\infty} p_{\ell_1,\ell_2} \exp\left(\ell_1 M_1 e^{-kt} + \ell_2 M_2\right) \exp\left(\left(\sqrt{\ell_1}\nu_1 e^{-kt} + \rho^{(D)}\sqrt{\ell_2}\nu_2\right)^2\right) \exp\left(\ell_2\nu_2^2(1-(\rho^{(D)})^2)\right) \tag{26}$$

and

$$e^{c(t)} = \boldsymbol{E}\left[e^{U^{D_2}(t)}\right] = \boldsymbol{E}\left[\sum_{p=1}^{P(t)} Y_p\right] = \exp\left(\psi t\left(\phi_Y(1)-1\right)\right), \tag{27}$$

where $\phi(1) = \exp\left(A + \frac{1}{2}\alpha^2\right)$ and $c(t) = \psi t\left(\phi_Y(1)-1\right)$ As done in Section 3.1, the no-arbitrage is found by choosing $h(t) = -a(t) - b(t) - c(t)$ where $b(t)$ can be computed numerically. Finally, after some algebra, our log-spot dynamics can be rewritten as

$$\begin{aligned}\log S_i(t) \stackrel{d}{=} & \log F(0,t) - b(t) - c(t) + e^{-kt}M_1 N_1(t) + M_2 N_2(t) + AP(t) + \\ & \frac{1}{2}\left[\left(\nu_1\sqrt{N_1(t)}e^{-kt} + \rho^{(D)}\nu_2\sqrt{N_2(t)}\right)^2 + \nu_2^2 N_2(t)\left(1-(\rho^{(D)})^2\right) + \alpha^2 P(t)\right] - \\ & \frac{1}{2}v^{(J)}_{N_1(t),N_2(t),P(t)} + \sqrt{v^{(J)}_{N_1(t),N_2(t),P(t)}}H.\end{aligned} \tag{28}$$

where

$$v^{(J)}_{N_1(t),N_2(t),P(t)} = \boldsymbol{V}\left[U^{(C)}(t)\right] + \left(\nu_1\sqrt{N_1(t)}e^{-kt} + \rho^{(D)}\nu_2\sqrt{N_2(t)}\right)^2 + \nu_2^2 N_2(t)\left(1-(\rho^{(D)})^2\right) + \alpha^2 P(t). \tag{29}$$

and H is a standard normal random variable.

4. Risk-neutral pricing formulas

4.1. *The European Plain Vanilla options case*

In order to simplify the calculations, we represent the price of a call option at time zero, $c(0)$, in terms of an abstract BS formula:

$$c(0) = \mathrm{BS}\left(P_0, K, r, T, v, q\right) \tag{30}$$

where P_0, K, r, T, v, q denote the initial price, strike, risk-free rate, maturity, terminal variance and dividend yield. Following the procedure shown in (12), getting a vanilla option pricing formula will then be a matter of plugging the terminal variance, the initial price and the dividend yield into the abstract formula after applying a conditioning argument on the Poisson probabilities.

- **GBM case**. Rearranging Equation (5) given the condition (7) we have:

$$\log S_i(T) = \log S_i(0) + N_i(T)\log M_i + P_i(T)\log A_i + \lambda_i(1-M_i)T + \psi_i(1-A_i)T \quad (31)$$
$$\left(r - \frac{v_i^{(J,N_i(T))}}{2}\right) + \sqrt{v_i^{(J,N_i(T))}}H_i.$$

The price of a call (put) option on underlying asset $i = 1$ given the GBM-plus-jumps market of Equation (1) is then

$$c(0) = \sum_{n_1,p_1=0}^{\infty} \pi_{n_1}(\lambda_1 T)\pi_{p_1}(\psi_1 T)\,\mathrm{BS}(S_{1,n_1,p_1}(0), K, r, T, v_{1,n_1,p_1}^{(J)}(T), 0) \quad (32)$$

where

$$S_{1,n_1,p_1}(0) = S_1(0)M_1^{n_1}A_1^{p_1}\exp\left[\lambda_1 T(1-M_1) + \psi_1 T(1-A_1)\right] \quad (33)$$

and $v_{1,n_1,p_1}^{(J)}(T)$ is defined in Equation (6).

- **GOU case**. In contrast, for the OU-plus-jumps market (8), $r = 0$, the initial price argument for the abstract BS formula is

$$S_{1,n_1,p_1}(0) = F_1(0,T)e^{\beta_{1,n_1,p_1}(T)}, \quad (34)$$

where $v_{1,n_1,p_1}^{(J)}(T)$ is defined in Equation (21) and

$$\beta_{1,n_1,p_1}(t) = -b_1(t) - c_1(t) + e^{-k_1 t}\left[n_1\left(\frac{1}{2}e^{-k_1 t}\nu_1^2 + M_1\right) + p_1\left(\frac{1}{2}e^{-k_1 t}\alpha_1^2 + A_1\right)\right]. \quad (35)$$

- **Schwartz-Smith case**. Assuming finally (22), a semi-closed form formula can be found following the procedure outlined in the GBM and GOU cases

$$c(0) = \sum_{n_1,n_2,p=0}^{\infty} p_{n_1,n_2}(T)\pi_p(\psi_T)BS(S_{n_1,n_2,p}(0), K, 0, T, v_{n_1,n_2,p}^{(J)}(T), 0). \quad (36)$$

where $p_{n_1,n_2}(T)$ is defined in section 2,

$$S_{n_1,n_2,p}(0) = F(0,t)e^{\beta_{n_1,n_2,p}(T)}, \quad (37)$$

$$\beta_{n_1,n_2,p}(t) = -b(t) - c(t) + e^{-kt}M_1 n_1 + M_2 n_2 + Ap + \quad (38)$$
$$\frac{1}{2}\left[\left(\nu_1\sqrt{n_1}e^{-kt} + \rho^{(D)}\nu_2\sqrt{n_2}\right)^2 + \nu_2^2 n_2\left(1-(\rho^{(D)})^2\right) + \alpha^2 p\right]$$

and $v_{n_1,n_2,p}^{(J)}(T)$ is defined in Equation (29).

Having obtained the risk-neutral conditions on each underlying asset, it is straightforward to obtain formulas for exchange options.

4.2. *Exchange options*

The application to spread options turns out to be the natural framework to compare our approach based on self-decomposable jumps, with other jump-diffusion cases. We start considering exchange options with zero-strike: building on the results of (15), with the same conditioning approach applied in the Section 4.1 we find that the price of an exchange option given the market (1) or (8) is

$$s(0) = \sum_{n_1,n_2,p_1,p_2=0}^{\infty} p_{n_1,n_2}(T)\pi_{p_1}(\psi_1 T)\pi_{p_2}(\psi_2 T)BS(S_{1,n_1,p_1}(0), S_{2,n_2,p_2}(0), 0, T, v^{(M)}_{n_1,n_2,p_1,p_2}(T), 0) \tag{39}$$

where $v^{(M)}_{n_1,n_2,p_1,p_2}(T) = v^{(J)}_{1,n_1,p_1}(T) + v^{(J)}_{2,n_2,p_2}(T) - 2\rho^{(J)}_{n_1,n_2,p_1,p_2}\sqrt{v^{(J)}_{1,n_1,p_1} v^{(J)}_{2,n_2,p_2}}$ is the spread terminal variance.

In the available literature different analytical approximations are available when the strike is not zero (see for instance (18, 1, 13)), moreover the extension to the jump diffusion case is also addressed in recent papers (see (17, 3, 4)).

For example, the methods discussed in (17) or (3) are applicable if the characteristic function of the underlying forming the spread is known analytically. Such approaches could also be used in our model but, since this mainly requires the knowledge of the characteristic function of the bivariate self-decomposable Poisson process in simple form, we will postpone this enquiry to future studies. For the time being the extension to the full spread option case will rely just on the truncation of the infinite series paired with one of the approximation procedures applicable to the log-normal case. Hence the computational cost depends, on one hand, on the truncation of the infinite series, as for the case with the zero strike, and on the other hand, on the computational effort needed by the approximation procedure chosen to calculate the spread option for the log-normal case for given n_1, n_2, p_1, p_2. Moreover, the Greeks can be obtained by simply using either the results of the BS model in the case of zero strike or again one of the numerical methods or approximation available for the log-normal case with strike different from zero.

The use of Monte Carlo methods is also possible by generating the 2-dim dependent and self-decomposable *rv*'s that constitute the Poisson process. The current framework however cannot for the time being cope with a multi-asset spread option case unless one considers that the third asset has no jump term. The problem of pricing the multi-assets spread options in any case can be tackled via simulations and analytical approximations as done in (19) and (25), or by applying moment matching and using one of the solutions available for two legs as in (24).

5. Numerical experiments

In this section we presents some numerical experiments assuming the GBM and GOU dynamics plus jumps as explained in the previous sections.

In order to better investigate the features of the self-decomposable correlated Poisson processes, we neglect for the time being the component of common jumps in the GBM and GOU and implement our numerical study focusing on the pair $N_1(t)$ and $N_2(t)$. In the following, we compare three different Poisson process models:

- **Independent jumps: here** $N_1(t)$ and $N_2(t)$ are independent Poisson processes, and the exchange option formula is

$$s(0) = \sum_{n_1,n_2=0}^{\infty} \pi_{n_1}(\lambda_1 T)\pi_{n_2}(\lambda_2 T)\,\mathrm{BS}(S_{1,n_1}(0), S_{2,n_2}(0), 0, T, v^{(M)}_{n_1,n_2}(T), 0) \tag{40}$$

- **Correlation via common jumps:** when instead $N_i(t) = N(t) + N_i^X$, $i = 1, 2$, where $N(t)$, with intensity λ and N_i^X are mutually independent Poisson processes, the exchange option formula is

$$s(0) = \sum_{\substack{n=0 \\ n_1,n_2 \geq n}}^{\infty} \pi_{n_1-n}(\lambda_1^X T)\pi_{n_2-n}(\lambda_2^X T)\pi_n(\lambda T) \times \mathrm{BS}(S_{1,n_1-n}(0), S_{2,n_2-n}(0), 0, T, v^{(M)}_{n_1-n,n_2-n}, 0) \quad (41)$$

- **Correlation via self-decomposability:**finally if we take $N_i(t)$, $i = 1, 2$ as described in the Section 4.2, the exchange option formula becomes

$$s(0) = \sum_{n_1,n_2=0}^{\infty} p_{n_1,n_2}(T)\,\mathrm{BS}(S_{1,n_1}(0), S_{2,n_2}(0), 0, T, v^{(M)}_{n_1,n_2}(T), 0) \quad (42)$$

  where the joint probabilities $p_{n_1,n_2}(T) = \boldsymbol{P}\{N_1(T) = n_1,\ N_2(T) = n_2\}$ are taken from the Section 2.

The payoff of the exchange options illustrated above considers the values of the two underlying at the same time T: other types of exchange options however look at the two underlying at different times, for instance the payoff could be $(S_1(T_1) - S_2(T_2))^+$, $T_2 < T_1$. In this case one needs to readapt the formulas by considering the probabilities at $p_{n_1 n_2} = \boldsymbol{P}\{N_1(T_1) = n_1,\ N_2(T_2) = n_2\}$: explicit formulas for these probabilities can be found in (8).

The case with GBM considers realistic parameters and is meant to study the exchange option values with different types of bivariate Poisson processes; the GOU case instead is based on real data from *EEX* and *Powernext* day-ahead prices. Hereafter we show that, compared to the other Poisson models we analyzed, modeling the dependence between Poisson process via self-decomposition gives more relevance to the dependence structure between the two markets, to the timing of the jumps and not only to their frequency. In particular, our numerical example with real data shows cases where the exchange option prices can assume values that are counterintuitive or may seem even wrong if one relies on pure linear correlation assumptions.

Finally, the computational cost depends on the choice to truncate the infinite series. In our numerical example, we adopt the strategy of truncating the terms higher than $n_{i,max} = 3\lfloor \lambda_i T \rfloor$ where $\lfloor \cdot \rfloor$ denotes the integer part.

5.1. *GBM: Applications to the exchange options*

We compare the exchange option values obtained using the equations (40)-(42) by changing the correlation between the two Poisson processes. In particular, assuming $N_i = N(t) + N_i^X(t)$ we have $\boldsymbol{Cov}(N_1(t), N_2(t)) = \boldsymbol{V}[N(t)] = \lambda t$, then the instantaneous correlation is $\boldsymbol{Corr}(N_1(t), N_2(t)) = \rho_{N_1 N_2} = \frac{\lambda}{\sqrt{\lambda_1 \lambda_2}}$ and is time-independent; in case of self-decomposable jumps this can be obtained numerically. It is apparent that in the case $\lambda_1 \neq \lambda_2$ a perfect correlation cannot be achieved. We consider two cases:

A) $\lambda_1 = \lambda_2 = 20$, and hence for self-decomposable jumps with $a < 1$ we always have $a\lambda_1 < \lambda_2$;
B) $\lambda_1 = 40$, $\lambda_2 = 20$, where for self-decomposable jumps we find either $a\lambda_1 < \lambda_2$ or $a\lambda_1 > \lambda_2$ respectively for $a < 0.5$ and $a > 0.5$.

Our choice for the parameters is shown in the Table 1: in both the cases we consider an at-the-money spread option with zero strike, $K = 0$ and maturity $T = 1$ so that we can use the exact Margrabe formula. One can also use suitable approximation techniques for the spread option value with non-zero strikes without changing the validity of our tests. We have also computed (see Table

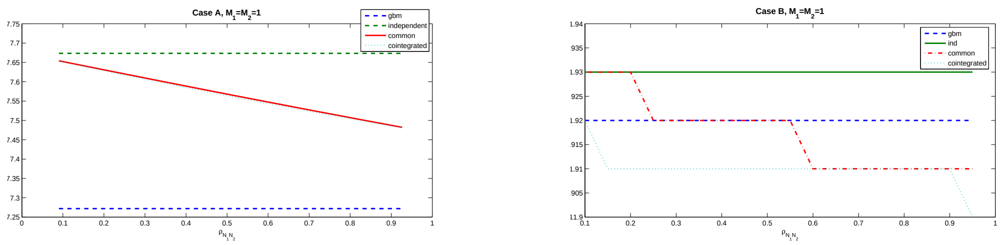
Table 1. Parameters of the GBM and compound Poisson processes

(a) Continuous Part.

	No Jump		With Jumps
	Case A	Case B	
$S_1(0)$	100	100	100
$S_2(0)$	100	100	100
σ_1	0.49	0.37	0.2
σ_2	0.35	0.23	0.15
$\rho^{(W)}(\%)$	96	60	80

(b) Discontinuous Part

	Case A	Case B
$\rho^{(D)}(\%)$	99	50
λ_1	20	40
λ_2	20	20
ν_1	0.10	0.05
ν_2	0.07	0.04
M_1	1.1	1.05
M_2	1.1	1.05



(a) Case A (b) Case B

Figure 1. Exchange option values in the cases A and B when $M_1 = M_2 = 1$

Table 2. Spread option values without jumps and independent compound Poisson processes.

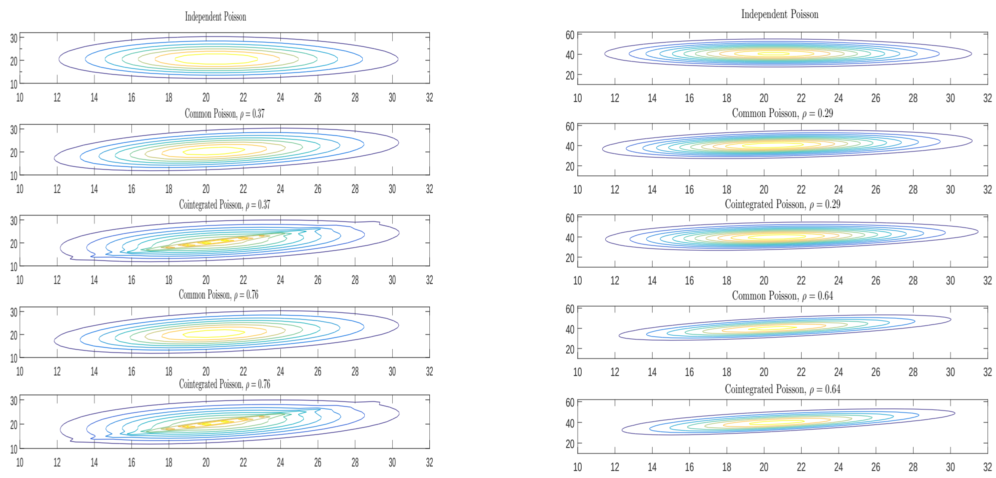
	No Jump		Independent Jumps	
	Case A	Case B	Case A	Case B
Option Value	7.27	11.92	25.23	19.27

Table 3. Comparison of computational times (msec).

Time (msec)	Independent	Common	SD
Case A	489	706	898
Case B	636	862	975

2) the exchange option value for a pure GBM with the parameters listed in the first column of (a) in Table 1: these parameters are suitably chosen in order to match the average spread terminal variance $v^{(M)}_{\lfloor \lambda_1 T \rfloor, \lfloor \lambda_2 T \rfloor}$, where $\lfloor \cdot \rfloor$ denotes the integer part.

The numerical calculations have been implemented in MATLAB 2015b on a 64bit computer with Processor with CPU of 2.3 GHz and with 8 GB RAM. Table 3 shows the computational times needed in all the three cases. We remark first of all that the outcomes of the equations (40) - (42) depend both on the values of the probabilities p_{n_1,n_2}, and on the values of the BS formulas separately, and that in their turn the probabilities p_{n_1,n_2} do not depend on the distribution of the jump size, while the BS formulas are independent from the structure of dependence between the Poisson processes. The Figure 1 clearly shows that the expected jump size has a relevant impact in the option value because under the assumption $M_1 = M_2 = 1$ the price is almost independent from the choice of the Poisson model. The Figure 2 on the other hand displays the differences among the joint probabilities of the Poisson processes. As expected, for independent Poisson, both in case A and B, the isolines of the contour plot of p_{n_1,n_2} resemble to a sort of ellipse whose axis are parallel to the X-Y axis. The positive correlation of the Poisson processes is on the other hand reflected in the fact that the axes are now rotated counterclockwise. However, in contrast to the Poisson process with common jumps, the one constructed via self-decomposition concentrate more probability mass around its expected value and when the difference between n_1 and n_2 is small.



(a) $\lambda_1 = \lambda_2 = 20$ (b) $\lambda_1 = 40$, $\lambda_2 = 20$

Figure 2. Contour plot of the joint probabilities of the Poisson processes.

Table 4. Spread option values with common and Self-decomposable (SD) compound Poisson.

	Case A		Option Value Case A		Case B		Option Value Case B	
a	$\rho_{N_1 N_2}(\%)$	λ	Common	SD	$\rho_{N_1 N_2}(\%)$	λ	Common	SD
0.1	9	1.80	24.30	24.22	7	2.09	18.87	18.87
0.15	14	2.71	23.76	23.64	11	3.13	18.66	18.67
0.2	18	3.63	23.20	23.05	15	4.16	18.45	18.46
0.25	23	4.55	22.63	22.44	18	5.19	18.25	18.26
0.3	27	5.47	22.04	21.81	22	6.21	18.04	18.05
0.35	32	6.40	21.42	21.16	26	7.23	17.83	17.83
0.4	37	7.34	20.78	20.48	29	8.24	17.61	17.62
0.45	41	8.29	20.11	19.78	33	9.25	17.40	17.40
0.5	46	9.24	19.41	19.05	36	10.25	17.18	17.18
0.55	51	10.20	18.68	18.29	40	11.25	16.97	16.96
0.6	56	11.17	17.90	17.49	43	12.24	16.75	16.74
0.65	61	12.16	17.08	16.64	47	13.23	16.53	16.51
0.7	66	13.15	16.20	15.74	50	14.21	16.30	16.29
0.75	71	14.17	15.25	14.78	54	15.19	16.08	16.06
0.8	76	15.20	14.21	13.75	57	16.16	15.85	15.83
0.85	81	16.26	13.06	12.61	61	17.13	15.62	15.60
0.9	87	17.36	11.74	11.33	64	18.09	15.38	15.37
0.95	93	18.53	10.14	9.82	67	19.05	15.15	15.14

The adoption of a common Poisson reduces instead the spectrum of the jump events: for instance in an extreme setting where $\lambda = \lambda_1 = \lambda_2$, $N_1(t)$ cannot jump more that $N_2(t)$, while this is not the case for the Poisson processes correlated via self-decomposition. Because of the fact that the impact of the BS formulas is common to the three case, the differences among the exchange option values is only attributable to the different type of dependence between the Poisson processes.

Table 4 then displays the results with an expected jump size different from zero: the intensity λ of $N(t)$ was chosen in such a way that the correlations for the *self-decomposable* and *common Poisson* models coincide. The effect of the correlation between the Poisson processes is remarkable in the sense that the value of the exchange option is decreasing when ρ_{N_1,N_2} is increasing, this behavior being in line with the intuition because the spread terminal variance decreases.

In the case A, the jump sizes can be almost perfectly correlated, and the exchange option values using the common Poisson setting is always higher than the values obtained within our model. This is indeed a consequence of the higher concentration of the probability isolines for the *self-decomposable* Poisson model; it is as if the exchange option has a lower terminal covariance.

On the other hand the choice of the Poisson model seems to bring no remarkable repercussion on the price of the exchange option in the configuration B. Given the parameters of Table 1 and the same ρ_{N_1,N_2}, the price of the exchange option seems to highly depend on the number of the jumps of both processes, rather than on the timing of their occurrence: this appears to be a likely account for the small detected differences.

5.2. *GOU: Applications to the power interconnectors*

In this section we apply our methodology to the pricing of power interconnectors between *EEX* and *Powernext* modeled as exchange options (*Powernext* minus *EEX*). To this end we assume that each dynamics behaves as a GOU plus a compound Poisson; as done for the GBM example we do not consider transport costs (no strike) and adopt the Margrabe formula in the equations (40) - (42). We will further suppose that the calculation date is the end of December 2015 with a historical time window of 2 years for the estimation period. We concentrate then on the power spread value for the first and second quarters (Q1, Q2) in 2016[1]. The choice of these maturities is justified by the fact that pure spot models are used for short maturities, while it is a common practice to use forward models for longer maturities and add a volatility premium to the forward volatility.

Since the expectations in the pricing formulas are in the risk neutral measure, a few remarks are necessary for the estimation of the parameters of the spot model. Some approaches propose indeed the assumption of living already in a risk-neutral world, and also of incorporating a market price of risk in the drift (see (14)). The main goal of our work, however, being not the parameters estimation, for sake of simplicity we will refrain from considering a market price of risk. As a consequence our estimation procedure will be split into two steps: at first, after filtering out the time-dependent components of each process, one can estimate the parameters of the one-dimensional processes, $\theta_i = (k_i, \sigma_i, \lambda_i, M_i, \nu_i)$. Then, as a second step, one can estimate the remaining joint parameters defined by the two-dimensional model[2]

Consider an equally spaced time grid $t_0, t_1, \ldots, t_T$ with $t_{i+1} - t_i = \Delta t$ and the Euler scheme of each *SDE* in (10)

$$U_i(t+1) = (1 - k_i \Delta t)\, U_i(t) + \sigma_i \sqrt{\Delta t}\, \epsilon_{i,t+1} + e^{-k_i t} \mathbb{1}_i(t+1) Y_i. \tag{43}$$

where

$$\mathbb{1}_i(t+1) = \begin{cases} 1, & \text{with probability } \lambda_i \Delta t \\ 0, & \text{with probability } 1 - \lambda_i \Delta t. \end{cases} \tag{44}$$

The transition density can then be seen as a combination of Gaussian densities:

$$p_i\left(U_i(t+1), t+1 | U_i(t), t\right) = (1 - \lambda_i \Delta t)\, \mathcal{N}\left(\mu_i^C(t), \sigma^C(t)\right) + \lambda_i \Delta t \mathcal{N}\left(\mu_i^J(t), \sigma^J(t)\right). \tag{45}$$

[1]The technique here discussed does not reflect the *UGC* view.
[2]An example of how to derive the parameters of the GOU process in (5) can be found at http://de.mathworks.com/help/fininst/simulating-electricity-prices-with-mean-reversion-and-jump-diffusion.html. Although set in a different context, this procedure can be adapted to our case.

where $\mathcal{N}(\mu, \sigma)$ denotes the density function of a Gaussian law $\mathfrak{N}(\mu, \sigma)$ and

$$\mu_i^C(t) = (1 - k_i \Delta t)\, U_i(t) \qquad \mu_i^J(t) = (1 - k_i \Delta t)\, U_i(t) + M_i e^{-k_i t}$$
$$\sigma_i^C(t) = \sigma_i \sqrt{\Delta t} \qquad (\sigma_i^J(t))^2 = \sigma_i^2 \Delta t + e^{-2k_i t} \nu_i^2$$

The parameters $\theta_i = (k_i, \sigma_i, \lambda_i, M_i, \nu_i)$ can then be calibrated by maximizing the log-likelihood function with the usual constrains on the parameters

$$\theta_i = argmax \sum_{t=0}^{T-1} \log \left(p_i \left(U_i(t+1), t+1 | U_i(t), t \right) \right). \tag{46}$$

The calibration for the two-dimensional process apparently depends on the model specifications listed in the Section 2 as explained below

- **Independent jumps**. The joint probability are simply

$$\begin{aligned} p_{0,0} &= (1 - \lambda_1 \Delta t)\, (1 - \lambda_2 \Delta t)\,, \qquad p_{1,0} = \lambda_1 \Delta t\, (1 - \lambda_2 \Delta t) \\ p_{0,1} &= (1 - \lambda_1 \Delta t)\, \lambda_2 \Delta t, \qquad p_{1,1} = 1 - p_{0,1} - p_{1,0} - p_{0,0} \end{aligned} \tag{47}$$

  The only two remaining parameters to estimate are $\rho^{(W)}$ and $\rho^{(J)}$ and can be obtained by maximizing the log-likelihood of the two dimensional process. The transition density is

$$\begin{aligned} p\left(U(t+1), t+1 | U(t), t\right) = {} & \mathcal{N}\left(\mu^{CC}(t), \Sigma^{CC}(t)\right) p_{0,0} + \mathcal{N}\left(\mu^{CJ}(t), \Sigma^{CJ}(t)\right) p_{0,1} \\ & + \mathcal{N}\left(\mu^{JC}(t), \Sigma^{JC}(t)\right) p_{1,0} + \mathcal{N}\left(\mu^{JJ}(t), \Sigma^{JJ}(t)\right) p_{1,1} \end{aligned} \tag{48}$$

  where

$$\mu^{CC}(t) = \left(\mu_1^C(t), \mu_2^C(t)\right), \Sigma^{CC}(t) = \begin{pmatrix} (\sigma_1^C)^2 & \rho^{(W)} \sigma_1^C \sigma_2^C \\ \rho^{(W)} \sigma_1^C \sigma_2^C & (\sigma_2^C)^2 \end{pmatrix},$$

$$\mu^{CJ}(t) = \left(\mu_1^C(t), \mu_2^J(t)\right), \Sigma^{CJ}(t) = \begin{pmatrix} (\sigma_1^C)^2 & \rho^{(W)} \sigma_1^C \sigma_2^C \\ \rho^{(W)} \sigma_1^C \sigma_2^C & (\sigma_2^J)^2 \end{pmatrix},$$

$$\mu^{JC}(t) = \left(\mu_1^J(t), \mu_2^C(t)\right), \Sigma^{JC}(t) = \begin{pmatrix} (\sigma_1^J)^2 & \rho^{(W)} \sigma_1^C \sigma_2^C \\ \rho^{(W)} \sigma_1^C \sigma_2^C & (\sigma_2^C)^2 \end{pmatrix},$$

$$\mu^{JJ}(t) \left(\mu_1^J(t), \mu_2^J(t)\right), \Sigma^{JJ}(t) = \begin{pmatrix} (\sigma_1^J)^2 & \rho^{(J)} \sigma_1^J \sigma_1^J \\ \rho^{(J)} \sigma_1^J \sigma_2^J & (\sigma_2^J)^2 \end{pmatrix}.$$

  We do not neglect the $o\left(\Delta t^2\right)$ terms because they are necessary to estimate $\rho^{(D)}$.
- **Correlation via common jumps**. One cannot detect the presence of the common Poisson process only looking at each log process independently. After some algebra one finds that the pair $(\mathbb{1}_i(t+1))$ is a 2-dimensional Bernoulli rv with:

$$\begin{aligned} p_{0,0} &= 1 - \left(\lambda_1^X + \lambda_2^X + \lambda\right) \Delta t, \qquad p_{1,0} = \lambda_1^X \Delta t \\ p_{0,1} &= \lambda_2^X \Delta t, \qquad p_{1,1} = \lambda \Delta t = 1 - p_{0,1} - p_{1,0} - p_{0,0}. \end{aligned} \tag{49}$$

Table 5. Market parameters for *EEX* and *Powernext*, standard errors are in parenthesis
(a) Parameters of the single underlyings.

Market	k	σ_i	μ_i	ν_i	λ_i
EEX	42.50(3.66)	1.66(0.028)	−0.10(0.018)	0.16(0.0069)	95.32(16.74)
Powernext	41.64(2.97)	1.52(0.013)	−0.06(0.019)	0.38(0.033)	56.74(8.01)

(b) Common parameters

Method	$\rho^{(W)}(\%)$	λ	a
Independent	43(5.5)	NA	NA
Common	43(5.5)	34.12(6.12)	NA
Self-decomp	43(5.5)	NA	0.44(0.09)

Table 6. *Powernext-EEX* interconnector prices

	Interconnector Value (EUR)		
	Independent	Common	Self-decomposable
Jan	289.51	289.64	310.08
Feb	287.40	287.40	290.81
Mar	263.17	263.17	263.37
Q2	405.05	405.05	405.06

At variance with the previous case, we neglect here the $o\left(\Delta t^2\right)$ terms, in the sense that we are neglecting the possibility that the Poisson processes $N_1^X(t)$ and $N_2^X(t)$ jump simultaneously in the unit of time Δt. The functional form of the transition density is similar that of equation (48) but with different probability weights.

- **Correlation via self-decomposability**
  - $a\lambda_1 > \lambda_2$: from the Section 2, up to $O(\Delta t^2)$ terms, we have

$$\begin{aligned} p_{0,0} &= 1-\lambda_1\Delta t, \qquad p_{1,0} = (\lambda_1-\lambda_2)\Delta t - \lambda_1\lambda_2\frac{1-a}{a}\Delta t^2 \\ p_{0,1} &= 0, \qquad p_{1,1} = \lambda_2\Delta t. \end{aligned} \tag{50}$$

  - $a\lambda_1 \le \lambda_2$: in a similar way, by neglecting the $o(\Delta t^2)$ terms (see Appendix B), we get

$$\begin{aligned} p_{0,0} &= 1-\left((1+a)\lambda_1+\lambda_2\right)\Delta t, \qquad p_{0,1} = (\lambda_2-a\lambda_1)\Delta t \\ p_{1,0} &= (1-a)\lambda_1\Delta t, \qquad p_{1,1} = a\lambda_1\Delta t. \end{aligned} \tag{51}$$

The results of the calibration are shown in Table 5. The expected jump sizes are both negative and their correlation ($\rho^{(D)} \approx 0$) is very small: as a consequence we will neglect it hereafter. Although the main purpose of our paper is not the parameters estimation, one can notice that the increase of renewable energy, on the other hand, has changed the statistics of the power day-ahead prices: even very negative day-ahead prices have been quoted on some days (for instance Christmas 2013 in *EEX*). Moreover the statistics of spot prices is still non-Gaussian, but big positive spikes are less frequent. In any case we are here considering log-prices: comparing the values of λ, λ_1, λ_2 and a or, we can conclude that the correlation between the two Poisson processes is of the order of 45%. In the case of Poisson processes with common jumps, the correlation is not time dependent, while in our self-decomposable case the correlation depends on time and it is not a, even if it can still give a reasonable order of magnitude for the correlation.

Table 6 shows instead the values of the power-spread in the quarters Q1 and Q2 and the detail of the first three months with the different dynamics. As shown in Section 4.2, the pricing formulas are double sums over probabilities and BS formulas. The three models differ in the Poisson probability terms, while the values of the BS terms are the same, hence the price difference is only attributable to the different 2-dimensional Poisson law. The results listed in the Table 6 show that the prices of the interconnector are similar for the cases with independent and common jumps for all the maturities. The model with self-decomposable jumps returns instead appreciably higher values for all the maturities with a larger difference in January and February whilst at longer maturities the difference reduces because of the effect of the mean reversion rate.

The fact that the prices obtained with our methodology are larger than that with independent jumps may seem counterintuitive, and also wrong. In fact, with the assumption of independent jumps the exchange option value increases with decreasing linear correlation between the Brownian motions. Our model instead, that is parameterized by a, implies a structure of dependence that is not based on the simple linear correlation and its effects are remarkable in the examples illustrated above. Indeed, if one relies on linear correlations only, given the fact that the pair of Poisson processes is positively correlated, one would indeed expect the price to be lower than the one obtained with uncorrelated jumps. On the other hand, in this numerical example the correlation induced by the 2-dimensional Poisson process with common jumps does not barely affect the exchange option value. Apparently, the cases of common and independent jumps give rise to comparable results because the different probability weights have the same effect on the overall exchange option prices.

In contrast, compared to the other approaches, our model gives more emphasis to the timing of the jumps, and not only to their frequency. In particular, the choice of common jumps imposes that a certain news to one of the two markets may cause an instantaneous propagation of the information and simultaneous jump of both spot prices, if the jumps belong to the common source, or no propagation if the news is associated to N_1^X or N_2^X. On the other hand, our solution has not this restriction: a shock to one of the two markets may have caused both prices to jump however at different times. That explains why the values of the exchange options obtained with our model are higher that those obtained with the model with common jumps although the linear correlations have similar numerical values. Since the difference among the prices obtained with a 2-dimensional Poisson process with independent or with a common jump is negligible we can conclude that, compared to these two other methodologies, our model gives more relevance to the interaction and the dependence between the two markets.

6. Conclusions and perspectives

Building on the concept of self-decomposability we have studied the use of pairs of co-dependent Poisson processes, first proposed in (8), to produce a model for energy derivatives, and in general to price exchange options. Due to the particular relationships among inter arrival times, with some abuse of terminology, we can think to this dependence as a form of cointegration among jumps.

Based on these observations, we propose to introduce a form of synaptic risk-interactions in analogy with what happens for the diffusion of information on a network: the individual reaction in every node propagates with a delay to other nodes, inducing then a new reaction and so on.

In the context of modeling energy markets and facilities, we have shown how to combine 2-dimensional compound Poisson processes with Geometric Brownian Motions and Geometric Ornstein-Uhlenbeck dynamics.

In contrast to Geometric Brownian motion plus jumps in (6, 2) where the macro-economic shocks to the system are modeled by common arrival jumps with correlated jump sizes, while independent shocks are added to represent the idiosyncratic components, we assume that the macro-economic shocks can impact each underlying asset, and its reaction can propagate to the other assets with random and correlated time delays. The relative timing of the correlated Poisson processes apparently allows for an enhanced flexibility of the model in the practical applications because we no longer have to rely on common shocks only.

In the Geometric Ornstein-Uhlenbeck case, we have adopted a dynamics for day-ahead prices that allows the derivation of simple (semi-)closed form formulas for plain vanilla options. Focusing then on the pricing of exchange options, we have compared the option prices calculated using our model with that derived from several different kinds of Poisson processes. We have found in particular that the formulas based on our coupling procedure can readily cope with a wide range of issues going well beyond the pure linear correlation modeling, and can answer several questions that arise in the financial context. Indeed, compared to the other methodologies we analyzed, modeling the dependence between Poisson process via self-decomposition gives more relevance to the dependence structure between two markets, to the timing of the jumps and not only to their frequency. Our numerical examples shows cases where the exchange option prices can assume values that are counterintuitive or may seem even wrong if one relies on pure linear correlation assumptions.

In this paper in fact we have considered just power interconnectors as energy facilities, but the model applicability can be extended without difficulty to other financial situations. Straightforward applications are in credit and insurance risk where our approach can answer questions regarding the time of contagion or time of propagation of certain information. In addition, the self-decomposability and the subordination techniques can be promising tools to study the dependence structure beyond the Gaussian-Itō world. In (8), for instance, it has already been detailed how to obtain dependent Erlang (Gamma) *rv*'s that can be used to create and simulate dependent variance gamma processes. Furthermore several recent papers (see (22, 11)) have studied the use of two sided Exponential-Polynomial-Trigonometric (ETP) density functions to option pricing: the EPT are distributions with a strictly proper rational characteristic function. Due to the fact that the Erlang and exponential distributions belong to this class, it will be worthwhile to investigate the use of self-decomposability to create dependence for this larger class of distributions.

Appendix A: Calculation of $\rho^{(J)}$

- The GBM plus Jumps Case.

$$\rho^{(J)}_{\ell_1,\ell_2,m_1,m_2} = Corr\left[\sigma_1 W_1(T) + \sqrt{\ell_1}\nu_1 Z_1 + \sqrt{m_1}\alpha_1 Y_1, \sigma_2 W_2(T) + \sqrt{\ell_2}\nu_2 Z_2 + \sqrt{m_2}\alpha_2 Y_2\right] =$$
$$= \frac{\rho^{(W)}\sigma_1\sigma_2 T + \rho^{(D)}\sqrt{\ell_1\ell_2}\nu_1\nu_2}{\sqrt{v^{(J)}_{1,\ell_1,m_1}(T) v^{(J)}_{2,\ell_2,m_2}(T)}}. \tag{A1}$$

- The Ornstein-Uhlenbeck plus Jumps Case.

$$\rho^{(J)}_{\ell_1,\ell_2,m_1,m_2} = Corr\left(L_1, L_2\right) =$$
$$= \frac{\frac{\rho^{(W)}\sigma_1\sigma_2}{2\sqrt{k_1k_2}}\sqrt{1-e^{-2k_1t}}\sqrt{1-e^{-2k_2t}} + \rho^{(D)}\sqrt{\ell_1\ell_2}\nu_1\nu_2 e^{-(k_1+k_2)t}}{\sqrt{v^{(J)}_{1,\ell_1,m_1}(T) v^{(J)}_{2,\ell_2,m_2}(T)}} \tag{A2}$$

where $L_i = \sigma_i \int_0^t e^{-k_i(t-s)} dW_i(s) + e^{-k_i t}\left(\nu_i\sqrt{\ell_i}Z_i + \sqrt{m_i}\alpha_i Y_i\right), i = 1, 2.$

Appendix B: Calculation of $p_{0,0}$, $p_{0,1}$, $p_{1,0}$ and $p_{1,1}$

From the results in (8) the joint *cdf* of $X_1 \sim \mathfrak{E}_1(\lambda_1)$ and $X_2 \sim \mathfrak{E}_1(\lambda_2)$ is

$$H(x_1, x_2) = \mathbb{1}_{\frac{\lambda_1}{\lambda_2}x_1 \wedge \frac{x_2}{a} \geq 0}\left[\left(1 - e^{-\left(\lambda_1 x_1 \wedge \frac{\lambda_2 x_2}{a}\right)}\right) - e^{-\lambda_2 x_2}\left(1 - e^{-(1-a)\left(\lambda_1 x_1 \wedge \frac{\lambda_2 x_2}{a}\right)}\right)\right] \tag{B1}$$

For Δt small we can assume that no more that one jump can occur hence:

$$\begin{aligned}
p_{1,1} &= \boldsymbol{P}(X_1 \leq \Delta t, X_2 \leq \Delta t) = H(\Delta t, \Delta t) \\
p_{1,0} &= \boldsymbol{P}(X_1 \leq \Delta t, X_2 \geq \Delta t) = F_1(\Delta t) - H(\Delta t, \Delta t) \\
p_{0,1} &= \boldsymbol{P}(X_1 \geq \Delta t, X_2 \leq \Delta t) = F_2(\Delta t) - H(\Delta t, \Delta t) \\
p_{0,0} &= \boldsymbol{P}(X_1 \geq \Delta t, X_2 \geq \Delta t) = 1 - p_{1,1} - p_{0,1} - p_{0,1}
\end{aligned}$$

Finally, the results in the Section 5.2 are obtained splitting between the cases $a\lambda_1 > \lambda_2$ and $a\lambda_1 \leq \lambda_2$.

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