

Non equilibrium densities of Nelson processes

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Abstract[‡]: In this paper we analyze the idea that arbitrary probability densities, solutions of the classical evolution equation associated by the Nelson stochastic mechanics to the quantum mechanics, always relax in time toward the quantum mechanical density $|\psi|^2$ derived from the Schrödinger equation. The analysis of a few general propositions and of some physical examples show that the choice of the Nelson stochastic flux is correct for a particular class of quantum states, but can not be adopted in general. This indicates that the question of the time relaxation of the solutions of the Kolmogorov equation associated to a quantum wave function toward the quantum densities is physically meaningful, even if a classical probabilistic model good for every quantum state is still not available. A few suggestion in this direction are finally discussed.

It has been known since longtime⁽¹⁾ that the Schrödinger equation

$$i\hbar\partial_t\psi(\mathbf{r}, t) = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}, t) + V(\mathbf{r}, t)\psi(\mathbf{r}, t), \quad (1)$$

can be analyzed in terms of the real functions $R(\mathbf{r}, t)$ and $S(\mathbf{r}, t)$ with

$$\psi(\mathbf{r}, t) = R(\mathbf{r}, t)e^{iS(\mathbf{r}, t)/\hbar} \quad (2)$$

so that by separating the real and imaginary parts, we have

$$\partial_t R^2 + \nabla \cdot \left(R^2 \frac{\nabla S}{m} \right) = 0, \quad (3)$$

$$\partial_t S + \frac{(\nabla S)^2}{2m} + V - \frac{\hbar^2}{2m} \frac{\nabla^2 R}{R} = 0, \quad (4)$$

where $R^2(\mathbf{r}, t) = |\psi(\mathbf{r}, t)|^2$ is interpreted as the density of a fluid with stream velocity

$$v(\mathbf{r}, t) = \frac{\nabla S}{m}. \quad (5)$$

The equation (3) expresses the conservation of the fluid while the equation (4) plays the role of a Hamilton-Jacobi equation for the velocity potential S in the presence of a quantum potential

$$-\frac{\hbar^2}{2m} \frac{\nabla^2 R}{R}$$

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which depends on the form of the wave function. It is important to remark now that, if we define

$$v_{(+)}(\mathbf{r}, t) = \frac{\nabla S}{m} + \frac{\hbar}{2m} \frac{\nabla R^2}{R^2} \quad (6)$$

the continuity equation (3) takes the form

$$\partial_t R^2 = \frac{\hbar}{2m} \nabla^2 R^2 - \nabla(R^2 v_{(+)}) \quad (7)$$

so that R^2 can always be considered as a particular solution of the evolution equation of the pdf's of a Markov process (forward Kolmogorov equation)

$$\partial_t f = \nu \nabla^2 f - \nabla(f v_{(+)}) \quad (8)$$

characterized by the velocity field $v_{(+)}$ and by a diffusion coefficient

$$\nu = \frac{\hbar}{2m}. \quad (9)$$

This points out a possible connection between the density R^2 of the Madelung fluid and the pdf of a suitable Markov process describing the random motion of a classical particle. However it must be remarked here that this connection looks purely formal, since, for exemple, while for a given $v_{(+)}$ we can determine an infinity of solutions of (8) (one for every initial condition $f(\mathbf{r}, 0) = f_0(\mathbf{r})$) which are pdf's of Markov processes, the quantum mechanics is characterized by the selection of just one particular solution $f = R^2$ among all the possibilities. In fact it must be emphasized that R^2 and $v_{(+)}$ are not independent: they are both derived from a ψ solution of (1) and hence are locked together by their common origin. In other words: not every couple of f , solution of (8), and $v_{(+)}$ can be considered as derived from the same solution of the Schrödinger equation (1) through the relation (6) and $f = R^2$.

A more convincing connection between quantum mechanics and classical random phenomena was achieved only later by means of the stochastic mechanics⁽²⁾: here the particle position is promoted to a stochastic Markov process $\xi(t)$ defined on some probabilistic space $(\Omega, \mathcal{F}, \mathbf{P})$ and taking values (for our limited purposes) in \mathbf{R}^3 . This process is characterized by a pdf $f(\mathbf{r}, t)$ and a transition pdf $p(\mathbf{r}, t | \mathbf{r}', t')$ and satisfies an Itô stochastic differential equation of the form

$$d\xi(t) = v_{(+)}(\xi(t), t) dt + d\eta(t) \quad (10)$$

where the velocity field $v_{(+)}$ plays the role of a dynamical variable, namely it is not given a priori, but it is subsequently determined on the basis of a variational principle; $\eta(t)$ is a Brownian process independent of ξ and such that

$$\mathbf{E}_t(d\eta(t)) = 0, \quad \mathbf{E}_t(d\eta(t) d\eta(t)) = 2\nu \mathbf{I} dt$$

where $d\eta(t) = \eta(t+dt) - \eta(t)$ (for $dt > 0$); ν is the diffusion coefficient; \mathbf{I} is the 3×3 identity matrix; and \mathbf{E}_t is the conditional expectation with respect to $\xi(t)$. We know that under fair analytical conditions on the velocity field $v_{(+)}$, the solution of (10) exists and is unique if we supplement our equation with the initial condition $\xi(0) = \xi_0$; moreover the pdf of the process satisfies the evolution equation (8) associated with the initial condition $f(\mathbf{r}, 0) = f_0(\mathbf{r})$ if $f_0(\mathbf{r})$ is the pdf of ξ_0 . Moreover every other solution of (8) (which satisfies the boundary and the non-negativity conditions to be a pdf) is propagated from its initial condition $f_0(\mathbf{r})$ following the prescription

$$f(\mathbf{r}, t) = \int_{\mathbf{R}^3} p(\mathbf{r}, t | \mathbf{r}', s) f(\mathbf{r}', s) d^3 \mathbf{r}', \quad (t > s), \quad (11)$$

since the transition pdf's play the role of the Green functions and are the solutions of (8) which satisfy (in the sense of the distributions) the initial conditions

$$p(\mathbf{r}, t | \mathbf{r}', 0) \rightarrow \delta(\mathbf{r} - \mathbf{r}'), \quad (t \rightarrow 0^+). \quad (12)$$

A suitable definition of the Lagrangian and of the stochastic action functional for the system described by means of the dynamical variables f and $v_{(+)}$ allows us to select, by means of the principle of stationarity of the stochastic action, the particular processes which simulates the quantum mechanics. More precisely the selected processes will have a drift velocity

$$v = v_{(+)} - \nu \frac{\nabla f}{f}$$

which, as required in (5), is always the gradient of a particular function $S(\mathbf{r}, t)$ solution of (4) with $R = \sqrt{f}$. Moreover it is possible to show that from f and S selected in this way we can always build a wave function

$$\psi(\mathbf{r}, t) = \sqrt{f(\mathbf{r}, t)} e^{iS(\mathbf{r}, t)/\hbar}$$

which satisfies the Schrödinger equation (1). In this formulation the foundations to interpret R^2 as a particular solution of a Kolmogorov equation for the pdf of Markov processes are well established. Of course we pay this by abandoning the idea of deterministic trajectories even if the stochastic mechanics keeps intact the description by means of continuous trajectories in space-time and recovers the paths as averages on the stochastic trajectories. However, as for the equation (8), for a given $v_{(+)}$ just one solution of the equation (10) (selected by a suitable initial condition) can in fact be interpreted as a Nelson process, since all the other solutions do not satisfy the principle of the stationarity of the stochastic action. This confronts us with the problem of explaining what to do with all the other solutions of the classical stochastic differential equation which do not fit the requirements to be interpreted as a Nelson process for a given $v_{(+)}$. In this paper we will analyze the idea that for every quantum wave function ψ it exists a stochastic flux, described by a family of transition pdf's $p(\mathbf{r}, t | \mathbf{r}', s)$, such that: a) the quantum pdf $|\psi|^2$ is correctly propagated by p ; b) every other pdf propagated by p approximates, in a suitable sense, the quantum pdf $|\psi|^2$ for $t \rightarrow +\infty$. In particular we will examine if and how the solutions of (8) selected by the stochastic mechanics to reproduce the quantum predictions attract other solutions which do not satisfy the stationary stochastic action principle and hence can not be considered as describing quantum systems ruled by the Schrödinger equation (1).

In what follows we will limit ourselves to the case of the one dimensional trajectories, so that the Markov processes $\xi(t)$ considered will always take values in \mathbf{R} . Since the set of all the probability density functions (pdf's) coincides with the set \mathcal{D} of all the non negative functions $f(x)$ of norm 1 in the Banach space $L^1(\mathbf{R})$ with norm

$$\|f\| = \int_{-\infty}^{+\infty} |f(x)| dx,$$

the time dependent pdf $f(x, t)$ of our stochastic processes $\xi(t)$ will be considered as trajectories on this subset \mathcal{D} . For Markov processes the transition pdf's $p(x, t | y, s)$ classified by means of the initial condition $\xi(s) = y$ (with $s < t$) are particular trajectories. In \mathcal{D} we can then introduce a metrics induced by the norm in $L^1(\mathbf{R})$:

$$\mathbf{d}(f, g) = \frac{1}{2} \int_{-\infty}^{+\infty} |f(x) - g(x)| dx.$$

Here the factor 1/2 guarantees that we always have $0 \leq \mathbf{d}(f, g) \leq 1$: the value 1 is attained when f and g have disjoint supports, and the value 0 when they coincide. In this framework we will say that the pdf $f(x, t)$ L^1 -approximates the pdf $g(x, t)$ (for $t \rightarrow +\infty$), and we will write

$$f(x, t) \stackrel{L^1}{\sim} g(x, t) \quad (t \rightarrow +\infty),$$

when $\mathbf{d}(f, g) \rightarrow 0$ ($t \rightarrow +\infty$). In particular we will say that f L^1 -converges toward g (for $t \rightarrow +\infty$) if the pdf $g(x)$ does not depend on the time t . If the stochastic processes $\xi(t)$ under examination are Markov processes (as happens in stochastic mechanics) satisfying the stochastic differential equation (10) with initial condition $\xi(0) = \xi_0$, their pdf will satisfy the one dimensional evolution equation

$$\partial_t f(x, t) = \nu \partial_x^2 f(x, t) - \partial_x (v_{(+)}(x, t) f(x, t)), \quad (13)$$

with the initial condition $f(x, 0) = f_0(x)$ if $f_0(x)$ is the pdf of ξ_0 .

It is now possible to show⁽³⁾ that if f and g are solutions of (14), the distance $\mathbf{d}(f, g)$ is a monotonic non-increasing function of the time t . Of course this is not enough to derive the consequence that this distance actually decreases, let alone the fact that it is infinitesimal when $t \rightarrow +\infty$. However this property is sufficient to prove that, since $\mathbf{d}(t)$ is a monotone and bounded function of t , the limit of $\mathbf{d}(t)$ for $t \rightarrow +\infty$ always exists and is finite. However, if we say that the family of the transition pdf's $p(x, t|y, 0)$ L^1 -approximates the pdf $g(x, t)$ in a *locally uniform way in y* (y - l.u.) for $t \rightarrow +\infty$, and we will write

$$p(x, t|y, 0) \stackrel{L^1}{\sim} g(x, t) \quad y \text{ - l.u.} \quad (t \rightarrow +\infty),$$

when for every $K > 0$ and for every $\epsilon > 0$ we can find a $T > 0$ such that $\mathbf{d}(p, g) < \epsilon$ for every $t > T$ and for every y such that $|y| \leq K$, it will be possible to show⁽³⁾ that, if the transition pdf's $p(x, t|y, 0)$ L^1 -approximate y - l.u. the pdf $g(x, t)$ for $t \rightarrow +\infty$, then every $f(x, t)$ solution of the evolution equation (14) L^1 -approximates $g(x, t)$ for $t \rightarrow +\infty$. In fact it is also possible to show that if the transition pdf's L^1 -approximate y - l.u. an arbitrary pdf g , then $\mathbf{d}(f_1, f_2) \rightarrow 0$ ($t \rightarrow +\infty$) for every f_1, f_2 solutions of (14). This means in particular that in the above stated conditions all the solutions of (13) globally tend to L^1 -approximate one another after a sufficiently long time. Vice-versa, if we can find two solutions f_1 and f_2 of (13) such that $\mathbf{d}(f_1, f_2)$ is not infinitesimal for $t \rightarrow +\infty$, then no pdf g can be L^1 -approximated y - l.u. by the family of the transition pdf's p .

We will discuss now our particular examples for systems reduced to a single non relativistic particle with a mass m , by remembering that the connection between the quantum mechanics and the stochastic mechanics is guaranteed if the diffusion coefficient and the Planck constant satisfy the relation (9). Since we will also consider quantum mechanical examples for non-stationary (time dependent) wave functions and velocity fields it will be useful recall that⁽³⁾ if the velocity field of the evolution equation (2) has the form

$$v_{(+)}(x, t) = -[b(t)x + c(t)]$$

with $b(t)$ and $c(t)$ continuous functions of time, then the fundamental solutions $p(x, t|y, 0)$ are normal pdf's $\mathcal{N}(\mu(t), \beta(t))$ where $\mu(t)$ and $\beta(t)$ are solutions of the equations

$$\mu'(t) + b(t)\mu(t) + c(t) = 0, \quad \beta'(t) + 2b(t)\beta(t) - 2\nu = 0 \quad (14)$$

with initial conditions $\beta(0) = 0$ and $\mu(0) = y$.

Let us consider first of all a simple harmonic oscillator with elastic constant k and classical (circular) frequency $\omega = \sqrt{k/m}$ and three possible wave functions obeying the Schrödinger equation: the (stationary) wave function of the ground state

$$\psi_0 = \left(\frac{1}{2\pi\sigma^2} \right)^{1/4} e^{-x^2/4\sigma^2} e^{-i\omega t/2}$$

the (stationary) wave function of the first excited state

$$\psi_1 = \left(\frac{1}{2\pi\sigma^2} \right)^{1/4} \frac{x}{\sigma} e^{-x^2/4\sigma^2} e^{-i3\omega t/2}$$

and the (non stationary) wave function of the oscillating coherent wave packet with initial displacement a

$$\psi_C = \left(\frac{1}{2\pi\sigma^2} \right)^{1/4} e^{-(x-a \cos \omega t)^2/4\sigma^2 - i[(4ax \sin \omega t - a^2 \sin 2\omega t)/8\sigma^2 + \omega t/2]}$$

where we have defined $\sigma^2 = \frac{\omega}{\omega}$. We then find for the quantum mechanical probability densities

$$\begin{aligned} R_0^2(x, t) &= f_0(x, t) = \frac{e^{-x^2/2\sigma^2}}{\sigma\sqrt{2\pi}} \\ R_1^2(x, t) &= f_1(x, t) = \frac{x^2}{\sigma^2} \frac{e^{-x^2/2\sigma^2}}{\sigma\sqrt{2\pi}} \\ R_C^2(x, t) &= f_C(x, t) = \frac{e^{-(x-a \cos \omega t)^2/2\sigma^2}}{\sigma\sqrt{2\pi}} \end{aligned}$$

and for the corresponding velocity fields

$$\begin{aligned} v_{(+)}^0(x, t) &= -\omega x \\ v_{(+)}^1(x, t) &= \omega \sigma \left(\frac{2\sigma}{x} - \frac{x}{\sigma} \right) \\ v_{(+)}^C(x, t) &= -\omega x + \omega a(\cos \omega t - \sin \omega t). \end{aligned}$$

This means that f_0 and f_C are respectively of the form $\mathcal{N}(0, \sigma^2)$ and $\mathcal{N}(a \cos \omega t, \sigma^2)$, and that the fundamental solutions of the corresponding evolution equation (13) can be calculated by means of (14) with

$$\begin{aligned} b_0(t) &= \omega, & c_0(t) &= 0 \\ b_C(t) &= \omega, & c_C(t) &= -\omega a(\cos \omega t - \sin \omega t) \end{aligned}$$

so that $p_0(x, t|y, 0)$ and $p_C(x, t|y, 0)$ will respectively be the normal pdf's $\mathcal{N}(\mu_0(t), \beta_0(t))$ and $\mathcal{N}(\mu_C(t), \beta_C(t))$ where

$$\begin{aligned} \beta_0(t) &= \sigma^2(1 - e^{-2\omega t}), & \mu_0(t) &= ye^{-\omega t} \\ \beta_C(t) &= \sigma^2(1 - e^{-2\omega t}), & \mu_C(t) &= a \cos \omega t + (y - a)e^{-\omega t}. \end{aligned}$$

For the first excited state, let us remark that the velocity field $v_{(+)}^1$ is no more linear in x so that the equations (14) does not hold. However the transition pdf can still be calculated in a closed form by means of an orthogonal polynomial expansion: since the velocity field $v_{(+)}^1$ presents a singularity in $x = 0$ the real axis will be effectively separated in two (positive and negative) semiaxis. As a consequence we are obliged to solve the evolution equation separately in the two semiaxis with boundary conditions imposed in $x = 0$ such that no transfer of probability would be possible between the two sectors. The result of this procedure is that the transition pdf will have the form

$$p_1(x, t|y, 0) = \Theta \left[\frac{x}{\mu(0)} \right] \frac{x}{\mu(t)} \left[\frac{e^{-[x-\mu(t)]^2/2\beta(t)}}{\sqrt{2\pi\beta(t)}} - \frac{e^{-[x+\mu(t)]^2/2\beta(t)}}{\sqrt{2\pi\beta(t)}} \right]$$

where

$$\beta(t) = \sigma^2(1 - e^{-2\omega t}); \quad \mu(t) = ye^{-\omega t},$$

and Θ is the Heavyside function

$$\Theta(x) = \begin{cases} 1, & \text{if } x \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

A second class of examples can be drawn from the wave functions of a free particle of mass m . In particular we will choose to examine the behavior of the (non stationary) wave function of a wave packet of minimal uncertainty centered around $x = 0$ with initial dispersion $\sigma^2 > 0$:

$$\psi_F(x, t) = \left(\frac{1}{2\pi\sigma^2\chi^2(t)} \right)^{1/4} e^{-x^2/4\sigma^2\chi(t)}$$

where $\chi(t) = 1 + i\omega t$, $\omega = \nu/\sigma^2$. In this case we have from (2)

$$R_F^2(x, t) = f_F(x, t) = \frac{e^{-x^2/2\sigma^2\alpha^2(t)}}{\sqrt{2\pi\sigma\alpha(t)}}$$

where $\alpha(t) = |\chi(t)| = \sqrt{1 + \omega^2 t^2}$. This means that f_F is normal of the form $\mathcal{N}(0, \sigma^2\alpha^2(t))$; moreover we get from (6) the velocity field

$$v_{(+)}^F(x, t) = -\frac{1 - \omega t}{1 + \omega^2 t^2}\omega x$$

and the fundamental solutions of the corresponding evolution equations (13) can then be calculated by means of (14) with

$$b_F(t) = \frac{1 - \omega t}{1 + \omega^2 t^2}\omega, \quad c_F(t) = 0.$$

As a consequence $p_F(x, t|y, 0)$ is a normal pdf $\mathcal{N}(\mu_F(t), \beta_F(t))$ where

$$\begin{aligned} \mu_F(t) &= y\sqrt{1 + \omega^2 t^2}e^{-\arctan \omega t} \\ \beta_F(t) &= \sigma^2(1 + \omega^2 t^2)(1 - e^{-2\arctan \omega t}). \end{aligned}$$

If now we calculate⁽³⁾ $\mathbf{d}(p_0, f_0)$, $\mathbf{d}(p_C, f_C)$ and $\mathbf{d}(p_F, f_F)$ we will find that (y - l.u.) $p_0 \stackrel{L^1}{\sim} f_0$ and $p_C \stackrel{L^1}{\sim} f_C$ ($t \rightarrow +\infty$) in the examples drawn from the harmonic oscillator, but that p_F will not L^1 -approximate f_F since $\mathbf{d}(p_F, f_F)$ turns out to be different from zero and still dependent on y in the limit $t \rightarrow +\infty$. For example, if $y = 0$ (so that both p_F and f_F will remain centered around $x = 0$ along all their evolution) we get $\mathbf{d} \sim 0.011$ in the limit $t \rightarrow +\infty$. It is also possible to show that in this case two transition pdf's with different initial conditions $y \neq y'$ will never L^1 -approximate one another as $t \rightarrow +\infty$, since

$$\mathbf{d}(p(x, t|y, 0), p(x, t|y', 0)) \rightarrow 2\Phi\left(\frac{|y - y'|}{2\sqrt{e^\pi - 1}}\right) - 1$$

which is zero if and only if $y = y'$, so that every solution of the evolution equation (13) L^1 -approximates the quantum mechanical pdf (for $t \rightarrow +\infty$) only in the examples of the harmonic oscillator but not in that of the free particle. As for the case of the first excited state, the analysis of the transition pdf p_1 shows that the node of the quantum pdf in $x = 0$ constitutes an impenetrable wall and hence that there is an infinity of equilibrium solutions, the quantum pdf f_1 being just the most symmetrical one. In fact it is easy to show that every solution of the form

$$f_\alpha = \Gamma(\alpha; x) f_1(x), \quad (0 \leq \alpha \leq 2)$$

with

$$\Gamma(\alpha; x) = \alpha \Theta(x) + (2 - \alpha) \Theta(-x)$$

are equilibrium solutions of the evolution equation with $v_{(+)}^1$ as velocity field. The quantum pdf corresponds to the case $\alpha = 1$ and it can be shown that, starting with an initial pdf $f_i(x)$, the asymptotic form of the pdf has the form f_α where

$$\alpha = 2 \int_0^{+\infty} f_i(x) dx$$

and hence is not in general coincident with the quantum pdf f_1 , unless $\alpha = 1$.

It is apparent from our examples that the Markov processes associated to a wave functions by the stochastic mechanics do not always exhibit the property of a global relaxation in time of the pdf's toward the quantum mechanical solution. The fact that the Nelson transition pdf's do not always L^1 -approximate one another also means that it is impossible to find a unique pdf g L^1 -approximated by them independently from y , and hence that the solutions of (13), at least in the cases of wave functions with nodes and of the free particles, will not in general tend to L^1 -approximate one another in time. Of course nothing forbids a priori, even in this case, that particular subsets of solutions can show the tendency to mutually L^1 -approximate and hence the field is open to investigations about, for instance, the possibility that some particular solution of (13) can be stable with respect to small perturbations of their initial conditions. In any case our examples show that, at least a significant set of systems and wave functions (that for stationary states without nodes) correctly relaxates in time toward the quantum pdf. It is not possible at present to state clearly and in a general way in which cases we realize the conditions for a global (or at least local) mutual L^1 -approximation of the solutions of (13). The examples discussed seem to indicate that the discriminating property is not the stationarity of the quantum mechanical wave function since also the square modulus of the non stationary, coherent, oscillating wave packet of the harmonic oscillator attracts in L^1 every other solution of (13). Even the main difference between two systems, namely the fact that their energy spectra are very different (the harmonic oscillator has a completely discrete spectrum and the free particle a completely continuous one) seems not to be relevant since the calculations about the case of harmonic oscillator stationary states with nodes indicates that, strictly speaking, even in this case there is no convergence to the quantum stationary pdf's.

However it must be pointed out that in this paper we have made the very particular choice of selecting the transition pdf's of the Nelson stochastic mechanics as a good candidate to the generation of the right stochastic flux, so that another possible conclusion could also be that the Nelson flux is not the right one. Hence we consider wide open the possibility that the right transition pdf's can be built in a different way. For example it is well known that in the Nelson stochastic mechanics the diffusive part of the stochastic differential equation (10) is given a priori: since the transition pdf which propagates a given time-dependent pdf $f(\mathbf{r}, t)$ is not uniquely determined (and is not, in general, observable in the stochastic mechanics), nothing forbids to find a diffusive flux, different from that of Nelson, which exhibits the required approximation property for every possible quantum wave function. In particular a possibility lies in a generalization of the stochastic mechanics where also the diffusive part of the stochastic differential equation ruling the process is dynamically determined in a way such that the approximation property is always satisfied. A few calculations with *ad hoc* chosen time dependent diffusion coefficients show that this is, at least in principle, possible. For example, if we suppose that

$$\nu(t) = \nu_N \Delta(t), \quad (\nu_N = \hbar/2m)$$

the evolution equation will take the form

$$\partial_t f(x, t) = \nu(t) \partial_x^2 f(x, t) - \partial_x (v_{(+)}(x, t) f(x, t)),$$

and, for the case of the minimum packet ψ_F of a free particle, we will have

$$v_{(+)}^F(x, t) = -b(t) x$$

with

$$b(t) = \frac{\Delta(t) - \omega t}{1 + \omega^2 t^2} \omega$$

so that $p_F(x, t|y, 0) \sim \mathcal{N}(\mu(t), \beta(t))$ where

$$\begin{aligned} \mu'(t) + b(t)\mu(t) + c(t) &= 0 \\ \beta'(t) + 2b(t)\beta(t) - 2\nu(t) &= 0 \end{aligned}$$

with initial conditions $\beta(0) = 0$ and $\mu(0) = y$. In particular, if $\Delta(t) = 2\omega t + 1$ (so that $\nu(0) = \nu_N$) we get

$$\begin{aligned} \mu(t) &= y \frac{e^{-\arctan \omega t}}{\sqrt{1 + \omega^2 t^2}} \\ \beta(t) &= \sigma^2 \left(1 + \omega^2 t^2 - \frac{e^{-2 \arctan \omega t}}{1 + \omega^2 t^2} \right). \end{aligned}$$

On the other hand, if $\Delta(t) = \omega^2 t^2 + \omega t + 1$ (so that we still have $\nu(0) = \nu_N$) we get

$$\begin{aligned} \mu(t) &= y e^{-\omega t} \\ \beta(t) &= \sigma^2 (1 + \omega^2 t^2 - e^{-2\omega t}). \end{aligned}$$

In both cases $\mu(t) \rightarrow 0$ ($t \rightarrow +\infty$) and $\beta(t)$ asymptotically behaves exactly like the quantum β_F , so that we have $p_F \stackrel{L^1}{\sim} f_F, y$ - l.u.

REFERENCES

1. L.de Broglie: *C.R.Acad.Sci.Paris* **183** (1926) 447;
L.de Broglie: *C.R.Acad.Sci.Paris* **184** (1927) 273;
L.de Broglie: *C.R.Acad.Sci.Paris* **185** (1927) 380;
D.Bohm: *Phys.Rev.* **85** (1952) 166, 180;
E.Madelung: *Z.Physik* **40** (1926) 332.
2. E.Nelson: *Phys.Rev.* **150** (1966) 1079;
E.Nelson: *Dynamical Theories of Brownian Motion* (Princeton U.P.; Princeton, 1967);
E.Nelson: *Quantum Fluctuations* (Princeton U.P.; Princeton, 1985);
F.Guerra: *Phys.Rep.* **77** (1981) 263;
F.Guerra and L.Morato: *Phys.Rev.* **D27** (1983) 1774;
F.Guerra and R.Marra: *Phys.Rev.* **D28** (1983) 1916;
F.Guerra and R.Marra: *Phys.Rev.* **D29** (1984) 1647.
3. N.Cufaro Petroni and F.Guerra: *Found.Phys.* **25** (1995) 297.