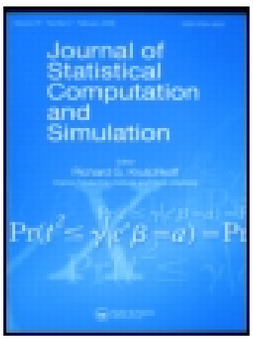

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Gamma-related Ornstein–Uhlenbeck processes and their simulation*

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ABSTRACT
We investigate the distributional properties of two Lévy-driven Ornstein–Uhlenbeck (OU) processes whose stationary distributions are the gamma law and the bilateral gamma law, respectively. The said distributions turn out to be related to the self-decomposable gamma and bilateral gamma laws, and their densities and characteristic functions are here given in closed form. Algorithms for the exact generation of such processes are accordingly derived with the advantage of being significantly faster than those available in the literature and therefore suitable for real-time simulations.

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1. Introduction and motivation

In the present paper, we study the distributional properties of the Gamma–Ornstein–Uhlenbeck process (Γ-OU) and the Bilateral Gamma–Ornstein–Uhlenbeck process (biΓ-OU). Our contribution consists in the derivation of the closed form of both the density and the characteristic function of such processes. In its turn, this main result enables us to obtain fast algorithms for their exact simulation, along with an unbiased transition density that can be used for parameter estimation.

As observed in Barndorff–Nielsen and Shephard [1], the Γ-OU process is a very tractable model that can be adopted in many potential applications. For instance, in the energy and in the commodity field, many authors (using sometimes different naming conventions) coupled a Γ-OU process or a combination of Γ-OU processes to a standard Gaussian-driven OU process to model day-ahead spot prices. Among others, Kluge [2] and Kjaer [3] apply such a combination to price swing options and gas storages whereas Benth and Pircalabu [4] apply a Γ-OU process to evaluate wind derivatives. The use of Γ-OU or biΓ-OU in the energy market is justified by the fact that gas and power prices exhibit strong mean-reversion and spikes. Other applications of the Γ-OU and biΓ-OU processes in finance can be found, among others, in Barndorff–Nielsen and Shephard [1] for the

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modelling of stochastic volatility, or in Schoutens and Cariboni [5] and Bianchi and Fabozzi [6] for the modelling of credit default risk and for the pricing of credit default swaps.

It is worthwhile noticing that the Variance Gamma distribution, that is widely used in financial applications since the introduction of the Variance Gamma processes by Madan and Seneta [7], is a special case of the bilateral gamma distribution. Finally, some applications of the Γ-OU processes beyond finance, can also be found in Brown et al. [8] and Gaver and Lewis [9] for the modelling of wind speed or computer failures, respectively.

Following Barndorff–Nielsen and Shephard [1], we consider a Lévy process $Z(t)$ and the generalized OU process defined by the *SDE*

$$\mathrm{d}X(t) = -kX(t)\,\mathrm{d}t + \mathrm{d}Z(t) \quad X(0) = X_0, \quad \boldsymbol{P} - a.s. \quad k > 0, \tag{1}$$

whose strong solution is

$$X(t) = X_0 e^{-kt} + \int_0^t e^{-k(t-s)}\mathrm{d}Z(s). \tag{2}$$

Up to indistinguishability the solution is unique (see Sato [10], Section 17). Here $Z(t)$ is called the Backward Driving Lévy Process (*BDLP*), and we will say that $X(t)$ is a D-OU process if D is the stationary law of $X(t)$. Now a well-known result (see, for instance, Cont and Tankov [11] or Sato [10]) is that, a given one-dimensional distribution D always is the stationary law of a suitable Lévy-driven OU process if and only if D is self-decomposable.

We recall that a law with probability density (*pdf*) $f(x)$ and characteristic function (*chf*) $\varphi(u)$ is said to be *self-decomposable* (see Sato [10] or Cufaro Petroni [12]) when for every $0 < a < 1$, we can find another law with *pdf* $g_a(x)$ and *chf* $\chi_a(u)$ such that

$$\varphi(u) = \varphi(au)\chi_a(u) \tag{3}$$

We will accordingly say that a random variable (*rv*) X with *pdf* $f(x)$ and *chf* $\varphi(u)$ is self-decomposable when its law is self-decomposable. This means that for every $0 < a < 1$ we can always find two *independent rv*'s, Y with the same law of X and Z_a, here called *a-remainder*, with *pdf* $g_a(x)$ and *chf* $\chi_a(u)$ such that

$$X \stackrel{d}{=} aY + Z_a. \tag{4}$$

It can be easily derived (see, for instance, Schoutens [13], page 68) that the BDLP $Z(t)$ of the Γ-OU process $X(t)$ is a compound Poisson with exponential jumps.

In this study, we also investigate the case where the stationary law is a bilateral gamma distribution that can be seen as the law of the difference of two independent gamma *rv*'s with different parameters. We show that in this case the *BDLP* $Z(t)$ is related to the difference of compound Poisson processes with exponentially distributed jumps.

We will prove that the laws of the Γ-OU process and of the biΓ-OU process at time t coincide with those of the a-remainder Z_a of a gamma and of a bilateral gamma distribution, respectively. This result allows to find the *pdf* and the *chf* of $X(t)$ in closed form because the a-remainder's of a gamma and a bilateral gamma distribution turn out to be manageable mixtures of other elementary distributions. As a consequence, we can design efficient and fast algorithms to exactly simulate Γ-OU and biΓ-OU processes, outperforming in so doing every other existing alternative (see Cont and Tankov [11] and Qu

et al. [14]). The numerical experiments we have conducted clearly show that the computational times of our approach are very small therefore, our solution is suitable for real-time simulations.

The paper is structured as follows: in Section 2, we study the distributional properties of a Γ-OU process showing that its transition law can be represented as a mixture of Polya or binomial mixtures and therefore, it can be seen as a compound sum of independent exponential *rv*'s or as an Erlang *rv* with a random index. These findings are instrumental to design the simulation algorithms illustrated in Subsection 2.3. Section 3 analyses the distributional properties of a biΓ-OU process and focuses on the case where the bilateral gamma law can be seen as the law of the difference of two independent gamma *rv*'s with equal parameters.

These results are then used in Subsection 3.2 to conceive the relative simulation algorithms. Section 4 illustrates the numerical experiments that we have conducted to compare the convergence and computational performance of our solutions to the approaches in Cont and Tankov [11] and in Qu at al. [14]. Finally, Section 5 concludes the paper with an overview of future inquiries and possible further applications.

Before proceeding, we introduce some notation relative to the various distributions that we consider in the paper. We write $\Gamma(\alpha, \beta)$ to denote the gamma distribution with shape parameter $\alpha > 0$ and rate parameter $\beta > 0$. In case $\alpha = n \in \mathbb{N}^*$, the gamma law coincides with Erlang law and is hereafter written $\mathcal{E}_n(\beta)$, of course, $\mathcal{E}_1(\beta)$ represents the exponential law with rate β. We write $\mathfrak{b}\Gamma(\alpha_1, \beta_1, \alpha_2, \beta_2)\,, \alpha_1 > 0, \alpha_2 > 0, \beta_1 > 0, \beta_2 > 0$, to denote the bilateral gamma law, for sake of brevity if $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$ we write $\mathfrak{b}\Gamma(\alpha, \beta)$ and we call such a law symmetric bilateral gamma. In general, when we add the superscript $^{++}$ to the symbol relative to a self-decomposable law we denote the law of the associated a-remainder; for instance $\Gamma_a^{++}(\alpha, \beta)$ denotes the law of the a-remainder of a $\Gamma(\alpha, \beta)$ law, and so on. We write $\mathcal{B}(n, p)\,, n \in \mathbb{N}^*, 0 < p < 1$ to denote the binomial distribution and $\overline{\mathcal{B}}(\alpha, p)\,,\ \alpha > 0,\ 0 < p < 1$ to denote the Polya distribution. Finally, we write $\mathcal{U}([0, 1])$ to denote a uniform distribution in $[0, 1]$ and $\mathcal{P}(\lambda)$ to denote the Poisson distribution with mean $\lambda > 0$.

2. Distributional properties of the Γ-OU process

Because, by definition, the stationary law of a Γ-OU process $X(t)$ with parameters (k, λ, β), hereafter denoted Γ-OU(k, λ, β), is a gamma law, it is natural to investigate how it is related to the law of the process at time t. We recall that the gamma distribution $\Gamma(\alpha, \beta)$ has the following *pdf* and *chf*

$$f_{\alpha,\beta}(x) = \frac{\beta}{\Gamma(\alpha)}(\beta x)^{\alpha-1}e^{-\beta x} \quad x > 0 \tag{5}$$

$$\varphi_{\alpha,\beta}(u) = \left(\frac{\beta}{\beta - iu}\right)^{\alpha} \tag{6}$$

The $\Gamma(\alpha, \beta)$ law is famously self-decomposable (see Grigelionis [15]), so that from (3) the law of its a-remainder $\Gamma_a^{++}(\alpha, \beta)$ has the *chf*

$$\chi_a(u; \alpha, \beta) = \frac{\varphi_{\alpha,\beta}(u)}{\varphi_{\alpha,\beta}(au)} = \left(\frac{\beta - iau}{\beta - iu}\right)^{\alpha} \tag{7}$$

On the other hand, according to an aforementioned result, a Γ-OU(k, λ, β) process, is the solution of (1) where the *BDLP* $Z(t)$ is a compound Poisson process

$$Z(t) = \sum_{n=1}^{N(t)} J_n$$

where from now on we use the convention that an empty sum equals 0 and

- $N(t)$ is a Poisson process with intensity λ;
- $(J_n)_{n\in\mathbb{N}^*}$ is a sequence of independent and identically distributed (*iid*) jump sizes independent of $N(t)$, each of which follows an exponential law with rate parameter β, i.e. $J_n \sim \mathcal{E}_1(\beta), \forall n \in \mathbb{N}^*$.

It turns out that $Z(t)$ is a subordinator, and that the solution (2) can be written as (see, for instance, Qu et al. [14])

$$X(t) = x_0 e^{-kt} + \sum_{n=1}^{N(t)} e^{-k(t-\tau_n)} J_n \tag{8}$$

where $(\tau_n)_{n\in\mathbb{N}^*}$ is the sequence of the jump times of the Poisson process $N(t)$ independent of $(J_n)_{n\in\mathbb{N}^*}$. Applying Lemma 15.1 in Cont and Tankov [11], after some calculations, one can find that the *chf* of $X(t)$, with $X(0) = 0$ $\boldsymbol{P} - a.s.$, is (see also Example 3.4.3 in Kluge [2] page 37)

$$\varphi(u, t) = \left(\frac{\beta - iue^{-kt}}{\beta - iu}\right)^{\frac{\lambda}{k}} \tag{9}$$

This coincides with the *chf* of the e^{-kt}-remainder of the gamma law $\Gamma(\lambda/k, \beta)$ whereas its stationary distribution

$$\varphi_s(u) = \left(\frac{\beta}{\beta - iu}\right)^{\frac{\lambda}{k}}$$

is instead recovered for $t \to +\infty$ and it coincides with the *chf* of the previous gamma law (see also Barndorff–Nielsen and Shephard [1], Grigelionis [15]). The above result can be also summarized by the following theorem whose proof is a straightforward application of the homogeneity of the Poisson process.

Theorem 2.1: *The chf of $X(t + s)$ conditional on $X(s)$ is given by*

$$\boldsymbol{E}\left[e^{iuX(t+s)}|X(s)\right] = e^{iuX(s)e^{-kt}} \times \left(\frac{\beta - iue^{-kt}}{\beta - iu}\right)^{\frac{\lambda}{k}} \tag{10}$$

An alternative version of Theorem 2.1 can be found in Qu et al [14].

Theorem 2.2: *The chf of $X(t+s)$ conditional on $X(s)$ is given by*

$$\boldsymbol{E}\left[e^{iuX(t+s)}|X(s)\right] = e^{iuX(s)e^{-kt}} \times e^{\lambda t\left(\varphi_{\tilde{J}}(u)-1\right)} \tag{11}$$

where

$$\varphi_{\tilde{J}}(u) = \int_0^1 \frac{\beta e^{ktv}}{\beta e^{ktv} - iu}\,\mathrm{d}v, \tag{12}$$

Theorem 2.2 implies then that

$$X(t+s) \stackrel{d}{=} X(s)e^{-kt} + \sum_{i=1}^{N} \tilde{J}_i \tag{13}$$

where N is distributed according to a Poisson distribution with mean λt and $\tilde{J}_i, i > 0$ are *iid rv*'s, all independent of N, distributed according to a uniform mixture of exponential laws with random parameter βe^{ktU} and $U \sim \mathcal{U}([0,1])$.

It is apparent now that the *chf* of a Γ-OU(k, λ, β) process at time t is that of the a-remainder of a $\Gamma(\alpha, \beta)$ law plus a constant when we take $a = e^{-kt}$ and $\alpha = \lambda/k$. Based on these observations we can then write

$$X(t+s) \stackrel{d}{=} aX(s) + Z_a, \quad a = e^{-kt} \tag{14}$$

that provides an alternative representation to Equation (13).

The moments of Z_a, as well as those of a Γ-OU process $X(t)$, can be obtained simply deriving the *chf* (10) that is much more treatable than that in (11); however, it is easier to work with the cumulants of Z_a that can be calculated with a straightforward application of the properties of the cumulant generating function (the logarithm of the moment generating function).

$$\kappa_n(Z_a) = (1 - a^n)\kappa_n(X) \tag{15}$$

where $\kappa_n(Z_a)$ and $\kappa_n(X)$ represent the n-th cumulant of the a-remainder and a gamma distributed *rv* X, respectively. We remark that (15) is applicable to the cumulants of the a-remainder of any self-decomposable distribution. After some algebra, it results that the expected value, the variance, the skewness and the kurtosis of Z_a are

$$\boldsymbol{E}[Z_a] = (1-a)\frac{\alpha}{\beta}, \tag{16}$$

$$\boldsymbol{V}[Z_a] = (1-a^2)\frac{\alpha}{\beta^2}, \tag{17}$$

$$\boldsymbol{Skew}[Z_a] = \frac{1-a^3}{(1-a^2)^{3/2}} \times \frac{2}{\sqrt{\alpha}}, \tag{18}$$

$$\boldsymbol{Kurt}[Z_a] = \frac{1+a^2}{1-a^2} \times \frac{6}{\alpha} + 3. \tag{19}$$

Of course, from Equation (14) $\boldsymbol{E}[X(t+s)|X(s)] = aX(s) + \boldsymbol{E}[Z_a]$ while the variance, the skewness and kurtosis of $X(t+s)$ given $X(s)$ and Z_a coincide because these quantities are

translation invariant. It is interesting to note that the laws $\Gamma(\alpha,\beta)$ and $\Gamma_a^{++}(\alpha,\beta)$ share the same summation and scaling properties.

Proposition 2.3: (1) *If* $Z_a \sim \Gamma_a^{++}(\alpha,\beta)$ *then for any* $c>0$,

$$c\,Z_a \sim \Gamma_a^{++}\left(\alpha,\frac{\beta}{c}\right).$$

(2) *If* $Z_{a,i} \sim \Gamma_a^{++}(\alpha_i,\beta)$, $i=1,\ldots,N$ *and independent then*

$$\sum_{i=1}^{N} Z_{a,i} \sim \Gamma_a^{++}\left(\sum_{i=1}^{N}\alpha_i,\beta\right).$$

Proof: The *chf* of cZ_a is

$$\chi_a(cu) = \left(\frac{\beta - iacu}{\beta - icu}\right)^{\alpha} = \left(\frac{\frac{\beta}{c} - iau}{\frac{\beta}{c} - iu}\right)^{\alpha},$$

which is the *chf* a $\Gamma_a^{++}(\alpha,\frac{\beta}{c})$ distributed *rv*.

The *chf* $\bar{\chi}_a(u)$ of $\sum_{i=1}^{N} Z_{a,i}$ is

$$\bar{\chi}_a(u) = \boldsymbol{E}\left[e^{iu\sum_{i=1}^{N} Z_{a,i}}\right] = \prod_{i=1}^{N}\left(\frac{\beta - iau}{\beta - iu}\right)^{\alpha_i} = \left(\frac{\beta - iau}{\beta - iu}\right)^{\sum_{i=1}^{N}\alpha_i},$$

that coincides with the *chf* of a $\Gamma_a^{++}(\sum_{i=1}^{N}\alpha_i,\beta)$ law and that concludes the proof. ■

2.1. Polya mixtures of gamma laws $\Gamma(\alpha,\beta)$

In order to further investigate the distributional properties of the law of the a-remainder and of the law of a Γ-OU process, we now consider a *rv* S distributed according to a *negative binomial, or Polya distribution*, namely such that

$$\boldsymbol{P}\{S=k\} = \binom{\alpha+k-1}{k}(1-p)^{\alpha}p^{k} \quad k=0,1,\ldots$$

where

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!} \quad \binom{\alpha}{0} = 1$$

denotes the generalized binomial coefficient. Note that, when $\alpha = n = 1,2,\ldots$ is a natural number, the Polya distribution $\overline{\mathcal{B}}(n,p)$ coincides with the so-called *Pascal distribution*, and in particular $\overline{\mathcal{B}}(1,p)$ is nothing else than the usual *geometric distribution* $(1-p)p^k$.

From the generalized binomial formula, it is possible to see now that its *chf* is

$$\varphi_S(u) = \sum_{k=0}^{\infty}\binom{\alpha+k-1}{k}(1-p)^{\alpha}p^{k}e^{iuk} = \left(\frac{1-p}{1-p\,e^{iu}}\right)^{\alpha}$$

where the series certainly converges because $|p\,e^{iu}| = p < 1$.

In addition, we remind that, given a countable set of *pdf* 's $p_1, p_2(x), \ldots$ with corresponding *cdf* 's $P_1(x), P_2(x), \ldots$ and *chf* 's *cdf* 's $\phi_1(x), \phi_2(x), \ldots$ and given a countable set of weights $w_1, w_2, \ldots$ such that $w_i \geq 0, i \in \mathbb{N}^*$ with $\sum_{i=1}^{\infty} w_i = 1$, a *countable mixture distribution* P is a law with *pdf* $p(x)$, *cdf* $P(x)$ and *chf* $\phi(x)$ that can be written as a convex combination as follows

$$p(x) = \sum_{i=1}^{\infty} w_i p_i(x), \quad P(x) = \sum_{i=1}^{\infty} w_i P_i(x) \quad \phi(x) = \sum_{i=1}^{\infty} w_i \phi_i(x).$$

Therefore each w_i can be seen as the probability $\boldsymbol{P}\{V = i\} = w_i$ of a certain discrete *rv* V. The distribution P is accordingly called a *V-weighted distribution* of laws $p_n, n \geq 1$.

As observed for instance in Panjer and Wilmott [16], by taking the *rv*

$$Z = \sum_{j=1}^{S} X_j$$

sum of a random number $S \sim \overline{\mathcal{B}}(\alpha, p)$ of *iid rv*'s $X_j, j \geq 1$ independent of S with the common *chf* $\varphi_X(u)$ we have indeed

$$\begin{aligned}
\varphi_Z(u) &= \boldsymbol{E}\left[e^{iuZ}\right] = \boldsymbol{E}\left[\boldsymbol{E}\left[e^{iuZ}\middle| S\right]\right] \\
&= \sum_{k=0}^{\infty} \binom{\alpha + k - 1}{k} (1-p)^{\alpha} p^k \, \boldsymbol{E}\left[e^{iu\sum_{j=0}^{k} X_j}\right] \\
&= (1-p)^{\alpha} \sum_{k=0}^{\infty} \binom{\alpha + k - 1}{k} p^k \varphi_X(u)^k = \left(\frac{1-p}{1-p\,\varphi_X(u)}\right)^{\alpha}
\end{aligned} \tag{20}$$

where again the series converges because $|p\,\varphi_X(u)| \leq p < 1$. This shows that the law of Z is an infinite Polya $\overline{\mathcal{B}}(\alpha, p)$-weighted mixture of laws $\varphi_X(u)^k$: if these laws also have a known *pdf*, then the law of Z too has an explicit representation as a mixture of *pdf* 's.

If on the other hand, $X_j \sim \mathcal{E}_1(\beta)$, the distribution of Z above can also be considered as an Erlang law $\mathcal{E}_S(\beta)$ with a Polya $\overline{\mathcal{B}}(\alpha, p)$-distributed random index S,

$$Z = \sum_{j=1}^{S} X_j \quad S \sim \overline{\mathcal{B}}(\alpha, p) \quad X_j \sim \mathcal{E}_1(\beta) \quad X_0 = 0, \ \boldsymbol{P} - a.s.$$

Based on these observations, we prove the following theorem that characterizes the law of the a-remainder of a $\Gamma(\alpha, \beta)$ distribution.

Theorem 2.4: *The law of the a-remainder $\Gamma_a^{++}(\alpha, \beta)$ is an infinite Polya $\overline{\mathcal{B}}(\alpha, 1-a)$-weighted mixture of Erlang laws $\mathcal{E}_k(\beta/a)$ with the following chf $\chi_a(u, \alpha, \beta)$ and density*

$g_a(x,\alpha,\beta)$

$$\chi_a(u,\alpha,\beta) = \sum_{k=0}^{\infty} \binom{\alpha+k-1}{k} a^\alpha (1-a)^k \left(\frac{\beta}{\beta - iau}\right)^k \tag{21}$$

$$g_a(x,\alpha,\beta) = a^\alpha \delta(x) \mathbb{1}_{x=0} + \sum_{k=1}^{\infty} \binom{\alpha+k-1}{k} a^\alpha (1-a)^k f_{k,\beta/a}(x) \mathbb{1}_{x>0} \tag{22}$$

where $\delta(x)$ is the Dirac delta function (a distribution or a generalized function) representing the probability density concentrated at 0.

Proof: By taking now $p = 1-a$ and $X \sim \mathcal{E}_1(\beta/a)$ with *chf*

$$\varphi_X(u) = \frac{\beta}{\beta - iau}$$

It is easy to see from (7) and (20) that

$$\left(\frac{\beta - iau}{\beta - iu}\right)^\alpha = \left(\frac{a(\beta - iau)}{\beta - iau - (1-a)\beta}\right)^\alpha = \left(\frac{a}{1-(1-a)\frac{\beta}{\beta - iau}}\right)^\alpha$$
$$= \sum_{k=0}^{\infty} \binom{\alpha+k-1}{k} a^\alpha (1-a)^k \left(\frac{\beta}{\beta - iau}\right)^k$$

that is the *chf* of an infinite Polya $\overline{\mathcal{B}}(\alpha, 1-a)$-weighted mixture of Erlang laws $\mathcal{E}_k(\beta/a)$.

Since on the other hand from (5) the *pdf* 's of the Erlang laws $\mathcal{E}_k(\beta/a)$ are known, also the *pdf* of the $\Gamma_a^{++}(\alpha,\beta)$ law is the following explicit mixture plus a degenerate in $x = 0$

$$g_a(x,\alpha,\beta) = a^\alpha \delta(x) \mathbb{1}_{x=0} + \sum_{k=1}^{\infty} \binom{\alpha+k-1}{k} a^\alpha (1-a)^k f_{k,\beta/a}(x) \mathbb{1}_{x>0}$$

that concludes the proof. ■

The above results give a closed-form representation of the transition density of a Γ-OU process.

Corollary 2.5: *The transition density $p(x, t+s|y, s)$ of the Γ-OU(k,λ,β) is*

$$p(x, t+s|y, s) = g_a\left(x - ay, \frac{\lambda}{k}, \beta\right), \quad a = e^{-kt}. \tag{23}$$

where $g_a(\cdot, \lambda/k, \beta)$ is defined in (22).

Although the parameters estimation is not the focus of our study, knowing the transition density in closed form gives a remarkable advantage compared to the results in Qu et al. [14] because one can write the log-likelihood and maximize it explicitly. Of course, in any practical applications, some series truncation rule must be adopted but it can, however, be easily fine tuned.

To this end, in discrete time, a Γ-OU process is equivalent to a GAR(1) auto-regressive process introduced by Gaver and Lewis [9] whose parameter estimation based on the EM algorithm has been discussed in Popovici and M. Dumitrescu [17] (for λ/k integer only, see next section). In the alternative, one could adopt the generalized method of moments using Equation (19) and obtain the associated Yule–Walker equations.

In addition, we remark that the corollary 2.1 in Zhang et al. [18] provides an alternative representation of the transition density of a Γ-OU in terms of a Poisson $\mathcal{P}(\alpha kt)$-weighted mixture of laws $h_n(x), n \geq 1$ where

$$h(x) = h_1(x) = \frac{e^{-\beta x} - e^{-\beta x/a}}{ktx}$$
$$h_n(x) = \int_0^x h(y) h_{n-1}(x-y)\, \mathrm{d}y.$$

Finally, it is also worthwhile noticing that an additional approach to find the desired transition density is based on the saddlepoint approximation detailed in Gatto [19] that also provides precise conditions under which the approximation holds. The main advantage of our representation is that the mixture is obtained in terms of Erlang laws so that, in contrast to the solution in corollary 2.1 in Zhang et al [18] and the results in Gatto [19], no numerical approximation or numerical integration is required neither for the parameters estimation nor for the path generation as shown in the next sections.

2.2. Binomial mixtures

It follows from the previous subsection that for $\alpha = n = 1, 2, \ldots$ the a-remainder of the Erlang laws $\Gamma(n, \beta) = \mathcal{E}_n(\beta)$ is an infinite mixture of Erlang $\mathcal{E}_k(\beta/a)$ with Pascal weights $\overline{\mathcal{B}}(n, 1-a)$, while for $n = 1$ the a-remainder of the exponential law $\Gamma(1, \beta) = \mathcal{E}_1(\beta)$ is an infinite mixture of Erlang $\mathcal{E}_k(\beta)$ with geometric weights $\overline{\mathcal{B}}(1, 1-a)$.

In these two cases, however, the following theorem shows that there is an alternative decomposition of the law of the a-remainder into a *finite*, binomial mixture of Erlang or in other words into an Erlang law $\mathcal{E}_S(\beta)$ with a binomial $\mathcal{B}(n, 1-a)$-distributed random index S, that is a sum

$$Z_a \stackrel{d}{=} \sum_{j=1}^{S} X_j \quad S \sim \mathcal{B}(n, 1-a)$$

of S *iid* exponentials $X_j \sim \mathcal{E}_1(\beta), j \geq 1$ independent of S.

Theorem 2.6: *The law of the a-remainder of the $\mathcal{E}_n(\beta)$ law is a finite mixture of Erlang $\mathcal{E}_k(\beta)$ with binomial weights $\mathcal{B}(n, 1-a)$ with the following chf $\chi_a(u, \alpha, \beta)$ and density $g_a(x, \alpha, \beta)$*

$$\chi_a(u, n, \beta) = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} (1-a)^k \left(\frac{\beta}{\beta - iu}\right)^k \tag{24}$$

$$g_a(x, n, \beta) = a^\alpha \delta(x) \mathbb{1}_{x=0} + \sum_{k=1}^{n} \binom{n}{k} a^{n-k} (1-a)^k f_{k,\beta}(x) \mathbb{1}_{x>0} \tag{25}$$

Proof: $\alpha = n$ we have indeed from (7)

$$\left(\frac{\beta - iau}{\beta - iu}\right)^n = \left(a + (1-a)\frac{\beta}{\beta - iu}\right)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}(1-a)^k \left(\frac{\beta}{\beta - iu}\right)^k \tag{26}$$

namely a finite mixture of Erlang $\mathcal{E}_k(\beta)$ with binomial weights $\mathcal{B}(n, 1-a)$. Once again, the *pdf* 's of the Erlang laws are known therefore the density is simply given by (25) that concludes the proof. ■

When $\lambda/k = n \in \mathbb{N}^*$ the above results lead to a closed form representation of the transition density of a Γ-OU process (or better Erlang-OU process) in terms of a finite sum of Erlang densities plus a degenerate term.

The ambiguity in the mixture representation above with respect to that of Theorem 2.4 is apparently allowed because, in general, a mixture decomposition is not unique.

Corollary 2.7: *The transition density $p(x, t+s|y, s)$ of the Γ-OU(k, λ, β) with $\lambda/k = n$ and $n \in \mathbb{N}^*$ is*

$$p(x, t+s|y, s) = g_a(x - ay, n, \beta), \quad a = e^{-kt}. \tag{27}$$

where $g_a(\cdot, n, \beta)$ is defined in (25).

The said binomial decomposition, however, while legitimate for $\alpha = n$, cannot be extended to the general case of $\alpha > 0$. From the generalized binomial formula, the following infinite decomposition of $\chi_a(u, \alpha, \beta)$ in (7)

$$\begin{aligned}\left(\frac{\beta - iau}{\beta - iu}\right)^\alpha &= \left(a + (1-a)\frac{\beta}{\beta - iu}\right)^\alpha = a^\alpha \left(1 + \frac{1-a}{a}\frac{\beta}{\beta - iu}\right)^\alpha \\ &= a^\alpha \sum_{k=0}^{\infty} \binom{\alpha}{k} \left(\frac{1-a}{a}\frac{\beta}{\beta - iu}\right)^k = \sum_{k=0}^{\infty} \omega_k(a, \alpha) \left(\frac{\beta}{\beta - iu}\right)^k \qquad (1)\end{aligned}$$

$$\omega_k(a, \alpha) = \binom{\alpha}{k} a^{\alpha-k}(1-a)^k \tag{28}$$

looks again as another infinite mixture of Erlang laws $\mathcal{E}_k(\beta)$, we must remark that first this expansion definitely converges exclusively when it is

$$\left|\frac{1-a}{a}\frac{\beta}{\beta - iu}\right| \leq \frac{1-a}{a} < 1$$

which, for $0 < a < 1$, only happens if $\frac{1}{2} \leq a < 1$; and second, and mainly, that although the infinite sequence of the $\omega_k(a, \alpha)$ sums up to one, the generalized binomial coefficients $\binom{\alpha}{k}$ take also negative values for $k > \alpha + 1$, and hence the $\omega_k(a, \alpha)$ do not always constitute a legitimate probability distribution. As a consequence, the decomposition (28) is not in general a true mixture, even if it holds mathematically whenever it converges. In other

words (as an alternative to (22)), the *pdf* of the a-remainder Z_a can always be represented also as the following combination – let us call it a *pseudo-mixture* – of Erlang *pdf* 's

$$g_a(x) = a^{\alpha}\delta(x)\mathbb{1}_{x=0} + \sum_{k\geq 1}^{\infty} \omega_k(a,\alpha) f_{k,\beta}(x)\mathbb{1}_{x>0}, \quad \frac{1}{2} \leq a < 1. \tag{29}$$

When α is an integer, we fall back into the hypothesis of Theorem 2.6 and Equation (29) becomes a true mixture of a finite number of terms.

2.3. Simulation algorithms

The results of the previous sections show that the *chf* (9) of an Γ-OU(k,λ,β) coincides with that of the a-remainder Z_a of a gamma law $\Gamma(\alpha,\beta)$ by simply taking $a = e^{-k\Delta t}$ and $\alpha = \lambda/k$. Algorithm 1 summarizes then the procedure to generate the skeleton of a Γ-OU(k,λ,β) process over a time grid $t_0, t_1, \ldots, t_M$, $\Delta t_m = t_m - t_{m-1}$, $m = 1,\ldots,M$.

Algorithm 1

1: $X_0 = X(0) = 0$
2: **for** $m = 1,\ldots,M$ **do**
3: $\quad \alpha \leftarrow \lambda/k, \quad a \leftarrow e^{-k\Delta t_m}$
4: $\quad b \leftarrow B \sim \mathcal{B}(\alpha, 1-a)$ ▷ Generate a Polya $(\alpha, 1-a)$ *rv*
5: $\quad z_a^m \leftarrow Z_a^{(m)} \sim \mathcal{E}_b\left({}^{\beta}/_{a}\right)$; ▷ Generate an Erlang *rv* with rate ${}^{\beta}/_{a}$
6: $\quad X(t_m) \leftarrow a\,X(t_{m-1}) + z_a^{(m)}$.
7: **end for**

The simulation of Z_a is very simple and it is applicable with no parameter constraints. It is worthwhile noticing that although Algorithm 1 resembles to that proposed in McKenzie [20], it has the advantage of simulating Erlang *rv*'s only. When in particular $\lambda/k = \alpha$ is an integer n, the steps four and five in Algorithm 1 can be replaced with those in Algorithm 2

Algorithm 2

4: $b \leftarrow B \sim \mathcal{B}(n, 1-a)$ ▷ Generate a Binomial *rv*
5: $z_a^m \leftarrow Z_a^{(m)} \sim \mathcal{E}_b(\beta)$; ▷ Generate an Erlang *rv* with rate β

This last assumption could be considered even when λ/k is not an integer namely, when $\lambda \gg k$, one could take the integer part $\lfloor \lambda/k \rfloor$. This approximation would be valid for an Γ-OU with either a low mean-reversion rate k or a high number of expected jumps per unit of time λ.

The simulation of the $Z_a^{(m)}$, could also be implemented starting from the representation (29) of their density. Over the usual time grid, the constraint $\frac{1}{2} \leq a < 1$ implies that $k < \log 2/\Delta t_m$. For instance, in energy markets and financial applications it is common to assume $\Delta t_m = 1/365$ or $\Delta t_m = 1/252$ that correspond to $k < 253$ or $k < 175$, respectively, values that virtually cover all the realistic market conditions.

Under this parameter constraint, we can conceive an acceptance–rejection procedure based on the method of Bignami and de Matteis [21] for pseudo-mixtures with non-positive terms (see also Devroye [22] page 74). Denoting indeed $\omega_k(a,\alpha)^+ = \max\{\omega_k(a,\alpha), 0\}$ and $\omega_k(a,\alpha)^- = \min\{\omega_k(a,\alpha), 0\}$, so that $\omega_k(a,\alpha) = \omega_k(a,\alpha)^+ + \omega_k(a,\alpha)^-$, the approach of Bignami and de Matteis [21] relies on the remark that from (29) we have

$$g_a(x) \leq \sum_{k\geq 0}^{\infty} \omega_k(a,\alpha)^+ f_{k,\beta}(x) = \bar{g}(x) = cg(x) \tag{30}$$

where

$$1 < c = \sum_{k\geq 0}^{\infty} \omega_k(a,\alpha)^+ < \infty \quad p_k = \frac{\omega_k(a,\alpha)^+}{c} \quad g(x) = \sum_{k\geq 0}^{\infty} p_k f_{k,\beta}(x) \tag{31}$$

so that $g(x)$ turns out to be a true mixture of Erlang laws, namely the *pdf* of

$$V = \sum_{i=0}^{S} X_i \sim \mathcal{E}_S(\beta) \quad X_i \sim \mathcal{E}_1(\beta) \quad \boldsymbol{P}\{S = k\} = p_k,$$

The generation of Z_a in the steps four and five in Algorithm 1 can then be replaced by the acceptance–rejection solution in Algorithm 3.

Algorithm 3 $\frac{1}{2} \leq (a < 1)$

1: Generate S with law $\boldsymbol{P}\{S = k\} = p_k, k = 0, \ldots, N$
2: **while** $u \leq \frac{g_a(\bar{z})}{\bar{g}(\bar{z})}$ **do**
3: $\quad u \leftarrow U \sim \mathcal{U}([0,1])$ ▷ Generate a uniform
4: $\quad \bar{z} \leftarrow \bar{Z} \sim \mathcal{E}_S(1)$ ▷ Generate a standard Erlang
5: **end while**
6: **return** βz

The computational performance of this algorithm can be assessed by observing that for relatively small values of α the probability $\boldsymbol{P}\{S\} = 0$ is high, hence V and Z_a turn out to be approximately degenerate, so that Z_a can be set to 0 as well because the acceptance condition is always satisfied. On the other hand, an efficient acceptance–rejection algorithm should have c of Equation (31) as close to 1 as possible because c roughly represents the expected number of iterations that are needed.

Note that for $0 < \alpha \leq 1$ and $^1/_2 \leq a < 1$ we always have $\omega_0(a,\alpha)^+ = a^\alpha \geq 0.5$ with the minimum value 0.5 attained for $a = 0.5,\ \alpha = 1$, which coincides with the simulation of $Z \sim \mathcal{E}_{\mathcal{B}(1,1-a)}(1)$ (see Cufaro Petroni and Sabino [23]). This means that the concentration of the weights $\omega_k(a,\alpha)$ is mainly around $\omega_0(a,\alpha)$ (which is a positive number) because in the said range of a, α the negative coefficients $\omega_k(a,\alpha)^-$ are rather negligible.

We benchmark the performance of our algorithms to two alternatives available in the literature. For instance, the exact sequential simulation of a Γ-OU process can be achieved using the simulation procedure introduced in Lawrence [24] that coincides with

Algorithm 4

$X_0 = X(0) = 0$
for $m = 1, \dots, M$ **do**
  Generate $n \sim \mathcal{P}(\lambda \Delta t_m)$, ▷ Poisson *rv* with intensity $\lambda \Delta t_m$
  Generate n *iid* uniform *rv's* $u_1, \dots, u_n$, $u_i \sim \mathcal{U}([0,1])$.
  Sort $u_{[1]} < \cdots < u_{[n]}$,
  $\tau_i \leftarrow \Delta t_m u_{[i]}$, $i = 1, \dots n$,
  Generate n *iid* $J_n \sim \mathcal{E}_1(\beta)$, ▷ Exponential *rv's* with scale β
  $X(t_m) \leftarrow X(t_{m-1}) e^{-k\Delta t_m} + \sum_{i=1}^{n} e^{-k(\Delta t_m - \tau_i)} J_i$.
end for

the modifying Algorithm 6.2 page 174 in Cont and Tankov [11] as detailed in Algorithm 4

Of course, the sorting can be avoided and step 5 can be implemented using by exponential spacing as explained in Devroye [22] page 213. Algorithm 4 does not directly rely on the statistical properties described by the *chf* (9), but it is rather based on the definition of the process (8). In contrast to Algorithm 4, our approach has the obvious advantage of not requiring to draw the complete skeletons of the jump times between two time steps.

The second alternative, summarized in Algorithm 5, is the exact simulation approach recently illustrated in Qu et al. [14] that is based on Theorem 2.2.

Algorithm 5

$X_0 = X(0) = 0$
for $m = 1, \dots, M$ **do**
  Generate $n \sim \mathcal{P}(\lambda \Delta t_m)$, ▷ Poisson *rv* with intensity $\lambda \Delta t_m$
  Generate n *iid* uniform *rv's* $u_1, \dots, u_n$, $u_i \sim \mathcal{U}([0,1])$.
  $\beta_i \leftarrow \beta e^{k\Delta t_m u_i}$, $i = 1, \dots, n$.
  Generate n *iid* $\tilde{J}_i \sim \mathcal{E}_1(\beta_i)$, $i = 1, \dots, n$, ▷ Exponential *rv*'s with random rate β_i
  $X(t_m) \leftarrow X(t_{m-1}) e^{-k\Delta t_m} + \sum_{i=1}^{n} \tilde{J}_i$.
end for

Algorithm 5 avoids simulating the jump times of the Poisson process as well but still requires additional steps compared to Algorithm 1 which, as we will show in Section 4, is by far the best performing alternative.

3. Distributional properties of the biΓ-OU process

The ƀΓ distribution with parameters $\alpha_1, \beta_1, \alpha_2, \beta_2$ has been explored by Küchler and Tappe [25] in the context of financial mathematics. It is worthwhile noticing that the Variance Gamma distribution, that is extensively used in financial applications, belongs to the set of bilateral gamma laws. A bilateral gamma distribution can be seen as the law of the difference $X^{(u)} - X^{(d)}$ of two independent *rv*'s $X^{(u)}, X^{(d)}$ with $X^{(u)} \sim \Gamma(\alpha_1, \beta_1)$ and

$X^{(u)} \sim \Gamma(\alpha_2, \beta_2)$. Therefore the *chf* of the $\mathfrak{b}\Gamma(\alpha_1, \beta_1, \alpha_2, \beta_2)$ distribution is

$$\varphi(v) = \left(\frac{\beta_1}{\beta_1 - iv}\right)^{\alpha_1} \left(\frac{\beta_2}{\beta_2 + iv}\right)^{\alpha_2} = (\varphi_u(v))^{\alpha_1} (\varphi_d(v))^{\alpha_2} \tag{32}$$

Küchler and S. Tappe [25] have shown that such distribution is self-decomposable therefore it is a suitable stationary law of a Lévy-driven OU process. Based of the definition of self-decomposable distributions, the *chf* of the a-remainder of such a $\mathfrak{b}\Gamma$ law is

$$\chi_a(v) = \left(\frac{\beta_1 - iav}{\beta_1 - iv}\right)^{\alpha_1} \left(\frac{\beta_2 + iav}{\beta_2 + iv}\right)^{\alpha_2}, \quad 0 < a < 1. \tag{33}$$

It means that the a-remainder of a $\mathfrak{b}\Gamma(\alpha_1, \beta_1, \alpha_2, \beta_2)$ can be seen as the difference $Z_a^{(u)} - Z_a^{(d)}$ of two independent *rv*'s where $Z_a^{(u)}$ and $Z_a^{(d)}$ are distributed according to the laws $\Gamma_a^{++}(\alpha_1, \beta_1)$ and $\Gamma_a^{++}(\alpha_2, \beta_2)$, respectively (with the same a).

Now consider a *BDLP* being the difference of two independent compound Poisson processes with exponential jumps namely, $Z(t) = \sum_{n=1}^{N_1(t)} U_n - \sum_{m=1}^{N_2(t)} D_n$. $N_1(t)$ and $N_2(t)$ are two independent Poisson processes with intensities λ_1 and λ_2, respectively, whereas, $(U_n)_{n\in\mathbb{N}^*}$ and $(D_m)_{m\in\mathbb{N}^*}$ are independent sequences, both independent of $N_1(t)$ and of $N_2(t)$, of *iid* exponential *rv*'s with parameters β_1 and β_2, respectively. It is easy to verify that the *chf* of a process $X(t)$ solution of (1) is simply the product of the *chf* 's of two independent Γ-OU processes with parameters (k, λ_1, β_1) and (k, λ_2, β_2), respectively. The stationary law is simply recovered for $t \to +\infty$ and coincides with a $\mathfrak{b}\Gamma$ law. Once again, as in the case of a Γ-OU process, the *chf* of a biΓ-OU process with parameters $k, \lambda_1, \beta_1, \lambda_2, \beta_2$, denoted bi$\Gamma$-OU$(k, \lambda_1, \beta_1, \lambda_2, \beta_2)$ at time t is that of the a-remainder of a $\mathfrak{b}\Gamma(\lambda_1/k, \beta_1, \lambda_2/k, \beta_2)$ law, namely, a $\mathfrak{b}\Gamma_a^{++}(\lambda_1/k, \beta_1, \lambda_2/k, \beta_2)$ law plus a constant ax_0 when we take $a = e^{-kt}$.

Knowing that the nth cumulant $\kappa_n(X)$ of the difference X of two independent *rv*'s $X^{(u)}$ and $X^{(d)}$ is $\kappa_n(X^{(u)}) + (-1)^n \kappa_n(X^{(d)})$, after some algebra we find

$$\boldsymbol{E}[Z_a] = (1 - a)\left(\frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2}\right) \tag{34}$$

$$\boldsymbol{V}[Z_a] = (1 - a^2)\left(\frac{\alpha_1}{\beta_1^2} + \frac{\alpha_2}{\beta_2^2}\right) \tag{35}$$

$$\boldsymbol{Skew}[Z_a] = \frac{1 - a^3}{(1 - a^2)^{3/2}} \times \frac{2(\alpha_1\beta_2^3 - \alpha_2\beta_1^3)}{(\alpha_1\beta_2^2 - \alpha_2\beta_1^2)^{3/2}} \tag{36}$$

$$\boldsymbol{Kurt}[Z_a] = \frac{1 + a^2}{1 - a^2} \times \frac{6(\alpha_1\beta_2^4 + \alpha_2\beta_1^4)}{(\alpha_1\beta_2^2 + \alpha_2\beta_1^2)^2} + 3. \tag{37}$$

The distributional properties of a $\mathfrak{b}\Gamma_a^{++}$ law are summarized by the following theorem.

Theorem 3.1: *The chf* $\chi_a(u,\alpha_1,\beta_1,\alpha_2,\beta_2)$ *and the pdf* $g_a(x,\alpha_1,\beta_1,\alpha_2,\beta_2)$ *of the* $\mathfrak{b}\Gamma_a^{++}(\alpha_1,\beta_1,\alpha_2,\beta_2)$ *law are*

$$\chi_a(v,\alpha_1,\beta_1,\alpha_2,\beta_2) = \sum_{n,m=0}^{\infty} b_n(a,\alpha_1)b_m(a,\alpha_2)\varphi_u^n(v)\varphi_d^m(v) \tag{38}$$

$$g_a(x,\alpha_1,\beta_1,\alpha_2,\beta_2) = a^{\alpha_1+\alpha_2}\delta(x)\mathbb{1}_{x=0} + \Bigg(a^{\alpha_1}(1-a)f_{n,\beta_1/a}(x) + a^{\alpha_2}(1-a)f_{m,\beta_2/a}(-x) + \sum_{n,m=1}^{\infty} b_n(a,\alpha_1)b_m(a,\alpha_2)f_{n,m,\beta_1/a,\beta_2/a}(x)\Bigg)\mathbb{1}_{x\neq 0}, \tag{39}$$

with

$$b_n(a,\alpha) = \binom{\alpha+n-1}{n}a^{\alpha}(1-a)^n$$

and

$$f_{n,m,\beta_1,\beta_2}(x) = \begin{cases} \dfrac{\beta_2 e^{\beta_2 x}}{(n-1)!}\left(\dfrac{\beta_1}{\beta_1+\beta_2}\right)^n \displaystyle\sum_{i=0}^{m-1}\frac{(n+m-i-2)!}{i!(m-i-1)!}\times \\ \left(\dfrac{\beta_2}{\beta_1+\beta_2}\right)^{m-i-1}(-\beta_2 x)^i \quad x<0, \\ \dfrac{\beta_1 e^{-\beta_1 x}}{(m-1)!}\left(\dfrac{\beta_2}{\beta_1+\beta_2}\right)^m \displaystyle\sum_{j=0}^{n-1}\frac{(n+m-j-2)!}{\ell!(n-j-1)!}\times \\ \left(\dfrac{\beta_1}{\beta_1+\beta_2}\right)^{n-j-1}(\beta_1 x)^j \quad x\geq 0 \end{cases}, \tag{40}$$

where $f_{n,m,\beta_1,\beta_2}(x)$ *represents the pdf of the difference* $E_u - E_d$ *of two independent Erlang distributed rv's* $E_u \sim \mathcal{E}_n(\beta_1)$ *and* $E_d \sim \mathcal{E}_d(\beta_2)$*, respectively.*

Proof: As already observed, the law $\mathfrak{b}\Gamma_a^{++}(\alpha_1,\beta_1,\alpha_2,\beta_2)$ is the law of the difference $Z_a^{(u)} - Z_a^{(d)}$ of two independent *rv*'s $Z_a^{(u)}$ and $Z_a^{(d)}$ distributed according to the $\Gamma_a^{++}(\alpha_1,\beta_1)$ and $\Gamma_a^{++}(\alpha_2,\beta_2)$, respectively. Hence, the *chf* in (38) is a simple consequence of Theorem 2.4.

On the other hand, the distributions of $Z_a^{(u)}$ and $Z_a^{(d)}$ can also be considered as two independent Erlang laws $\mathcal{E}_{S_u}(\beta_1)$ and $\mathcal{E}_{S_d}(\beta_2)$ with two independent Polya $\overline{\mathcal{B}}(\alpha_1,1-a)$ and $\overline{\mathcal{B}}(\alpha_2,1-a)$ distributed random indexes S_u and S_d, respectively. Hence, given $S_u = n$ and $S_d = m$ the distribution can be seen as the law of the difference of two independent Erlang *rv*'s whose *pdf* (40) is known in closed form (see also Simon [26] page 28). Combining all these observation leads to the conclusion that the *pdf* of Z_a is given by (39). ■

Corollary 3.2: *The transition density* $p(x,t+s|y,s)$ *of a* biΓ*-OU law with parameters* $\alpha_1 = \lambda_1/k$*,* β_1*,* $\alpha_2 = \lambda_2/k$ *and* β_2 *is*

$$p(x,t+s|y,s) = g_a\left(x-ay,\frac{\lambda_1}{k},\beta_1,\frac{\lambda_2}{k},\beta_2\right), \quad a = e^{-kt}. \tag{41}$$

where $g_a(x-ay,\frac{\lambda_1}{k},\beta_1,\frac{\lambda_2}{k},\beta_2)$ *is defined in* (39).

We now study the case where the *BDLP* $Z(t)$ is a compound Poisson whose jumps are now distributed according to a double exponential law that is mixture of a positive exponential *rv* $U \sim \mathcal{E}_1(\beta_1)$ and a negative exponential *rv* $-D \sim \mathcal{E}_1(\beta_2)$ with mixture parameters p and $q = 1-p$ with the following *pdf* and *chf*

$$f_{\beta_1,\beta_2,p}(x) = p\beta_1 e^{-\beta_1 x}\mathbb{1}_{x\geq 0} + (1-p)\beta_2 e^{\beta_2 x}\mathbb{1}_{x<0} \tag{42}$$

$$\varphi_{\beta_1,\beta_2,p}(v) = p\frac{\beta_1}{\beta_1 - iv} + (1-p)\frac{\beta_2}{\beta_2 + iv} = p\varphi_u(v) + (1-p)\varphi_d(v). \tag{43}$$

Theorem 3.3: *Let* $X(t)$ *be the solution of* (1) *where the BDLP is a compound Poisson whose jumps are distributed according to the law with pdf and chf in* (43) *and* (42)*, respectively, then the chf of* $X(t+s)$ *conditional on* $X(s)$ *is given by*

$$\boldsymbol{E}\left[e^{ivX(t+s)}|X(s)\right] = e^{ivX(s)e^{-kt}} \times \left(\frac{\beta_1 - ive^{-kt}}{\beta_1 - iv}\right)^{\frac{p\lambda}{k}} \times \left(\frac{\beta_2 + ive^{-kt}}{\beta_2 + iv}\right)^{\frac{(1-p)\lambda}{k}} \tag{44}$$

Proof: Based on the results of Dassios and Jang [27] and Kluge [2], the logarithm of *chf* of $X(t+s)$ conditional on $X(s)$ is given by

$$\log \boldsymbol{E}\left[e^{ivX(t+s)}|X(s)\right] = ivX(s)e^{-kt} + \lambda\int_0^t \left(\varphi_J(ve^{-kw}) - 1\right)\mathrm{d}w,$$

where $\varphi_J(v)$ is the *chf* of the double exponential in (43), therefore we have

$$\begin{aligned}\log \boldsymbol{E}\left[e^{ivX(t+s)}|X(s)\right] = {} & ivX(s)e^{-kt} + p\lambda\int_0^t \left(\varphi_u(ve^{-kw}) - 1\right)\mathrm{d}w \\ & + (1-p)\lambda\int_0^t \left(\varphi_d(ve^{-kw}) - 1\right)\mathrm{d}w,\end{aligned}$$

hence solving the integrals in the second and third terms we have

$$\boldsymbol{E}\left[e^{ivX(t+s)}|X(s)\right] = e^{iuX(s)e^{-kt}} \times \left(\frac{\beta_1 - ive^{-kt}}{\beta_1 - iv}\right)^{\frac{p\lambda}{k}} \left(\frac{\beta_2 + ive^{-kt}}{\beta_2 + iv}\right)^{\frac{(1-p)\lambda}{k}}, \tag{45}$$

that concludes the proof. ■

The stationary law is simply recovered for $t \to +\infty$ and coincides with a $\mathfrak{b}\Gamma$ law as summarized by the following corollary.

Corollary 3.4: *The stationary law of* $X(t)$ *is a* $\mathfrak{b}\Gamma$ *law with parameters* $\alpha_1 = p\lambda/k$, β_1, $\alpha_2 = (1-p)\lambda/k$ *and* β_2 *with* $0 < p < 1$.

Once again, the *chf* of a biΓ-OU process at time t is that of the a-remainder of a $\mathfrak{b}\Gamma(p\lambda/k, \beta_1, (1-p)\lambda/k, \beta_2)$ law plus a constant ax_0 when we take $a = e^{-kt}$.

3.1. Symmetric biΓ-OU process

The results of the previous subsection simplify when the stationary law of the biΓ-OU process is the law of the difference of two independent gamma *rv*'s with $\alpha = \alpha_1 = \alpha_2$ and $\beta = \beta_1 = \beta_2$. For such a stationary law the *BDLP* coincides with a compound Poisson whose jumps are distributed according to a centred Laplace law. A simple consequence of Theorem 3.1 is given in the following corollary

Corollary 3.5: *The chf and the pdf of the a-remainder of a symmetric* $\mathfrak{b}\Gamma(\alpha, \beta)$ *are*

$$\chi_a(u) = \sum_{k=0}^{\infty} \binom{\alpha + k - 1}{k} a^{2\alpha} \left(1 - a^2\right)^k \left(\frac{\beta^2}{\beta^2 + a^2 u^2}\right)^k$$

$$g_a(x, \alpha, \beta) = a^{2\alpha}\delta(x)\mathbb{1}_{x=0} + \sum_{k=1}^{\infty} \binom{\alpha + k - 1}{k} a^{2\alpha} \left(1 - a^2\right)^k \bar{f}_{n,\beta/a}(x)\mathbb{1}_{x\neq 0}$$

where

$$\bar{f}_{n,\beta}(x) = \frac{\beta}{2^n(n-1)!} (\beta|x|)^{n-1} e^{-\beta|x|} \sum_{k=1}^{n-1} \frac{(n-1+k)!}{k!(n-1-k)!(2\beta)^k|x|^k} \tag{46}$$

represents the pdf of a symmetric bilateral Erlang law, namely the law of the difference $E_u - E_d$ *of two independent Erlang distributed rv's having the same parameters n and* β.

Once again the law of the a-remainder is an infinite Polya $\overline{\mathcal{B}}(\alpha, 1 - a^2)$-weighted mixture of bilateral Erlang laws with parameter β/a. Hence, taking $a = e^{-kt}$ and $\alpha = \frac{\lambda}{2k}$, the law of a symmetric biΓ-OU at time t coincides with the *chf* of the a-remainder law of a gamma difference whose transition density $p(x, t + s|y, s) = g_a(x - ay, \frac{\lambda}{2k}, \beta)$. In addition,

$$\boldsymbol{E}[Z_a] = 0 \tag{47}$$

$$\boldsymbol{V}[Z_a] = (1 - a^2) \times \frac{2\alpha}{\beta} \tag{48}$$

$$\boldsymbol{Skew}[Z_a] = 0 \tag{49}$$

$$\boldsymbol{Kurt}[Z_a] = \frac{1 + a^2}{1 - a^2} \times \frac{3}{\alpha} + 3, \tag{50}$$

$\boldsymbol{E}[X(t)] = ax_0$ while the variance, the skewness and kurtosis of $X(t)$ and Z_a coincide because these quantities are translation invariant. Finally, we remark that for $\alpha = n \in \mathbb{N}^*$ it is straightforward to extend Theorem 2.6 and to represent the law of the a-remainder of a symmetric $\mathfrak{b}\Gamma$ law as a binomial mixture of bilateral Erlang distributions. It suffices to replace $\beta/(\beta - iu)$ in (24) with $\beta^2/(\beta^2 + u^2)$ and $f_{n,\beta}(x)$ in (25) with $\bar{f}_{n,\beta}(x)$. Finally, the representation based on the generalized binomial theorem at the end of subsection 2.2 can also be extended to the case of symmetric $\mathfrak{b}\Gamma$ laws replacing $\omega_k(a, \alpha)$ with $\omega_k(a^2, \alpha)$ under the constrain $\frac{1}{\sqrt{2}} \leq a < 1$ due to the fact that the parameters of the Polya distribution in Corollary 3.5 are α and now $1 - a^2$.

3.2. Simulation algorithms

We have seen that the law of the a-remainder $\mathfrak{b}\Gamma_a^{++}(\beta_1,\alpha_1,\beta_2,\alpha_2)$ coincides with that of the difference of the independent a-remainder's $\Gamma_a^{++}(\beta_1,\alpha_1)$ and $\Gamma_a^{++}(\beta_2,\alpha_2)$, respectively. We have also observed that such a distribution coincides with the law at time t of the biΓ-OU$(k,\lambda_1,\beta_1,\lambda_2,\beta_2)$ process if one sets $\alpha_1=\lambda_1/k$ $\alpha_2=\lambda_2/k$ and $a=e^{-kt}$. Based on Theorems 2.1 and 2.2 then, the simulation of the increment of such a process consists of nothing less than implementing the algorithms detailed in Section 2.3 two times. To this end, for sake of brevity, the detailed steps are not repeated here.

Instead, we here detail some simulation algorithms tailored to the symmetric case. For instance, because of Corollary 3.5, the implementation steps of Algorithm 1 can be replaced by those in Algorithm 6.

Algorithm 6

1: $X_0=X(0)=0$
2: **for** $m=1,\dots,M$ **do**
3: $\alpha\leftarrow\lambda/2k,\quad a\leftarrow e^{-k\Delta t_m}$
4: $b\leftarrow B\sim\overline{\mathcal{B}}(\alpha,1-a^2)$ ▷ Generate a Polya $(\alpha,1-a^2)$ *rv*
5: $z_a^{(r)}\leftarrow Z_a^{(r)}\sim\mathcal{E}_b\left({}^{\beta}/_{a}\right), r\in\{u,d\}$; ▷ Generate two independent Erlang *rv*'s with rate β/a
6: $z_a^m=z_a^{(u)}-z_a^{(d)}$
7: $X(t_m)\leftarrow a\,X(t_{m-1})+z_a^{(m)}$.
8: **end for**

In addition, one can adapt Algorithm 3 to the case of a symmetric biΓ-OU observing that

$$g_a(x)\le\sum_{k\ge0}^{\infty}\omega_k(a^2,\alpha)^+\bar{f}_{k,\beta}(x)=\bar{g}(x)=cg(x) \tag{51}$$

where

$$1<c=\sum_{k\ge0}^{\infty}\omega_k(a^2,\alpha)^+<\infty\quad p_k=\frac{\omega_k(a^2,\alpha)^+}{c}\quad g(x)=\sum_{k\ge0}^{\infty}p_k\bar{f}_{k,\beta}(x) \tag{52}$$

therefore $g(x)$ is the *pdf* of a true mixture of symmetric bilateral Erlang laws, namely, it is *pdf* of the random sum

$$V=\sum_{i=0}^{S}\left(X_i^{(u)}-X_i^{(d)}\right),\quad X_0^{(r)}=\boldsymbol{P}-a.s.,\quad \boldsymbol{P}\{S=k\}=p_k, r\in\{u,d\},$$

where $(X_i^{(u)})_{i\in\mathbb{N}}$ and $(X_i^{(d)})_{i\in\mathbb{N}}$ are independent sequences, independent of S, of *iid* exponentially distributed jumps sizes with rate β. The adaptation simply consists of replacing the forth step with Algorithm 7 and using the *pdf* 's in (51) and (52).

In addition, Algorithm 4 can also be extended simply replacing the seventh step by those shown in Algorithm 8.

Algorithm 7 ($\frac{1}{\sqrt{2}} \leq a < 1$)

5: $\bar{z}^{(r)} \leftarrow \bar{Z}^{(r)} \sim \mathcal{E}(S,1), r \in \{u,d\}$ ▷ Generate two independent standard Erlang *rv*'s.
6: $\bar{z} \leftarrow \bar{z}^{(u)} - \bar{z}^{(d)}$

Algorithm 8

7: Generate n *iid* $J_i^{(r)} \sim \mathcal{E}_1(\beta), i = 1,\ldots,n, r \in \{u,d\}$, ▷ Generate two sets of independent exponential *rv*'s with random rate β
8: $J_i \leftarrow J_i^{(u)} - J_i^{(d)}$

Finally, the following theorem extends the approach in Qu et al. [14] to the case of a symmetric biΓ-OU process and avoids having to run Algorithm 5 twice.

Theorem 3.6:

$$\boldsymbol{E}\left[e^{iuX(t+s)}|X(s)\right] = e^{iuX(s)e^{-kt}} \times e^{\lambda t\left(\varphi_{\tilde{L}}(u)-1\right)} \tag{53}$$

where

$$\varphi_{\tilde{L}}(u) = \int_0^1 \frac{\beta^2 e^{2kt\nu}}{\beta^2 e^{2kt\nu} + u^2}\,\mathrm{d}\nu \tag{54}$$

the right-hand side in (54) is then the *chf* of compound Poisson whose jumps are independent copies $\tilde{J}_i$ distributed according to a uniform mixture of centred Laplace laws with random parameter βe^{ktU} with $U \sim \mathcal{U}([0,1])$.

Proof: From Theorems 2.1 and 2.2 we know that

$$e^{\frac{\lambda t}{2}\left(\varphi_{\tilde{J}}(u)-1\right)} = \left(\frac{\beta - iue^{-kt}}{\beta - iu}\right)^{\frac{\lambda}{2k}}$$

where $\varphi_{\tilde{J}}(u)$ is defined in (12) then from Theorems 3.1 and 3.3 with $p = 1/2$, $\beta_1 = \beta_2$, we have

$$\boldsymbol{E}\left[e^{iuX(t+s)}|X(s)\right] = e^{iuX(s)e^{-kt}} \times e^{\frac{\lambda t}{2}t\left(\varphi_{\tilde{J}}(u)+\varphi_{\tilde{J}}(-u)-2\right)} = e^{iuX(s)e^{-kt}} \times e^{\lambda t\left(\frac{\varphi_{\tilde{J}}(u)+\varphi_{\tilde{J}}(-u)}{2}-1\right)}$$

on the other hand, we observe that

$$\frac{\varphi_{\tilde{J}}(u)+\varphi_{\tilde{J}}(-u)}{2} = \frac{1}{2}\int_0^1 \left(\frac{\beta e^{kt\nu}}{\beta e^{t\nu} - iu} + \frac{\beta e^{kt\nu}}{\beta e^{kt\nu} + iu}\right)\mathrm{d}\nu = \int_0^1 \frac{\beta^2 e^{kt\nu}}{\beta^2 e^{kt\nu} + u^2}\,\mathrm{d}\nu$$

that concludes the proof. ■

It turns out that a symmetric biΓ-OU can be simulated as detailed in Algorithm 9

Algorithm 9

for $m = 1, \dots, M$ **do**
  Generate $n \sim \mathcal{P}(\lambda \Delta t_m)$, ▷ Poisson *rv* with intensity $\lambda \Delta t_m$
  Generate n *iid* uniform *rv*'s $u_1, \dots, u_n$, $u_i \sim \mathcal{U}([0,1])$.
  $\beta_i^{(r)} \leftarrow \beta e^{k \Delta t_m u_i}, i = 1, \dots, n, r \in \{u, d\}$.
  Generate n *iid* $\tilde{J}_i^{(r)} \sim \mathcal{E}_1(\beta_i^{(r)}), i = 1, \dots, n,$ ▷ Generate two sets of independent exponential *rv*'s with random rate $\beta_i^{(r)}$
  $\tilde{J}_i \leftarrow \tilde{J}_i^{(u)} - \tilde{J}_i^{(d)}$
  $X(t_m) \leftarrow X(t_{m-1}) e^{-k \Delta t_m} + \sum_{i=1}^{n} \tilde{J}_i$.
end for

4. Simulation experiments

In this section, we compare the performance of the Algorithms detailed in Subsection 2.3 for the Γ-OU process and in Subsection 3.2 for the biΓ-OU process. The performance is ranked in terms of convergence and in terms of CPU times. In this comparison, we have decided to exclude the analysis relative to new Algorithm 2 because its applicability is covered by Algorithm 1. Their computational cost is also similar because both of them rely on the simulation of known and simple discrete *rv*'s and on the generation of Erlang *rv*'s although with different parameters.

All the simulation experiments in the present paper have been conducted using *MATLAB R2019a* with a 64-bit Intel Core i5-6300U CPU, 8GB.[1] As an additional validation, the comparisons of the simulation computational times have also been performed with *R* and *Python* leading to similar conclusions. We first consider a Γ-OU process with parameters $(k, \lambda, \beta, x_0) = (36, 10, 3, 0)$ and we only simulate one time step at $\Delta t = 1/365$. We observe that Algorithm 1 is still suitable because $0.5 \leq a < 1$ ($a = e^{-k \Delta t} \approx 0.9061$) where we have truncated the series in (30) and (31) at the 40th term.

In realistic examples, one could estimate the parameters relying on the closed form of the transition densities of the process, using the generalized method of moments or the least squares method. The idea of coupling a Γ-OU process with a Gaussian-OU process is common in the modelling of energy prices (see among others for instance, Kjaer [3] and Kluge [2]), indeed, the choice of the parameters above is motivated by the fact that these numbers look like realistic values that can be adopted for the pricing of energy facilities. Applications of the Γ-OU process to portfolio selection or to credit risk can also be found in Schoutens and Cariboni [5] and Bianchi and Fabozzi [6]. Beyond the financial world, such processes have been used to model computer failures (see Gaver and Lewis [9]) or wind speed (see Brown et al. [8]).

Table 1 reports the CPU times in seconds of all the approaches and compares the Monte Carlo estimated values of the true $\boldsymbol{E}[X(t)]$, $\boldsymbol{V}[X(t)]$, $\boldsymbol{Skew}[X(t)]$ and $\boldsymbol{Kurt}[X(t)]$ at a single time point $t = 1/365$. Varying the number of simulations N_S, we can conclude that all the algorithms are equally convergent, although it seems that a large number of simulations is required to achieve a good estimate of the kurtosis. On the other hand, their computational performance is quite different. Figure 1(a,b) clearly show that Algorithm 1 by far outperforms all other approaches. It provides a remarkable improvement in terms of computational time that is at least 30 times smaller than that of any other alternative

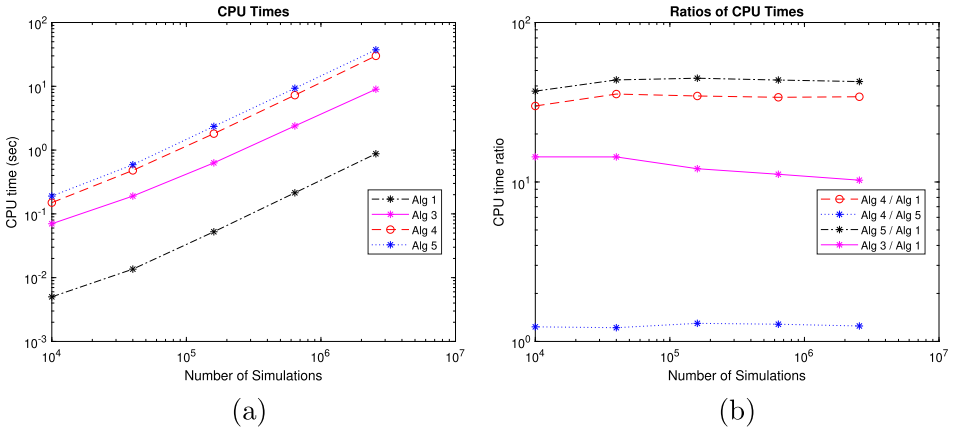


Figure 1. Γ-OU with $(k, \lambda, \beta, x_0) = (36, 10, 3, 0)$, $\Delta t = 1/365$. (a) CPU times in seconds and (b) Ratios CPU times.

available in the literature. With our computer generating $N_S = 2,560,000$ values of $X(t)$ at time $t = 1/365$ does not even take a second in contrast to several seconds using the other alternatives. Note that even the new Algorithm 3 is faster than Algorithms 4 and 5 although based on a acceptance–rejection method, but unfortunately, it is only applicable under the constraint $0.5 \leq e^{-k\Delta t} < 1$. To conclude, it seems that Algorithms 4 and 5 exhibit similar CPU times.

Of course, the superior performances of Algorithm 1 with respect to all the alternatives becomes even more remarkable when the entire trajectory over a time grid is simulated. To this end, we generate the skeleton of the process on an equally space time grid $t_1, \ldots, t_M$ with $M = 4$ and $\Delta t = 1/4$. In order to better highlight the difference in performance among the approaches, we have here chosen the same parameters set as in Qu et al. [14], $(k, \lambda, \beta, x_0) = (0.5, 1, 1, 10)$.

The results in Table 2 confirm that our proposal provides the smallest CPU times making Algorithm 1 very attractive for real-time calculations. We remark that the CPU times in Table 2 are relative to the simulation of the entire trajectory with four time steps while instead, the estimated statistics refer to the process at the last time point $t = 1$. Refining the time grid with a smaller time step will increase the overall computational times almost linearly making all alternatives to Algorithm 1 not competitive for real-time applications. It is also worthwhile noticing that our implementation, although based on a less powerful computer, returns smaller CPU times than those reported in Qu et al. [14] relative these authors' approach. Finally, as described in Section 3, the simulation of a biΓ-OU process can be obtained by repeating the algorithms above two times therefore, we can extrapolate the same conclusions with regards to the biΓ-OU case.

We conclude this section illustrating the results of the numerical experiments relative to a symmetric biΓ-OU process where we have chosen the same set of parameters selected for the Γ-OU process. $\boldsymbol{E}[X(t)] = x_0 e^{-kt}$ and the skewness is zero therefore in Table 3 we show the CPU times in seconds and the Monte Carlo estimated values of the true $\boldsymbol{V}[X(t)]$ and ***Kurt*** $[X(t)]$ at time $t = 1/365$ only. We also remark that Algorithm 7 is also applicable because $\sqrt{2}/2 \leq a < 1$ using both parameter sets. The conclusions are very much in line

Table 1. CPU times in seconds and comparison among the true $\boldsymbol{E}[X(t)]$, $\boldsymbol{V}[X(t)]$, $\boldsymbol{Skew}[X(t)]$ and $\boldsymbol{Kurt}[X(t)]$ of a Γ-OU process at time $t = 1/365$ with $(k, \lambda, \beta, x_0) = (36, 10, 3, 0)$ and their relative estimated values with N_S MC scenarios using Algorithms 1, 3, 4 and 5.

		$\boldsymbol{E}[X(t)] = 0.0087$		$\boldsymbol{V}[X(t)] = 0.0055$		$\boldsymbol{Skew}[X(t)] = 12.83$		$\boldsymbol{Kurt}[X(t)] = 222.71$	
Algorithm 1									
N_S	CPU	MC	error %	MC	error %	MC	error %	MC	error %
10, 000	0.0050	0.0089	2.12	0.0052	6.42	11.25	12.30	157.64	27.25
40, 000	0.0136	0.0091	4.27	0.0060	8.89	12.41	3.24	198.06	8.60
160, 000	0.0524	0.0084	3.76	0.0053	4.92	13.01	1.44	227.59	5.02
640, 000	0.2139	0.0086	1.07	0.0054	1.95	12.83	0.01	220.47	1.74
2, 560, 000	0.8757	0.0088	1.42	0.0056	1.81	12.69	1.09	220.91	1.94
Algorithm 3									
10, 000	0.0718	0.0083	4.70	0.0049	12.22	12.04	6.16	189.14	12.72
40, 000	0.1948	0.0093	6.71	0.0062	11.60	12.67	1.25	209.80	3.19
160, 000	0.6349	0.0092	5.53	0.0060	8.49	12.48	2.72	205.54	5.15
640, 000	2.3906	0.0088	0.63	0.0055	0.11	12.69	1.08	216.85	0.07
2, 560, 000	8.9852	0.0087	0.19	0.0055	0.07	12.74	0.71	217.53	0.38
Algorithm 4									
10, 000	0.15	0.0077	11.23	0.0044	20.21	13.46	4.93	248.17	14.52
40, 000	0.48	0.0085	2.77	0.0050	9.54	11.92	7.08	182.57	15.75
160, 000	1.81	0.0084	3.18	0.0053	4.46	13.18	2.72	237.77	9.72
640, 000	7.26	0.0087	0.01	0.0055	0.05	12.79	0.30	220.48	1.74
2, 560, 000	29.98	0.0086	0.64	0.0054	1.45	12.83	0.01	222.45	2.65
Algorithm 5									
10, 000	0.19	0.0090	3.33	0.0056	0.94	11.45	10.71	160.49	25.94
40, 000	0.59	0.0090	3.93	0.0059	6.70	12.28	4.24	191.69	11.54
160, 000	2.35	0.0085	1.79	0.0053	3.66	12.83	0.02	224.77	3.72
640, 000	9.32	0.0086	0.75	0.0056	0.52	13.23	3.17	220.55	1.77
2, 560, 000	37.32	0.0088	0.65	0.0056	0.73	12.71	0.89	217.25	0.25

with what found for a Γ-OU process. As expected, all the approaches are equally convergent and the CPU times are higher that those for the Γ-OU case because all the solutions require additional steps. From Figure 2(a,b), one can observe that Algorithm 6 is by far the fastest solution and Algorithm 7, even if based on an acceptance–rejection method, is once more a faster solution than Algorithms 8 and 9. On the other hand, these last two approaches seem to be equally fast with the former slightly outperforming the approach in Qu et al. adapted to the symmetric biΓ-OU process.

In Table 4, we also report the results of generating the trajectory of a symmetric biΓ-OU with the same parameters and time grid of the Γ-OU case. The values in Table 4 once more confirm that our newly developed simulation approach, detailed Algorithm 6, exhibits high accuracy as well as efficiency and in particular, largely outperforms any other alternative.

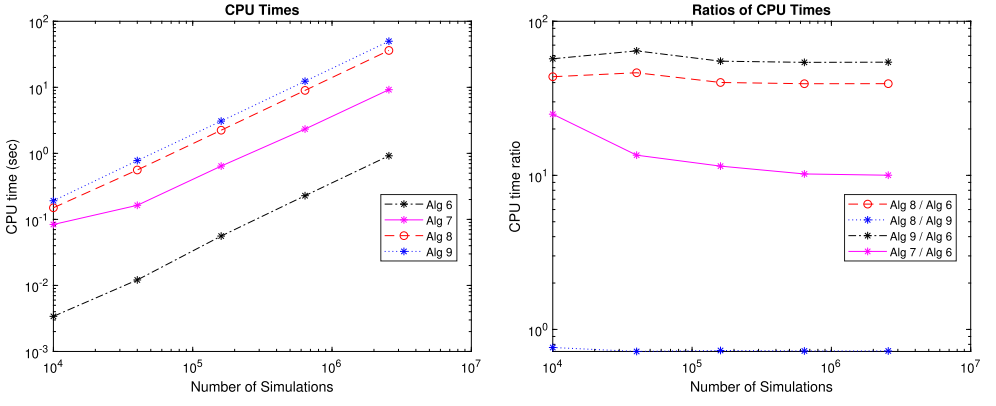
5. Conclusions and future inquiries

In this paper, we have studied the distributional properties of the Γ-OU process and its bilateral counterpart biΓ-OU process. We have shown that the *chf* 's and the *pdf* 's of such laws can be represented in closed form as a mixture of known and tractable laws, namely, a mixture of a Polya or a Binomial distribution.

As a simple consequence, we can design exact and efficient algorithms to generate the trajectory of a Γ-OU and a biΓ-OU process. Our numerical experiments have illustrated

Table 2. CPU times in seconds and comparison among the true $\boldsymbol{E}[X(t)]$, $\boldsymbol{V}[X(t)]$, $\boldsymbol{Skew}[X(t)]$ and $\boldsymbol{Kurt}[X(t)]$ of a Γ-OU process at time $t = 1$ with $(k, \lambda, \beta, x_0) = (0.5, 1, 1, 10)$ and their relative estimated values with N_S MC scenarios using Algorithms 1, 3, 4 and 5.

		$\boldsymbol{E}[X(t)] = 6.8522$		$\boldsymbol{V}[X(t)] = 1.2642$		$\boldsymbol{Skew}[X(t)] = 2.1861$		$\boldsymbol{Kurt}[X(t)] = 9.4919$	
Algorithm 1									
N_S	CPU	MC	error %	MC	error %	MC	error %	MC	error %
10, 000	0.0203	6.8612	0.13	1.2922	2.17	2.1615	−1.14	9.0429	−4.97
40, 000	0.0262	6.8580	0.08	1.3048	3.11	2.2173	1.41	9.6260	1.39
160, 000	0.1752	6.8476	−0.07	1.2647	0.04	2.2338	2.14	9.9624	4.72
640, 000	0.7359	6.8531	0.01	1.2680	0.30	2.1955	0.43	9.6146	1.28
2, 560, 000	3.1980	6.8520	0.00	1.2625	−0.13	2.1904	0.20	9.5559	0.67
Algorithm 3									
10, 000	0.238	6.8358	−0.24	1.1968	−5.63	2.1230	−2.97	8.7994	−7.87
40, 000	0.721	6.8537	0.02	1.2628	−0.11	2.1580	−1.30	9.3910	−1.07
160, 000	2.531	6.8549	0.04	1.2636	−0.05	2.1902	0.19	9.6369	1.50
640, 000	14.451	6.8526	0.01	1.2655	0.10	2.1828	−0.15	9.4517	−0.43
2, 560, 000	60.933	6.8512	−0.01	1.2602	−0.32	2.1903	0.19	9.5654	0.77
Algorithm 4									
10, 000	0.41	6.8664	0.21	1.2755	0.89	2.1181	−3.21	8.9754	−5.76
40, 000	1.37	6.8423	−0.14	1.2279	−2.96	2.1568	−1.36	9.2733	−2.36
160, 000	5.53	6.8578	0.08	1.2830	1.47	2.1674	−0.86	9.1683	−3.53
640, 000	22.09	6.8518	−0.01	1.2673	0.25	2.1968	0.49	9.5983	1.11
2, 560, 000	96.19	6.8523	0.00	1.2637	−0.04	2.1847	−0.07	9.5011	0.10
Algorithm 5									
10, 000	0.44	6.8499	−0.03	1.2870	1.77	2.3304	6.19	10.879	12.75
40, 000	1.63	6.8431	−0.13	1.2270	−3.03	2.1825	−0.16	9.5881	1.00
160, 000	6.54	6.8515	−0.01	1.2620	−0.18	2.1988	0.58	9.6740	1.88
640, 000	29.53	6.8535	0.02	1.2688	0.36	2.1898	0.17	9.5400	0.50
2, 560, 000	100.99	6.8525	0.00	1.2655	0.10	2.1871	0.05	9.4898	−0.02



(a) (b)

Figure 2. Symmetric biΓ-OU with $(k, \lambda, \beta, x_0) = (36, 10, 3, 0)$, $\Delta t = 1/365$. (a) CPU times in seconds and (b) Ratios CPU times.

that our strategy has a remarkable computational advantage and cuts the simulation time down by a factor larger than 30 compared to the existing alternatives available in the literature. In particular, due to the very small computational times, they are well suitable for real-time applications.

Table 3. CPU times in seconds and comparison among the true $\boldsymbol{V}[X(t)]$ and $\boldsymbol{Kurt}[X(t)]$ of a symmetric biΓ-OU process at time $t = 1/365$ with $(k, \lambda, \beta, x_0) = (36, 10, 3, 0)$ and their relative estimated values with N_S MC scenarios using Algorithms 6, 7, 8 and 9.

	$\boldsymbol{V}[X(t)] = 0.0055$			$\boldsymbol{Kurt}[X(t)] = 222.71$		$\boldsymbol{V}[X(t)] = 0.0055$			$\boldsymbol{Kurt}[X(t)] = 222.71$	
	Algorithm 6					Algorithm 7				
N_S	CPU	MC	error %	MC	error %	CPU	MC	error %	MC	error %
10, 000	0.0034	0.0062	12.9	141.39	34.8	0.0837	0.0066	20.1	218.45	0.8
40, 000	0.0121	0.0055	0.97	202.53	6.54	0.1634	0.0057	2.96	176.28	18.66
160, 000	0.0559	0.0054	2.83	207.09	4.44	0.6415	0.0059	6.20	227.93	5.18
640, 000	0.2284	0.0054	2.35	228.70	5.53	2.3321	0.0057	2.79	217.46	0.34
2, 560, 000	0.9188	0.0055	1.15	211.83	2.25	9.2060	0.0055	0.39	218.86	0.99
	Algorithm 8					Algorithm 9				
10, 000	0.15	0.0063	13.2	266.74	23.1	0.1915	0.0052	6.4	190.45	12.1
40, 000	0.56	0.0052	5.14	203.09	6.28	0.7769	0.0059	6.94	212.10	2.13
160, 000	2.24	0.0055	0.26	246.67	13.82	3.0793	0.0060	8.11	223.25	3.02
640, 000	8.98	0.0054	1.48	216.01	0.32	12.3615	0.0056	0.58	224.35	3.06
2, 560, 000	36.16	0.0055	0.54	223.38	0.52	49.8659	0.0055	0.31	217.83	0.52

Table 4. CPU times in seconds and comparison among the true $\boldsymbol{V}[X(t)]$ and $\boldsymbol{Kurt}[X(t)]$ of a symmetric biΓ-OU process at time $t = 1$ with $(k, \lambda, \beta, x_0) = (0.5, 1, 1, 10)$ and their relative estimated values with N_S MC scenarios using Algorithms 6, 7, 8 and 9.

	$\boldsymbol{V}[X(t)] = 1.2642$			$\boldsymbol{Kurt}[X(t)] = 9.4919$		$\boldsymbol{V}[X(t)] = 1.2642$			$\boldsymbol{Kurt}[X(t)] = 9.4919$	
	Algorithm 6					Algorithm 7				
N_S	CPU	MC	error %	MC	error %	CPU	MC	error %	MC	error %
10, 000	0.0157	1.2539	−0.82	9.306	−1.99	0.2502	1.2920	2.15	8.966	−5.87
40, 000	0.0193	1.2148	−4.07	9.370	−1.31	0.9680	1.2478	−1.32	9.249	−2.62
160, 000	0.1098	1.2598	−0.35	9.498	0.06	3.4811	1.2619	−0.19	9.450	−0.44
640, 000	0.4614	1.2617	−0.20	9.509	0.18	14.713	1.2576	−0.53	9.536	0.47
2, 560, 000	3.1008	1.2650	0.06	9.537	0.47	61.448	1.2648	0.05	9.509	0.18
	Algorithm 8					Algorithm 9				
10, 000	0.6143	1.2844	1.6	10.60	10.5	0.7889	1.2325	−2.57	8.222	−15.4
40, 000	2.2989	1.2950	2.38	9.668	1.83	3.1908	1.2771	1.01	9.168	−3.54
160, 000	9.2519	1.2578	−0.51	9.510	0.19	12.327	1.2564	−0.63	9.582	0.94
640, 000	36.601	1.2617	−0.20	9.579	0.91	48.689	1.2622	−0.16	9.422	−0.74
2, 560, 000	147.79	1.2628	−0.12	9.455	−0.39	199.71	1.2651	0.06	9.498	0.06

Moreover, although not the focus of our study, knowing the density in closed form and having simple formulas for the cumulants of the distribution, one could conceive a parameter estimation procedure based on likelihood methods and on the generalized method of moments using the analogy with the GAR(1) auto-regressive processes introduced in Gaver and Lewis [9] and discussed in Lawrence [28]. Of course, in any practical applications, some series truncation rule must be adopted as well as the generalization to time-dependent parameters is still open. These investigations will then be the focus of future inquires.

From the mathematical point of view, Sabino [29] has observed that the law of a Lévy-driven Ornstein–Uhlenbeck process at time t is always related to the law of the a-remainder of its relative self-decomposable stationary law: for any OU process $X(t) \stackrel{d}{=} a X_0 + Z_a$, $a = e^{-kt}$ where Z_a is the a-remainder of the stationary law of the process. It

would then be interesting to investigate the consequence of this observation and under which conditions one could extend for instance, the results in Zhang and Zhang [30], Zhang [31], Kawai and Masuda [32], Bianchi et al. [33] and the results recently published in Grabchak [34] relative to Tempered Stable Ornstein–Uhlenbeck processes to other Lévy-driven Ornstein–Uhlenbeck processes.

In addition, future studies could cover the extension to a multidimensional framework with correlated Poisson processes as those introduced for instance in Lindskog and McNeil [35] or in Cufaro Petroni and Sabino [23,36]. A last topic deserving further investigation is the time-reversal simulation of the Γ-OU and biΓ-OU processes generalizing the results of Pellegrino and Sabino [37] and Sabino [38] to the case of Γ-OU and biΓ-OU processes.

Note

1. The relative codes are available at https://github.com/piergiacomo75/GammaOUBiGammaOU

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References

[1] Barndorff-Nielsen OE, Shephard N. Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. J R Stat Soc Ser B. 2001;63(2):167–241.
[2] Kluge T. Pricing swing options and other electricity derivatives. Technical report, University of Oxford, 2006. PhD Thesis. Available at http://perso-math.univ-mlv.fr/users/bally.vlad/publications.html.
[3] Kjaer M. Pricing of swing options in a mean reverting model with jumps. Appl Math Finance. 2008;15(5–6):479–502.
[4] Benth FE, Pircalabu A. A non-Gaussian Ornstein-Uhlenbeck model for pricing wind power futures. Appl Math Finance. 2018;25(1):36–65.
[5] Schoutens W, Cariboni J. Lévy processes in credit risk. Chichester: John Wiley and Sons; 2010.
[6] Bianchi ML, Fabozzi FJ. Investigating the performance of non-Gaussian stochastic intensity models in the calibration of credit default swap spreads. Comput Econom. 2015 Aug;46(2):243–273.
[7] Madan DB, Seneta E. The variance gamma (v.g.) model for share market returns. J Business. 1990;63(4):511–24.
[8] Brown BG, Katz RW, Murphy AH. Time series models to simulate and forecast wind speed and wind power. J Climate Appl Meteorol. 1984;23:1184–1195.
[9] Gaver DP, Lewis PAW. First-order autoregressive gamma sequences and point processes. Adv Appl Probab. 1980;12(3):727–745.
[10] Sato K. Lévy processes and infinitely divisible distributions. Cambridge: Cambridge U.P.; 1999.
[11] Cont R, Tankov P. Financial modelling with jump processes. London: Chapman and Hall; 2004.
[12] Cufaro-Petroni N. Self-decomposability and self-similarity: a concise primer. Physica A Stat Mech Appl. 2008;387(7–9):1875–1894.
[13] Schoutens W. Lévy processes in finance: pricing financial derivatives. Chichester: John Wiley and Sons Inc; 2003.

[14] Qu Y, Dassios A, Zhao H. Exact simulation of gamma-driven ornstein–Uhlenbeck processes with finite and infinite activity jumps. J Operat Res Soc. 2019;1–14.
[15] Grigelionis B. On the self-decomposability of Euler's gamma function. Lithuanian Math J. 2003;43(3):295–305.
[16] Panjer HH, Willmot GE. Finite sum evaluation of the negative binomial-exponential model. ASTIN Bull. 1981;12(2):133–137.
[17] Popovici G, Dumitrescu M. Estimation on a GAR(1) process by the EM algorithm. Econom Qual Control. 2010;22(2):165–174.
[18] Zhang S, Zhang X, Sun S. Parametric estimation of discretely sampled gamma-OU processes. Sci China Ser A Math. 2006;49(9):1231–1257.
[19] Gatto R. A saddlepoint approximation to the distribution of inhomogeneous discounted compound Poisson processes. Methodol Comput Appl Probab. 2010;12:533–551.
[20] McKenzie E. Innovation distribution for gamma and negative binomial autoregressions. Scandinavian J Stat Theory Appl. 1987;14(1):79–85.
[21] Bignami A, de Matteis A. A Note on sampling from combination of distribution. J Inst Math Appl. 1971;8:80–81.
[22] Devroye L. Non-Uniform random variate generation. New York: Springer-Verlag; 1986.
[23] Cufaro-Petroni N, Sabino P. Coupling Poisson processes by self-decomposability. Mediterranean J Math. 2017;14(2):69.
[24] Lawrance AJ. The innovation distribution of a gamma distributed autoregressive process. Scandinavian J Stat. 1982;9:234–236.
[25] Küchler U, Tappe S. Bilateral gamma distributions and processes in financial mathematics. Stoch Process Their Appl. 2008;118(2):261–283.
[26] Simon MK. Probability distributions involving gaussian random variables. 2006. (The springer international series in engineering and computer science).
[27] Dassios A, Jang J. Pricing of Catastrophe Reinsurance and Derivatives Using the Cox Process with Shot Noise Intensity. Finance Stoch. 2003;7(1):73–93.
[28] Lawrance AJ. Some autoregressive models for point processes. In Bartfai P and Tomko J, editors. Point processes and queueing problems (colloquia mathematica societatis János Bolyai 24). Vol. 24, Amsterdam: North Holland; 1980. p. 257–275.
[29] Sabino P. Exact simulation of variance gamma-related ou processes: application to the pricing of energy derivatives. Appl Math Finance. 2020;1–21.
[30] Zhang S, Zhang X. On the Transition Law of Tempered Stable Ornstein-Uhlenbeck Processes. J Appl Probab. 2009;46(3):721–731.
[31] Zhang S. Exact simulation of tempered stable Ornstein-Uhlenbeck processes. J Stat Comput Simul. 2011;81(11):1533–1544.
[32] Kawai R, Masuda H. Exact discrete sampling of finite variation tempered stable Ornstein-Uhlenbeck processes. Monte Carlo Methods Appl. 2011;17(3):279–300.
[33] Bianchi ML, Rachev ST, Fabozzi FJ. Tempered stable Ornstein-Uhlenbeck processes: a practical view. Commun Stat Simulat Comput. 2017;46(1):423–445.
[34] Grabchak M. On the Simulation for General Tempered Stable Ornstein-Uhlenbeck Processes. J Stat Comput Simul. 2020;90(6):1057–1081.
[35] Lindskog F, McNeil J. Common poisson shock models: applications to insurance and credit risk modelling. ASTIN Bull. 2003;33(2):209–238.
[36] Cufaro-Petroni N, Sabino P. Pricing exchange options with correlated jump diffusion processes. Quant Finance. 2020;20(11):1811–1823.
[37] Pellegrino T, Sabino P. Enhancing least squares monte carlo with diffusion bridges: an application to energy facilities. Quant Finance. 2015;15(5):761–772.
[38] Sabino P. Forward or backward simulation? A comparative study. Quant Finance. 2020;20(7):1213–1226.