Asymptotic behaviour of densities for Nelson processes†

Nicola Cufaro Petroni*

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In a recent paper[(1)] an idea of Bohm ad Vigier[(2)] about the possible decay of every initial probability density function (pdf), whose evolution is ruled by the quantum Fokker-Planck equation, toward the quantum mechanical pdf has been discussed in the light of the stochastic mechanics. The causal interpretation of the quantum mechanics[(3)] is based on the idea that a non relativistic particle of mass m, whose wave function obeys the Schrödinger equation

$$i\hbar\partial_t\psi(\mathbf{r},t) = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r},t) + V(\mathbf{r},t)\psi(\mathbf{r},t)\,, \tag{1}$$

is a classical object following a continuous and causally defined trajectory with a well defined position and accompanied by a physically real wave field ψ which contributes to determine its motion. If we write down (1) in terms of the real functions $R(\mathbf{r},t)$ and $S(\mathbf{r},t)$ with

$$\psi(\mathbf{r},t) = R(\mathbf{r},t)\,\mathrm{e}^{iS(\mathbf{r},t)/\hbar} \tag{2}$$

and separate real and imaginary parts, we have

$$\partial_t R^2 + \nabla\left(R^2\frac{\nabla S}{m}\right) = 0\,, \tag{3}$$

$$\partial_t S + \frac{(\nabla S)^2}{2m} + V - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} = 0\,, \tag{4}$$

where $R^2(\mathbf{r},t) = |\psi(\mathbf{r},t)|^2$ is interpreted as the density of a fluid with stream velocity

$$v(\mathbf{r},t) = \frac{\nabla S}{m}\,. \tag{5}$$

It is important to remark now that, if we define

$$v_{(+)}(\mathbf{r},t) = \frac{\nabla S}{m} + \frac{\hbar}{2m}\frac{\nabla R^2}{R^2} \tag{6}$$

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* Dipartimento di Fisica dell' Universitá and I.N.F.N.; via Amendola 173, 70126 Bari, Italy; E-mail CUFARO@BARI.INFN.IT

the continuity equation (3) takes the form

$$\partial_t R^2 = \frac{\hbar}{2m}\nabla^2 R^2 - \nabla(R^2 v_{(+)}) \tag{7}$$

so that R^2 can also be considered as a particular solution of the evolution equation (Fokker-Planck equation)

$$\partial_t f = \nu\nabla^2 f - \nabla(f v_{(+)}) \tag{8}$$

for the pdf's of a Markov process characterized by the velocity field $v_{(+)}$ and by a diffusion coefficient

$$\nu = \frac{\hbar}{2m}\,. \tag{9}$$

This points out a possible connection between the density R^2 and the pdf of a suitable Markov process describing the random motion of a classical particle. In fact the causal interpretation is obliged to add some randomness to its deterministic description in order to reproduce the statistical predictions of the quantum mechanics and hence it identifies the function $R^2 = |\psi|^2$ with the pdf of an ensemble of particles. But, since this addition is made by hand, is it easy for the critics of the model to argue that "it should be possible to have an arbitrary probability distribution (a special case of which is the function $P = \delta(x - x_0)$ representing a particle in a well defined location) that is at least in principle independent of the ψ field and dependent only on our degree of information concerning the location of the particle"[(2)]. The physical idea of Bohm and Vigier was that, even if our ensemble of quantum systems is described by an arbitrary initial pdf, this will decay in time to an ensemble with pdf $|\psi|^2$, because of the random fluctuations arising from the interactions with a subquantum medium: "no matter what the initial probability distribution may have been (for example a delta function) it will eventually be given by $P = |\psi|^2$".

A more convincing connection between quantum mechanics and classical random phenomena is achieved by means of the stochastic mechanics[(4)]: here the particle position is promoted to a stochastic Markov process $\xi(t)$ defined on some probabilistic space $(\Omega, \mathcal{F}, \mathbf{P})$ and taking values (for our limited purposes) in $\mathbf{R}^3$. This process is characterized by a pdf $f(\mathbf{r}, t)$ and a transition pdf $p(\mathbf{r}, t|\,\mathbf{r}', t')$ and satisfies an Itô stochastic differential equation of the form

$$d\xi(t) = v_{(+)}\big(\xi(t), t\big)\,dt + d\eta(t) \tag{10}$$

where $v_{(+)}$ is a velocity field which plays the role of a dynamical variable not given a priori but subsequently determined on the basis of a variational principle, and $\eta(t)$ is a Brownian process independent of ξ and such that

$$\mathbf{E}_t\big(d\eta(t)\big) = 0\,, \qquad \mathbf{E}_t\big(d\eta(t)\,d\eta(t)\big) = 2\nu\,\mathbf{I}\,dt$$

where $d\eta(t) = \eta(t+dt) - \eta(t)$ (for $dt > 0$), ν is the diffusion coefficient, $\mathbf{I}$ is the 3×3 identity matrix and $\mathbf{E}_t$ are the conditional expectations with respect to $\xi(t)$. A suitable definition of the Lagrangian and of the stochastic action functional for the system described by the dynamical variables f and $v_{(+)}$ allows us to select, by means of the principle of stationarity of the action, the particular processes which reproduce the quantum mechanics. In this formulation the idea proposed by Bohm and Vigier can be checked as a property of the solutions of the Fokker-Planck equations with the field $v_{(+)}$ derived according to (6) from the wave functions solutions of (1).

In what follows we will limit ourselves to the case of the one dimensional trajectories, so that the Markov processes $\xi(t)$ considered will always take values in $\mathbf{R}$ and we will introduce a metrics induced by the norm in $L^1(\mathbf{R})$:

$$\mathbf{d}(f,g) = \frac{1}{2}\int_{-\infty}^{+\infty} |f(x) - g(x)|\, dx\,.$$

If the stochastic processes $\xi(t)$ under examination are Markov processes (as happens in stochastic mechanics) satisfying the equation (10) with initial condition $\xi(0) = \xi_0$, their pdf will satisfy the one dimensional evolution equation

$$\partial_t f(x,t) = \nu\partial_x^2 f(x,t) - \partial_x\left(v_{(+)}(x,t)f(x,t)\right), \tag{11}$$

with the initial condition $f(x,0) = f_0(x)$ if $f_0(x)$ is the pdf of ξ_0.

Definition 1: We will say that the pdf $f(x,t)$ L^1-*approximates* the pdf $g(x,t)$ (for $t \to +\infty$), and we will write

$$f(x,t) \overset{L^1}{\sim} g(x,t) \qquad (t \to +\infty)\,,$$

when

$$\mathbf{d}(f,g) \to 0 \qquad (t \to +\infty)\,.$$

In particular we will say that f L^1-*converges* toward g (for $t \to +\infty$) if the pdf $g(x)$ does not depend on the time t.

We will examinate next a few properties of the concept of L^1-approximation for processes satisfying the equation (10). First of all we can prove[1] the following proposition

Proposition 1: If f and g are solutions of (11), the distance $\mathbf{d}(f,g)$ is a monotonic non-increasing function of the time t.

In order to examinate the conditions that are sufficient to make the distance $\mathbf{d}(f,g)$ actually tend to zero when $t \to +\infty$, let us now introduce the following definition:

Definition 2: We will say that the family of the transition pdf's $p(x,t|\,y,0)$ L^1-*approximates* the pdf $g(x,t)$ in a *locally uniform way in* y (y - l.u.) for $t \to +\infty$, and we will write

$$p(x,t|\,y,0) \overset{L^1}{\sim} g(x,t) \qquad y \text{ - l.u.} \qquad (t \to +\infty)\,,$$

when for every $K > 0$ and for every $\epsilon > 0$ we can find a $T > 0$ such that

$$\mathbf{d}(p,g) = \mathbf{d}\big(p(x,t|\,y,0), g(x,t)\big) < \epsilon$$

for every $t > T$ and for every y such that $|y| \leq K$.

We can then prove[(1)] the following proposition:

Proposition 2: If $p(x,t|\,y,0) \overset{L^1}{\sim} g(x,t)\,,\; y$ - l.u.$\,,\; (t \to +\infty)\,,$ where p is the transition pdf of (11) and g an arbitrary pdf, then we have that $f(x,t) \overset{L^1}{\sim} g(x,t)\,,\; (t \to +\infty)\,,$ for every $f(x,t)$ solution of (11), and hence that $f_1(x,t) \overset{L^1}{\sim} f_2(x,t)\,,\; (t \to +\infty)\,,$ for every $f_1(x,t)$ and $f_2(x,t)$ solutions of (11).

In order to discuss a few examples in detail it will be useful to derive a formula to calculate the L^1-distance among the pdf's $\mathcal{N}(m,\sigma^2)$ of *normal* random variables , namely pdf's of the form

$$g_{m,\sigma}(x) = \frac{\mathrm{e}^{-(x-m)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}$$

with real m and $\sigma > 0$. In the following we will indicate with the symbol

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-y^2/2}\,dy$$

the usual *error function* and we will also pose $\mathbf{d}(a,b;\,p,q) = \mathbf{d}(g_{a,p}, g_{b,q})$. With the previous notations we can now prove[(1)] that, if $p > q$ we have

$$\mathbf{d}(a,b;\,p,q) = \left[\Phi\left(\frac{x_1-b}{q}\right) - \Phi\left(\frac{x_2-b}{q}\right)\right] - \left[\Phi\left(\frac{x_1-a}{p}\right) - \Phi\left(\frac{x_2-a}{p}\right)\right] \qquad (12)$$

where

$$x_1 = \frac{aq^2 - bp^2 - qp\sqrt{(a-b)^2 + 2(q^2-p^2)\ln(q/p)}}{q^2-p^2}$$

$$x_2 = \frac{aq^2 - bp^2 + qp\sqrt{(a-b)^2 + 2(q^2-p^2)\ln(q/p)}}{q^2-p^2}\,;$$

if $p = q$ and $a \neq b$ we have

$$\mathbf{d}(a,b;\,p,p) = 2\Phi\left(\frac{|b-a|}{2p}\right) - 1\,;$$

and finally, if $p = q$ and $a = b$ we have $\mathbf{d}(a,a;\,p,p) = 0$. This will be useful since in our examples both the transition pdf's and the pdf's derived from the quantum mechanical wave functions are normal, as indicated by the following proposition[(1)] :

Proposition 3: If the velocity field of the evolution equation (2) has the form

$$v_{(+)}(x,t) = -b(t)x - c(t)$$

with $b(t)$ and $c(t)$ continuous functions of time, then the fundamental solutions $p(x,t|\,y,0)$ are normal pdf's $\mathcal{N}\big(\mu(t),\beta(t)\big)$ where $\mu(t)$ and $\beta(t)$ are solutions of the equations

$$\mu'(t) + b(t)\mu(t) + c(t) = 0\,; \qquad \beta'(t) + 2b(t)\beta(t) - 2\nu = 0$$

with initial conditions $\beta(0) = 0$ and $\mu(0) = y$.

We will discuss now our examples for systems reduced to a single non relativistic particle with a mass m. Let us consider first of all a simple harmonic oscillator with elastic constant k and classical (circular) frequency $\omega = \sqrt{k/m}$ and two possible wave functions obeying the Schrödinger equation: the (stationary) wave function of the ground state

$$\psi_0(x,t) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/4} \mathrm{e}^{-x^2/4\sigma^2}\,\mathrm{e}^{-i\omega t/2}$$

and the (non stationary) wave function of the oscillating coherent wave packet with initial displacement a

$$\psi_C(x,t) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/4} \exp\left[-\frac{(x-a\cos\omega t)^2}{4\sigma^2} - i\left(\frac{4ax\sin\omega t - a^2\sin 2\omega t}{8\sigma^2} + \frac{\omega t}{2}\right)\right]$$

where we have defined $\sigma^2 = \nu/\omega$. From the position (2) we find for our wave functions that

$$R_0^2(x,t) = f_0(x,t) = \frac{\mathrm{e}^{-x^2/2\sigma^2}}{\sigma\sqrt{2\pi}}\,; \qquad R_C^2(x,t) = f_C(x,t) = \frac{\mathrm{e}^{-(x-a\cos\omega t)^2/2\sigma^2}}{\sigma\sqrt{2\pi}}$$
$$S_0(x,t) = -\frac{1}{2}\hbar\omega t\,; \qquad S_C(x,t) = -\frac{1}{2}\hbar\omega t - \hbar\frac{4ax\sin\omega t - a^2\sin\omega t}{8\sigma^2}$$

and hence we can calculate from (6) the corresponding velocity fields

$$v_{(+)}^0(x,t) = -\omega x\,; \qquad v_{(+)}^C(x,t) = -\omega x + \omega a(\cos\omega t - \sin\omega t)\,.$$

This means that f_0 and f_C are respectively of the form $\mathcal{N}(0,\sigma^2)$ and $\mathcal{N}(a\cos\omega t,\sigma^2)$, and that the fundamental solutions $p_0(x,t|\,y,0)$ and $p_C(x,t|\,y,0)$ of (11) can be calculated by means of Proposition 3, and they will respectively have the form $\mathcal{N}\big(\mu_0(t),\beta_0(t)\big)$ and $\mathcal{N}\big(\mu_C(t),\beta_C(t)\big)$ where

$$\beta_0(t) = \sigma^2\big(1-\mathrm{e}^{-2\omega t}\big)\,, \qquad \mu_0(t) = y\mathrm{e}^{-\omega t}$$
$$\beta_C(t) = \sigma^2\big(1-\mathrm{e}^{-2\omega t}\big)\,, \qquad \mu_C(t) = a\cos\omega t + (y-a)\mathrm{e}^{-\omega t}\,.$$

A second class of examples can be drawn from the wave functions of a free particle of mass m. In particular we will choose to examinate the behavior of the (non stationary) wave function of a wave packet of minimal uncertainty centered around $x = 0$ with initial dispersion $\sigma^2 > 0$:

$$\psi_F(x,t) = \left(\frac{1}{2\pi\sigma^2\chi^2(t)}\right)^{1/4} \mathrm{e}^{-x^2/4\sigma^2\chi(t)}$$

where $\chi(t) = 1 + i\omega t\,,\ \omega = \nu/\sigma^2$. In this case we have from (2)

$$R_F(x,t) = f_F(x,t) = \frac{\mathrm{e}^{-x^2/2\sigma^2\alpha^2(t)}}{\sqrt{2\pi}\sigma\alpha(t)}\,; \qquad S_F(x,t) = \frac{\hbar}{2}\left(\frac{\omega t x^2}{2\sigma^2\alpha^2(t)} - \arctan\omega t\right)$$

where $\alpha(t) = |\chi(t)| = \sqrt{1+\omega^2 t^2}$. This means that f_F is normal of the form $\mathcal{N}\big(0, \sigma^2\alpha^2(t)\big)$. Moreover the velocity field is

$$v_{(+)}^F(x,t) = -\frac{1-\omega t}{1+\omega^2 t^2}\omega x$$

and the fundamental solutions $p_F(x,t|\,y,0)$ of (11) can then be calculated by means of Proposition 3, and they will have the form $\mathcal{N}\big(\mu_F(t), \beta_F(t)\big)$ where

$$\mu_F(t) = y\sqrt{1+\omega^2t^2}\mathrm{e}^{-\arctan\omega t}\,; \qquad \beta_F(t) = \sigma^2(1+\omega^2t^2)\big(1 - \mathrm{e}^{-2\arctan\omega t}\big)\,.$$

We can now use (12) in order to calculate $\mathbf{d}(p_0, f_0)$, $\mathbf{d}(p_C, f_C)$ and $\mathbf{d}(p_F, f_F)$: a long but simple calculation will show that (y - l.u.) $p_0 \overset{L^1}{\sim} f_0\,,\ p_C \overset{L^1}{\sim} f_C$ $(t \to +\infty)$ in the examples drawn from the harmonic oscillator, but that p_F will not L^1-approximate f_F since $\mathbf{d}(p_F, f_F)$ turns out to be different from zero and still dependent on y in the limit $t \to +\infty$. In particular two transition pdf's with different initial conditions $y \neq y'$ will never L^1-approximate one another as $t \to +\infty$, unless $y = y'$. Hence every solution of the evolution equation (11) L^1-approximates the quantum mechanical pdf (for $t \to +\infty$) only in the examples of the harmonic oscillator but not in that of the free particle.

It is apparent from our examples that the Markov processes associated to the quantum mechanical wave functions by the stochastic mechanics do not always exhibit the behavior required by the Bohm and Vigier hypothesis. In fact the calculations show that, in order to recover the property of a global relaxation in time of the pdf's toward the quantum mechanical solution, we must restrict ourselves to a particular set of physical systems. In any case at least for a significant set of systems and wave functions the Bohm and Vigier property holds in the L^1-metrics if we adopt the transition pdf suggested by the Nelson stochastic mechanics, and hence it can be surely stated that their original idea posed an interesting an physically well grounded problem.

The fact that the Nelson transition pdf's do not always L^1-approximate one another also means that it is impossible to find a unique pdf g L^1-approximated by them independently from y, and hence that the solutions of (11) in the discussed free particle case will not globally tend to

L^1-approximate one another in time. Of course nothing forbids a priori, even in this case, that particular subsets of solutions can show the tendency to mutually L^1-approximate and hence the field is open to investigations about, for instance, the possibility that some particular solution of (11) can be stable with respect to small perturbations of their initial conditions: which in some minimal sense was the essential intention of the Bohm and Vigier proposal. It is not possible at present to state clearly and in a general way in which cases we realize the conditions for a global (or at least local) mutual L^1-approximation of the solutions of (11). Even the main difference between our systems, namely the fact that their energy spectra are very different (the harmonic oscillator has a completely discrete spectrum and the free particle a completely continuous one) seems not to be relevant since the first calculations about the case of harmonic oscillator stationary states *with nodes* indicate that there is not convergence to the quantum stationary pdf's. However it must be pointed out that we have made the very particular choice of selecting the transition pdf's of the Nelson stochastic mechanics as a good candidate to the generation of the right stochastic flux exhibiting the Bohm and Vigier property in some suitable sense. As a consequence another possible conclusion could also be that the Nelson flux is not the right candidate to represent, in the general case, the interpretative scheme of Bohm and Vigier. Hence we consider wide open the possibility that the right transition pdf's can be built in a different way. For example, since the transition pdf which propagates a given time-dependent pdf $f(\mathbf{r}, t)$ is not uniquely determined (and are not, in general, observable in the stochastic mechanics), nothing forbids to find a diffusive flux, different from that of Nelson, which exhibites the Bohm and Vigier property for every possible quantum wave function. In particular a possibility lies in a generalization of the stochastic mechanics where also the diffusive part of the stochastic differential equation ruling the process is dynamically determined in a way such that the Bohm and Vigier property is always satisfied. The first calculations with *ad hoc* chosen time dependent diffusion coefficients show that this is, at least in principle, possible.

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