Singular densities, test particles and generalized Hamiltonian systems in general relativity

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\begin{abstract}
We derive the motion equations and the structure equations of neutral and charged test particles, starting from the gravitational field equations. The method consists in the application of conservation laws to singular tensor densities, which represent regions of strong matter concentration. Moreover, a Hamiltonian formulation of the particle equations is given, in the form of implicit differential equations generated by Hamiltonian Morse families.
\end{abstract}

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1. Introduction

In this paper we describe a general method for deriving both the equations of structure and the equations of motion of test particles in Einstein's gravitational theory. We will consider the following problem: is it possible to derive the particle equations starting from the gravitational field equations? The problem was already considered in [1,2] with further developments in [3]. Since then it has been attracting the attention of many researchers who have been studying its different facets and its possible generalizations to different theories [4].

Particles are physically represented by regions of strong concentration of matter. In particular we will idealize particles as matter concentrated along world lines. This constraint is mathematically realized by making use of singular tensor densities on the space–time manifold of general relativity.

Local conservation laws of the energy–momentum tensor density play a crucial role in the derivation of the equations of both structure and motion of particles. Indeed, the Bianchi identities applied to the gravitational equations imply that the covariant divergence of the energy–momentum tensor density vanishes identically. It is just a consequence of these conservation equations that the world line of a test particle cannot be arbitrarily assigned, but must be a geodesic of the space–time manifold. Moreover, the structure itself of the tensor density along the geodesic is completely determined by the conservation laws: the tensor must be proportional to the particle velocity and the mass is conserved.

The same procedure is applied to charged particles in the presence of both gravitational and electromagnetic fields. Here, conservation laws force the invariance of both the mass and the charge of the particle and yield the correct equations...
of motion. The method works also for the case of massless particles which must follow light-like geodesics. In this case the role of the mass is played by a positive Lagrange multiplier related to the arbitrary parameterization of the geodesic.

Note that the resulting mechanical Lagrangians are not hyperregular and, even more, for massless particles are given by a Morse family depending on additional parameters. In these cases one cannot apply the standard Legendre transformation in order to obtain a Hamiltonian formulation of the dynamics. However, there is a general theory that allows us to deal with degenerate Lagrangians [5,6]. By applying a generalized Legendre transformation one can construct Hamiltonian equations as implicit differential equations, generated by a Hamiltonian Morse family [7]. We will look at the above construction in detail for the cases at hand and exhibit the Hamiltonian systems for neutral and charged particles in general relativity. As we will see, the Hamiltonian systems obtained are not given by a Hamiltonian defined on a constraint submanifold, i.e. by a Dirac system [8,9]. More general Hamiltonian systems have to be considered.

We will discuss this method by always using coordinate expressions, but we will always refer to the geometrical objects that are behind it.

The organization of the paper is as follows. In Section 2 we consider neutral particles. After introducing singular tensor densities, we show how motion and structure equations are derived from conservation laws. In Section 2.2 we construct the Hamiltonian Morse family and the implicit differential equation that it generates.

The above derivations and constructions are then applied to massless particles in Section 2.3 and a unified treatment of massive and massless particles is given in Section 2.4. In Section 3 the case of charged particles is considered.

2. Dynamics of neutral particles

The space–time of General Relativity is a manifold $M$ of four dimensions with a metric tensor $g_{\alpha\beta}$ of signature $-2$ representing the gravitational field. Throughout the paper coordinates $x^\alpha$ with $\kappa = 0, 1, 2, 3$ will be used and repeated indices will always imply a sum. Einstein’s field equations

$$G^{\kappa\lambda} = \frac{8\pi G}{c^4} T^{\kappa\lambda}, \quad G^{\kappa\lambda} = \left( R^{\kappa\lambda} - \frac{1}{2} g^{\kappa\lambda} R \right) \sqrt{-g}$$  \hspace{1cm} (1)

relate the curvature tensor $G^{\kappa\lambda}$ to the matter energy–momentum tensor density $T^{\kappa\lambda}_m$. The energy–momentum tensor density $T^{\kappa\lambda}_m$ is defined by the relation

$$\delta_S S_m = -\frac{1}{2c^4} \int_{\Omega} T^{\kappa\lambda}_m \delta g_{\kappa\lambda} d^4x,$$  \hspace{1cm} (2)

where $S_m$ denotes the action of a material system and the variation $\delta_S$ is done with respect to variations $\delta g_{\kappa\lambda}$ vanishing on the boundary $\partial\Omega$ with their first derivatives. By recalling that

$$\delta_S S_g = \frac{c^4}{16\pi G} \int_{\Omega} G^{\kappa\lambda} \delta g_{\kappa\lambda} d^4x, \quad S_g = -\frac{c^4}{16\pi G} \int_{\Omega} R \sqrt{-g} d^4x,$$  \hspace{1cm} (3)

it is easy to see that Einstein’s equations derive by a variational principle applied to the sum of the action of the gravitational field $S_g$ and the material system $S_m$, namely $\delta_S (S_g + S_m) = 0$.

The Bianchi identities imply that the covariant derivative of $G^{\kappa\lambda}$ must vanish, $\nabla_\kappa G^{\kappa\lambda} = 0$, and, therefore, the energy–momentum tensor density of matter must satisfy the continuity condition

$$\nabla_\kappa T^{\kappa\lambda}_m = 0.$$  \hspace{1cm} (4)

In the following we will consider matter whose distribution is concentrated on a world line and thus we will be interested in singular energy–momentum tensor densities. They are described in terms of the Dirac $\delta$ function characterized by the equality

$$\int f(\xi^\sigma) \delta(\xi^\sigma - \xi^\sigma) d^4x = f(\xi^\sigma)$$  \hspace{1cm} (5)

for all test functions $f$ with compact support. Under coordinate transformations the $\delta$ function transforms as a scalar density and its covariant derivative is the tensor density

$$\nabla_\kappa \delta(\xi^\sigma - \xi^\sigma) = \partial_\kappa \delta(\chi^\sigma - \xi^\sigma) + \Gamma^{\kappa\lambda}_{\mu\nu} \delta(\chi^\sigma - \xi^\sigma).$$  \hspace{1cm} (6)

Let us first start with matter concentrated at a single event with coordinates $\xi^\kappa$. Its singular energy–momentum tensor density reads $T^{\kappa\lambda}_m = t^{\kappa\lambda} \delta(\chi^\sigma - \xi^\sigma)$. The integrability condition $\nabla_\kappa T^{\kappa\lambda}_m = 0$ implies

$$\int f_\kappa \nabla_\kappa T^{\kappa\lambda}_m d^4x = -\int \nabla_\kappa f_\kappa t^{\kappa\lambda} \delta(\chi^\sigma - \xi^\sigma) d^4x = -\nabla_\kappa f_\kappa (\xi^\sigma) t^{\kappa\lambda} = 0$$  \hspace{1cm} (7)

for all test fields $f_\kappa$ with compact support. Hence, $t^{\kappa\lambda} = 0$. This means that matter concentrated at a single event is not compatible with Einstein equations.
2.1. Matter distributed along a world line

We will now consider matter distributed along a time-like world line and show that the continuity Eq. (4) is sufficient to completely characterize the structure of the energy–momentum tensor density and the world line itself.

Now, let γ be an oriented time-like world line. A curve $\xi: \mathbb{R} \to M$, with $\gamma = \text{im}(\xi)$, will be called a parameterization of γ. Matter distributed along γ is represented by the singular energy–momentum density

$$ T^\gamma_m = c^2 \int_{-\infty}^{\infty} t^\gamma_m(s) \| \xi \|^2 \delta(\chi^\gamma - \xi^\gamma(s)) \, ds, $$

where the factor $\| \xi \| = \sqrt{g(\xi, \xi, \xi, \xi)}$ is used to make the tensor $t^\gamma_m(s)$ independent of the choice of parameterization $\xi$ of $\gamma$. We will see later that the requirement of an orientation for the world line yields sensible physical results. The tensor $t^\gamma_m(s)$ is decomposed into components parallel and orthogonal to the world line $\gamma$:

$$ t^\gamma_m(s) = m(s) \frac{\dot{\xi}^\gamma(s)}{\| \xi \|^2} + m^\parallel(s) \frac{\dot{\xi}^\parallel(s)}{\| \xi \|^2} + m^\perp(s), $$

$$ m(s) = t^\gamma_m \left( \delta^\mu_{\lambda} - \frac{\dot{\xi}^\mu s}{\| \xi \|^2} \right) \left( \delta^\nu_{\lambda} - \frac{\dot{\xi}^\nu s}{\| \xi \|^2} \right). $$

The integrability condition $\nabla_\lambda T^\gamma_m = 0$ is transformed into

$$ \int_{-\infty}^{\infty} \frac{d}{ds} \left( m(s) \frac{\dot{\xi}^\gamma(s)}{\| \xi \|^2} + m^\parallel(s) \right) \delta(\chi^\gamma - \xi^\gamma(s)) \, ds + \int_{-\infty}^{\infty} \nabla_\lambda \left( (m^\parallel(s) \xi^\parallel + m^\gamma(s)) \| \xi \| \right) \delta(\chi^\gamma - \xi^\gamma(s)) \, ds = 0. $$

By integrating (12) over a test field $f_\lambda(\xi^\gamma)$ and making use of the arbitrariness of both $f_\lambda(\xi^\gamma(s))$ and their covariant derivatives $\nabla_\lambda f_\lambda(\xi^\gamma(s))$ in all directions orthogonal to $\dot{\xi}^\gamma$, one obtains

$$ \frac{d}{ds} \left( m(s) \frac{\dot{\xi}^\gamma(s)}{\| \xi \|^2} + m^\parallel(s) \right) = 0, \quad m^\parallel(s) \frac{\dot{\xi}^\parallel(s)}{\| \xi \|^2} + m^\gamma(s) = 0. $$

These equations imply that

$$ m^\gamma = 0, \quad m^\parallel = 0, \quad \frac{dm}{ds} = 0, \quad \frac{\delta}{ds} \left( \frac{\dot{\xi}^\gamma}{\| \xi \|^2} \right) = 0. $$

The final conclusion is that the energy–momentum tensor density of matter distributed along a world line $\gamma$ must be of the form

$$ T^\gamma_m = c^2 \int_{-\infty}^{\infty} \frac{m(s) \dot{\xi}^\gamma(s)}{\| \xi \|^2} \delta(\chi^\gamma - \xi^\gamma(s)) \, ds. $$

The world line $\gamma$ must be a geodesic and the parameter $m$ must be constant. The physical system is a point particle and the parameter $m$ is the mass of the particle.

2.2. The generalized Hamiltonian system

We will now focus on the dynamics related to the energy–momentum tensor density (15). First, note that it derives from the action

$$ S_m = -mc \int_{-\infty}^{\infty} \int_{\Omega} \sqrt{g_{\alpha\beta}(\xi^\gamma(s)) \dot{\xi}^\gamma(s)} \dot{\xi}^\gamma(s) \delta(\chi^\gamma - \xi^\gamma(s)) \, d^4x \, ds, $$

where the integration extends over the domain $\Omega$, since we are not interested in the variation of the gravitational field. This, in turns, reads

$$ S_m = -mc \int_{-\infty}^{\infty} \sqrt{g_{\alpha\beta}(\xi^\gamma(s)) \dot{\xi}^\gamma(s) \dot{\xi}^\gamma(s)} \, ds. $$

This is nothing but the mechanical action of a relativistic particle of mass $m$, whose Lagrangian is

$$ L(\xi^\gamma, \dot{\xi}^\gamma) = mc \sqrt{g_{\alpha\beta}(\xi^\gamma(s)) \dot{\xi}^\gamma(s) \dot{\xi}^\gamma(s)}, \quad g_{\alpha\beta}(\xi^\gamma(s)) \dot{\xi}^\gamma(s) \dot{\xi}^\gamma(s) > 0, $$

(18)
and whose Lagrange equations are
\[ p_k = \frac{\partial L}{\partial \dot{\xi}^k} = m c g_{k\lambda} \frac{\dot{\xi}^k}{\| \dot{\xi} \|}, \quad \dot{p}_k = \frac{\partial L}{\partial \dot{\xi}^k} = m c g_{k\nu} \Gamma^\nu_{\mu\nu} \frac{\dot{\xi}^\nu}{\| \dot{\xi} \|}, \] (19)
or
\[ p_k = m c g_{k\lambda} \frac{\dot{\xi}^k}{\| \dot{\xi} \|}, \quad \frac{\delta p_k}{\delta s} = \dot{p}_k - \Gamma^\nu_{\lambda\nu} p_\nu \dot{\xi}^\lambda = 0. \] (20)

Notwithstanding the singularity of the Lagrangian (18), it is possible to give a Hamiltonian formulation by applying the generalized Legendre transformation [7,10,6]. Since the Lagrangian (18) is singular, one could try to give a Hamiltonian formulation in terms of a Dirac-constrained system [8]. We will see that this is not the case.

We recall that a Dirac system is a couple \((C, H)\), where \(C \subset P\) is a submanifold of the symplectic phase space \((P, \omega)\) and \(H : C \to \mathbb{R}\) is a differentiable function. The corresponding implicit Hamiltonian equation is the set [11,8]
\[ D_{\text{Dirac}} = \left\{ w \in TP : p = \tau_p(w) \in C, \forall u \in T_p C, \omega(u, w) = dH(u) \right\}, \] (21)
where \(\tau_p : TP \to P\) denotes the canonical tangent bundle projection. We remark that if the constraint submanifold \(C\) is given by \(C = K^{-1}(0)\), where \(K : P \to V\) is a submersion onto a vector space \(V\) of dimension \(s\), and \(H = H|_C\) is the restriction to \(C\) of a differentiable function \(H : P \to \mathbb{R}\), one gets that the Dirac system (21) is generated by the Hamiltonian
\[ G : P \times V^* \to \mathbb{R} : (p, \lambda) \mapsto \tilde{H}(p) + \langle \lambda, K(p) \rangle \] (22)
which is a Morse family with respect to the fibration \(pr : P \times V^* \to P\). In coordinates, from the Hamiltonian Morse family
\[ G(\xi^\mu, p_\nu, \lambda) = H(\xi^\mu, p_\nu) + \lambda_j K^j(\xi^\mu, p_\nu), \] (23)
with \(1 \leq j \leq s\), we get
\[ D_{\text{Dirac}} = \left\{ (\xi^\mu, p_\nu, \dot{\xi}^\mu, \dot{p}_\nu) : K^j(\dot{\xi}^\mu, \dot{p}_\nu) = 0, \dot{\xi}^\mu = \frac{\partial H}{\partial p_\mu}(\xi^\mu, p_\nu), \dot{p}_\nu = -\frac{\partial H}{\partial \xi^\nu}(\xi^\mu, p_\nu) \right\}. \] (24)

We will now show that the dynamics of a particle in general relativity cannot be described by a Dirac system. Indeed, time-like velocities and time-like momenta belong to the submanifolds
\[ TM^+ = \{(\xi^\mu, \xi^\nu)| g_{\lambda \sigma} \xi^\lambda \xi^\sigma > 0\}, \quad T^* M^+ = \{(\xi^\mu, p_\nu)| g^{\lambda \nu} p_\lambda p_\nu > 0\} \] (25)
of \(TM\) and \(T^* M\), respectively. The singular Lagrangian (18) is
\[ L : TM^+ \to \mathbb{R} : (\xi^\mu, \dot{\xi}^\mu) \mapsto L(\xi^\mu, \dot{\xi}^\mu) = m c \sqrt{g_{\lambda \sigma} (\dot{\xi}^\lambda \dot{\xi}^\sigma)} \] (26)
and the Legendre mapping is given by
\[ L : TM^+ \to T^* M^+: (\xi^\mu, \dot{\xi}^\mu) \mapsto \left( \xi^\mu, p_\nu = \frac{\partial L}{\partial \dot{\xi}^\mu} \right) = \left( \xi^\mu, m c g_{\lambda \mu} \frac{\dot{\xi}^\mu}{\| \dot{\xi} \|} \right), \] (27)
whose range is the mass shell, characterized by \(p = \sqrt{g^{\lambda \nu} p_\lambda p_\nu} = m c\).

At variance with the (hyper)regular case of nonrelativistic mechanics, the mapping (27) is not invertible. Indeed, it is not injective, because
\[ \frac{\partial L}{\partial \dot{\xi}^\mu}(\xi^\mu, \omega \dot{\xi}^\lambda) = \frac{\partial L}{\partial \dot{\xi}^\mu}(\xi^\mu, \dot{\xi}^\lambda), \quad \forall \omega \neq 0. \] (28)
The graph of the Legendre mapping \(L\) is still a submanifold of the product fibration \(TM^+ \times_M T^* M^+\).

Let us consider the energy function \(E : TM^+ \times_M T^* M^+ \to \mathbb{R}\),
\[ E(\xi^\mu, \dot{\xi}^\mu, p_\mu) = p_\mu \dot{\xi}^\mu - L(\xi^\mu, \dot{\xi}^\mu) = p_\mu \dot{\xi}^\mu - m c \sqrt{g_{\lambda \sigma} (\dot{\xi}^\lambda \dot{\xi}^\sigma)}. \] (29)
It is a Morse family with respect to the fibration
\[ \zeta : TM^+ \times_M T^* M^+ \to T^* M^+: (\xi^\lambda, \dot{\xi}^\lambda, p_\mu) \mapsto (\dot{\xi}^\lambda, p_\mu), \] (30)
since it satisfies the maximal rank condition. The critical set of this family is given by
\[ \text{Cr}(E; \zeta) = \left\{ (\xi^\lambda, \dot{\xi}^\lambda, p_\mu) \in TM^+ \times_M T^* M^+: \frac{\partial E}{\partial \dot{\xi}^\lambda}(\xi^\lambda, \dot{\xi}^\lambda, p_\mu) = 0 \right\} \]
\[ = \left\{ (\xi^\lambda, \dot{\xi}^\lambda, p_\mu) : p_\mu = m c g_{\lambda \mu} \frac{\dot{\xi}^\mu}{\| \dot{\xi} \|} \right\} = \text{graph } L \] (31)
and coincides with the graph of the Legendre mapping. However, it is not the image of a section of $\zeta$, since $\mathcal{L}$ is not invertible. Nevertheless, the Hamiltonian equation is given by the implicit differential equation

\[
D_H = \left\{ (\xi^\mu, p_\mu, \dot{\xi}^\mu, \dot{p}_\mu) \in T^*T^+M^+ : \exists (v^\nu), (\zeta^\lambda, \dot{v}^\sigma, p_\nu) \in \text{Cr}(E; \zeta), \right. \\
\left. \dot{\xi}^\mu = \frac{\partial E}{\partial p_\mu}(\xi^\mu, v^\sigma, p_\nu), \dot{p}_\nu = -\frac{\partial E}{\partial \xi^\nu}(\xi^\mu, v^\sigma, p_\nu) \right\} \\
= \left\{ (\zeta^\lambda, \dot{v}^\sigma, p_\nu) : \exists (v^\nu), (\xi^\mu, p_\mu, \|\dot{\xi}\|) \right. \\
\left. (\xi^\mu, p_\mu, \dot{\xi}^\mu, \dot{p}_\nu) = \frac{\partial L}{\partial \xi^\nu}(\xi^\mu, v^\sigma, p_\nu), \right. \\
\left. \dot{\xi}^\mu = \frac{\partial E}{\partial p_\mu}(\xi^\mu, v^\sigma, p_\nu), \dot{p}_\nu = -\frac{\partial E}{\partial \xi^\nu}(\xi^\mu, v^\sigma, p_\nu) \right\} . \tag{32}
\]

By looking at the last line of (32), note that formally $D_H$ has the standard form of Hamilton equations. In particular the first equation represents the usual relation between canonical momenta in terms of positions and velocities, while the last two equations are the coupled Hamilton equations. However, unless the Lagrangian is (hyper)regular, the velocities cannot be eliminated and the equation cannot be written as an explicit differential equation, that is as the image of a vector field. In general, they play the role of Lagrange multipliers.

Note now that, in our case of a relativistic particle, there are too many Lagrange multipliers $v^\sigma$, more than expected. In fact, the range of the Legendre transformation (27) is the mass shell, a manifold of codimension 1. Therefore one expects that the dynamics is generated by a one-parameter Morse family of Hamiltonian functions. This expectation is correct. In fact, consider the fibration

\[
\zeta_1 : TM^+ \times_M T^*M^+ \to T^*M^+ \times \mathbb{R}_+ : (\xi^\mu, \dot{\xi}^\mu, p_\mu) \mapsto (\xi^\mu, p_\mu, \|\dot{\xi}\|).
\]

The function $E$ is a Morse family with respect to $\zeta_1$ and, by noting that

\[
\ker T\zeta_1 = \{ (\xi^\mu, \dot{\xi}^\mu, p_\mu, 0, \delta \dot{\xi}^\nu, 0) | g_{\mu\nu} \dot{\xi}^\mu \delta \dot{\xi}^\nu = 0 \},
\]

one can easily find the critical set

\[
\text{Cr}(E; \zeta_1) = \{ (\xi^\mu, \dot{\xi}^\mu, p_\mu) : p_\mu = v g_{\mu\nu} \dot{\xi}^\nu, \nu \neq 0 \} . \tag{35}
\]

We note now that $\text{Cr}(E; \zeta_1)$ is the union of the images of two sections of $\zeta_1$. Indeed, its intersection with the fiber over $(\xi^\mu, p_\mu, \lambda)$ is

\[
\text{Cr}(E; \zeta_1) \cap \zeta_1^{-1}(\xi^\mu, p_\mu, \lambda) = \left\{ (\xi^\mu, \dot{\xi}^\mu, p_\mu) : \dot{\xi}^\mu = \pm \lambda \frac{g^{\sigma\mu} p_\mu}{\|p\|} \right\} = \text{im } \sigma_+ \cup \text{im } \sigma_- , \tag{36}
\]

where $\sigma_\pm : T^*M^+ \times \mathbb{R}_+ \to TM^+ \times_M T^*M^+$,

\[
\sigma_\pm(\xi^\mu, p_\mu, \lambda) = \left( \xi^\mu, \pm \lambda \frac{g^{\sigma\mu} p_\mu}{\|p\|}, p_\mu \right) . \tag{37}
\]

Therefore, one can define two functions $H_\pm = E \circ \sigma_\pm : T^*M^+ \times \mathbb{R}_+ \to \mathbb{R}$,

\[
H_\pm(\xi^\mu, p_\mu, \lambda) = \lambda \left( \sqrt{g^{\sigma\mu} p_\mu} \pm mc \right) . \tag{38}
\]

These are two Morse families of the trivial fibration

\[
\text{pr} : T^*M^+ \times \mathbb{R}_+ \to T^*M^+ : (\xi^\mu, p_\mu, \lambda) \mapsto (\xi^\mu, p_\mu) \tag{39}
\]

with critical sets

\[
\text{Cr}(H_\pm; \text{pr}) = \{ (\xi^\mu, p_\mu, \lambda) : \sqrt{g^{\sigma\mu} p_\mu} = \pm mc \} . \tag{40}
\]

Note that $\text{Cr}(H_+; \text{pr})$ is the inverse image of the mass shell $\|p\| = mc$. The Hamilton equation, generated by $H = H_+$ is

\[
D_H = \left\{ (\zeta^\lambda, \dot{v}^\sigma, p_\nu) : \exists \lambda, \frac{\partial H}{\partial \zeta^\lambda}(\zeta^\lambda, \dot{v}^\sigma, p_\nu, \lambda) = 0, \dot{\zeta}^\lambda = \frac{\partial H}{\partial p_\mu}(\zeta^\lambda, \dot{v}^\sigma, p_\nu, \lambda), \dot{p}_\nu = -\frac{\partial H}{\partial \zeta^\lambda}(\zeta^\lambda, \dot{v}^\sigma, p_\nu, \lambda) \right\} . \tag{41}
\]

Since $\text{Cr}(H_-; \text{pr}) = \emptyset$, the Hamilton equation generated by $H_-$ is empty.

Summarizing, the generalized Legendre transformation applied to the singular Lagrangian (18) results in the Hamiltonian Morse family

\[
H(\xi^\mu, p_\mu, \lambda) = \lambda \left( \sqrt{g^{\sigma\mu}(\xi^\sigma)p_\mu} - mc \right) . \quad \lambda > 0, \ g^{\sigma\mu}(\xi^\sigma)p_\mu > 0. \tag{42}
\]

The Morse parameter $\lambda$ is treated as a Lagrange multiplier although it is restricted to $\mathbb{R}_+$. The generalized Hamilton equations

\[
\frac{\partial H}{\partial \lambda} = \sqrt{g^{\sigma\mu}(\xi^\sigma)p_\mu} - mc = 0, \quad \dot{\xi}^\mu = \frac{\partial H}{\partial p_\mu} = \lambda g^{\sigma\mu} p_\mu \sqrt{g^{\sigma\mu} p_\mu}, \tag{43}
\]

\[
\dot{p}_\mu = -\frac{\partial H}{\partial \xi^\mu} = \lambda \Gamma^\nu_{\mu\rho} g^{\sigma\nu} \frac{p_\rho p_\mu}{\sqrt{g^{\sigma\nu} p_\rho}}.
\]
are equivalent to the Lagrange equations (19). Equivalence is easily established by using the equality \( \lambda = \| \dot{\xi} \| \) derived from Eq. (43).

Note that the Hamiltonian (42) is a Morse family with respect to a fibration which is not a vector bundle. This implies that if we plug the Dirac-constrained Hamiltonian

\[
G(\xi^\alpha, p_\alpha, \lambda) = \lambda \left( \sqrt{g^{\mu\nu}(\xi^\alpha)} p_\mu p_\nu - mc \right)
\]

(44)

with the unrestricted Lagrange multiplier \( \lambda \in \mathbb{R} = V^* \) into the Hamilton equations (24) we reobtain Eq. (43), but, since the sign of \( \lambda \) is undefined, one only gets that \( \lambda = \pm \| \dot{\xi} \| \). Therefore, Dirac-constrained equations are not equivalent to the Lagrange equations (19). The dynamics of a relativistic particle cannot be presented as a Dirac constrained Hamiltonian system.

From the physical point of view, the restriction to positive values of \( \lambda \) is appealing since solutions are oriented lines. Their orientation reflects the distinction between particles and antiparticles introduced by Feynman and Stueckelberg [12, 13], where the positive oriented world lines describe particles, while the negative oriented lines are associated to antiparticles. On the other hand, the Dirac system provides an incomplete description of relativistic dynamics: the distinction between particles and antiparticles is lost.

2.3. Massless particles

A world line \( \gamma \) of a massless particle is light-like. Let \( \xi : \mathbb{R} \rightarrow M \) be a parameterization of \( \gamma \). The energy–momentum tensor density of matter distributed along \( \gamma \) is a singular density

\[
T_m^{\lambda \mu} = c^2 \int_{-\infty}^{\infty} \epsilon^{\lambda \mu}(s) \delta(\nu^\alpha - \xi^\alpha(s))ds.
\]

(45)

It is not possible to make the tensor \( \epsilon^{\lambda \mu}(s) \) independent of the choice of parameterization by using the factor \( \| \dot{\xi} \| \) since \( \| \dot{\xi} \| = 0 \). However, it is possible to decompose the tensor \( \epsilon^{\lambda \mu}(s) \) into components parallel and orthogonal to the world line \( \gamma \):

\[
\epsilon^{\lambda \mu}(s) = \mu(s) \dot{\xi}^\lambda \dot{\xi}^\mu + \mu^\kappa(s) \dot{\xi}^\kappa + \mu^{\kappa \lambda}(s) \dot{\xi}^\kappa + \mu^{\kappa \lambda}(s),
\]

(46)

where, as will be seen in a while, \( \mu > 0 \). Such a decomposition is not unique. If an orthogonal basis \( u_0^\alpha, u_1^\alpha, u_2^\alpha \) and \( u_3^\alpha \) is chosen with \( u_0^\alpha = \dot{\xi}^\alpha \), \( u_1^\alpha \) light-like, and \( u_2^\alpha \) and \( u_3^\alpha \) space-like, then the tensor \( \epsilon^{\lambda \mu}(s) \) can be expressed as a combination \( \epsilon^{\lambda \mu}(s) = \sigma^{\alpha \beta} u_\alpha^\alpha u_\beta^\beta \) and the decomposition (46) is obtained by setting \( \mu = \sigma^{00}, \mu^\kappa = \sigma^{0i} u_i^\kappa, \mu^{\kappa \lambda} = \sigma^{ij} u_i^\kappa u_j^\lambda \), with \( i, j = 1, 2, 3 \).

An analysis of the equation \( \nabla_\gamma T_m^{\lambda \mu} = 0 \), similar to that in Section 2.1 leads to the equations

\[
T_m^{\lambda \mu} = c^2 \int_{-\infty}^{\infty} \mu(s) \dot{\xi}^\lambda \dot{\xi}^\mu \delta(\nu^\alpha - \xi^\alpha(s))ds, \quad \frac{\delta}{\delta s} (\mu(s) \dot{\xi}^\lambda) = 0.
\]

(47)

These results are independent of the choice of the base \( (u_\alpha^\alpha) \). Parameterization independence requires that if a new parameter \( s' \) is introduced by \( s = \sigma(s') \), then \( \mu(s) \) must be replaced by \( \mu'(s') = \mu(\sigma(s')) (\frac{ds}{ds'})^{-1} \). The Eq. (47) written in the form \( \frac{d\dot{\xi}^\lambda}{ds} = -\frac{1}{\mu} \frac{d\mu}{ds} \dot{\xi}^\lambda \) indicates that \( \gamma \) is a geodesic. If \( s \) is an affine parameter, then \( \delta \dot{\xi}^\lambda / ds = 0 \). If \( s \) is not affine, then an affine parameter \( s' = \sigma^{-1}(s) \) can be found by solving the second order differential equation \( \frac{d^2 s}{ds'^2} (\mu(s) \frac{ds'}{ds'}) = 0 \) for the function \( \sigma(s') \).

The energy–momentum tensor density (47) derives from the action

\[
S_m = -c \int_{-\infty}^{\infty} \int_\Omega \frac{\mu(s)}{2} g_{\lambda \mu}(\xi^\kappa(s)) \dot{\xi}^\lambda(s) \dot{\xi}^\mu(s) \delta(\nu^\alpha - \xi^\alpha(s))d^4xds,
\]

(48)

where the parameter \( \mu \) is a Morse parameter treated as a Lagrange multiplier restricted to \( \mathbb{R}_+ \). Since we are not interested in the variation of the gravitational field, Eq. (48) can be integrated over the domain \( \Omega \) and we get the mechanical action

\[
S_m = -c \int_{-\infty}^{\infty} \frac{\mu(s)}{2} g_{\lambda \mu}(\xi^\kappa(s)) \dot{\xi}^\lambda(s) \dot{\xi}^\mu(s)ds
\]

(49)

whose mechanical Lagrangian is in the form of the Morse family

\[
L(\mu, \dot{\xi}^\lambda, \dot{\xi}^\alpha) = \frac{\mu c}{2} g_{\lambda \mu}(\xi^\alpha(s)) \dot{\xi}^\lambda \dot{\xi}^\mu, \quad \mu > 0, \dot{\xi}^\alpha \neq 0.
\]

(50)

The Lagrange equations

\[
\frac{\partial L}{\partial \mu} = \frac{c}{2} g_{\lambda \mu}(\xi^\alpha(s)) \dot{\xi}^\lambda, \quad p_\mu = \frac{\partial L}{\partial \dot{\xi}^\mu} = \mu c g_{\lambda \mu} \dot{\xi}^\mu, \quad \dot{p}_\mu = \frac{\partial L}{\partial \ddot{\xi}^\mu} = T_{\lambda \mu} p_\lambda \dot{\xi}^\mu,
\]

(51)
can be recast in the form

\[ g_{\xi^\lambda}(\xi^\sigma) \dot{\xi}^\mu \dot{\xi}^\nu = 0, \quad p_\xi = \mu g_{\xi^\lambda}(\dot{\xi}^\lambda), \quad \frac{\delta p_\xi}{\delta s} = \dot{p}_\xi - \Gamma^{\mu}_{\xi^\lambda} p_\xi \dot{\xi}^\lambda = 0. \]  

(52)

Also in this case, by following a path analogous to that described in the previous section one obtains a Hamiltonian given by the Morse family

\[ H(\mu, \xi^\mu, p_\xi) = \frac{1}{2\mu c} g^{\mu \lambda}(\xi^\sigma) p_\xi p_\lambda, \quad \mu > 0, \quad p_\xi \neq 0. \]  

(53)

and the generalized Hamilton equations

\[ \frac{\partial H}{\partial \mu} = - \frac{1}{2\mu c^2} g^{\mu \lambda} p_\xi p_\lambda = 0, \quad \dot{\xi}^\mu = \frac{\partial H}{\partial p_\xi} = \frac{1}{\mu c} g^{\mu \lambda} p_\lambda, \quad \dot{p}_\xi = - \frac{\partial H}{\partial \xi^\mu} = \Gamma^{\mu}_{\xi^\lambda} p_\xi \dot{\xi}^\lambda. \]  

(54)

with \( \mu > 0 \) are equivalent to the Lagrange equations (51). As in the case of massive particles, Eq. (54) are not the Hamilton equations of a Dirac-constrained system (24), where the Stueckelberg distinction between particles and antiparticles would be lost. Even more, the dynamics of massless particles offers a rare example of a Lagrangian Morse family which is not an example of a Dirac-constrained Hamiltonian system.

2.4. Particles with mass \( m \geq 0 \)

An uniform treatment of particles with mass \( m > 0 \) and \( m = 0 \) is possible. The action is the integral

\[ S_m = -c \int_{-\infty}^{\infty} \int_\Omega \left( \mu(s) \frac{1}{2} g_{\xi^\lambda}(\xi^\sigma) \dot{\xi}^\mu \dot{\xi}^\lambda + \frac{m^2 c^2}{2\mu(s)} \right) \delta(\xi^\sigma - \xi^\sigma(s)) \, d^4 x \, ds \]  

(55)

with a Morse parameter \( \mu > 0 \). The energy–momentum tensor density derived from this action is the singular density

\[ T^{\mu \lambda}_m = c^2 \int_{-\infty}^{\infty} \mu(s) \dot{\xi}^\mu \dot{\xi}^\lambda \delta(\xi^\sigma - \xi^\sigma(s)) \, ds. \]  

(56)

The Lagrange equations

\[ \frac{\partial L}{\partial \mu} = \frac{c}{2} g_{\xi^\lambda}(\xi^\sigma) \dot{\xi}^\mu \dot{\xi}^\lambda - \frac{m^2 c^2}{2\mu} = 0, \quad p_\xi = \frac{\partial L}{\partial \dot{\xi}^\mu} = c \mu g_{\xi^\lambda}(\dot{\xi}^\lambda), \quad \dot{p}_\xi = \frac{\partial L}{\partial \dot{\xi}^\mu} = \Gamma^{\mu}_{\xi^\lambda} p_\xi \dot{\xi}^\lambda \]  

(57)

are derived from the Lagrangian Morse family

\[ L(\mu, \xi^\mu, \dot{\xi}^\mu) = \frac{c \mu}{2} g_{\xi^\lambda}(\xi^\sigma) \dot{\xi}^\mu \dot{\xi}^\lambda + \frac{m^2 c^2}{2\mu}, \quad \mu > 0, \quad \dot{\xi}^\mu \neq 0. \]  

(58)

The Hamiltonian is the Morse family

\[ H(\mu, \xi^\mu, p_\xi) = \frac{1}{2\mu c} (g^{\mu \lambda}(\xi^\sigma) p_\xi p_\lambda - m^2 c^2), \quad \mu > 0, \quad p_\xi \neq 0 \]  

(59)

and the generalized Hamilton equations are

\[ \frac{\partial H}{\partial \mu} = - \frac{1}{2\mu c^2} (g^{\mu \lambda} p_\xi p_\lambda - m^2 c^2) = 0, \quad \dot{\xi}^\mu = \frac{\partial H}{\partial p_\xi} = \frac{1}{\mu c} g^{\mu \lambda} p_\lambda, \quad \dot{p}_\xi = - \frac{\partial H}{\partial \dot{\xi}^\mu} = \Gamma^{\mu}_{\xi^\lambda} p_\xi \dot{\xi}^\lambda. \]  

(60)

If \( m > 0 \), then the Morse parameter \( \mu \) can be eliminated from the Lagrangian (58) and from the Eqs. (57), by using the equality \( \mu = m/\sqrt{g_{\xi^\lambda}(\dot{\xi}^\lambda)} \), derived from (57). However, the Hamiltonian always remains a Morse family, since the Lagrangian obtained is not regular. Incidentally, note that the energy–momentum tensor density can be rewritten in the form

\[ T^{\mu \lambda}_m = c \int_{-\infty}^{\infty} g^{\mu \lambda} p_\mu \dot{\xi}^\lambda \delta(\xi^\sigma - \xi^\sigma(s)) \, ds. \]  

(61)

3. Dynamics of charged particles

Let us now consider the dynamics of charged matter in the presence of gravitational and electromagnetic fields. We will follow the same path used in Section 2.1. Let \( S_m \) denote the action of a charged material system. The current \( J^\mu \) is defined by
\[ \delta_S m = -\frac{1}{c^2} \int_\Omega J^\kappa \delta A_\kappa \, d^4 x, \]  
where the variations are taken over the electromagnetic potential \( A_\kappa \). The Maxwell field equations
\[ \nabla_\kappa H^{\kappa \lambda} = \frac{4\pi}{c} J^\lambda, \quad H^{\kappa \lambda} = g^{\kappa \mu} g^{\lambda \nu} F_{\mu \nu} \sqrt{-g}, \quad F_{\kappa \lambda} = \nabla_\kappa A_\lambda - \nabla_\lambda A_\kappa, \]  imply the continuity conditions
\[ \nabla_\kappa J^\kappa = 0. \]  
On the other hand, the equations of the gravitational field
\[ G^{\kappa \lambda} + \frac{2k}{c^4} \left( g^{\kappa \mu} F_{\mu \nu} - \frac{1}{4} g^{\kappa \lambda} F_{\mu \nu} F^{\mu \nu} \right) = \frac{8\pi k}{c^4} T^{\kappa \lambda}_m \]  in the presence of charged matter imply the continuity relations
\[ \nabla_\kappa T^{\kappa \lambda}_m - \frac{1}{c^2} g^{\kappa \mu} F_{\mu \nu} J^\nu = 0. \]  
Let us now consider the case of charged matter distributed along a world line \( \gamma \). Its current is described by the singular vector density
\[ J^\kappa = c \int_{-\infty}^{\infty} t^\kappa (s) \| \dot{\xi} \| \delta (\chi^\kappa - \xi^\kappa (s)) \, ds. \]  
The vector \( t^\kappa \) can be decomposed in the following way
\[ t^\kappa = e^\kappa \frac{\dot{\xi}^\kappa}{\| \dot{\xi} \|} + e^\kappa, \quad e = t^\kappa \frac{\dot{\xi}^\kappa}{\| \dot{\xi} \|}, \quad e^\kappa = t^\kappa \left( \delta^\kappa - \frac{\dot{\xi}_\lambda \dot{\xi}^\lambda}{\| \dot{\xi} \|^2} \right). \]  
By plugging (68) into (67) and using the continuity relation (64) one gets
\[ e^\kappa (s) = 0 \quad \text{and} \quad \frac{de(s)}{ds} = 0. \]  
Hence,
\[ J^\kappa = ec \int_{-\infty}^{\infty} \dot{\xi}^\kappa (s) \delta (\chi^\kappa - \xi^\kappa (s)) \, ds. \]  
By using this expression for the current and the analogous expression for the energy–momentum density \( T^{\kappa \lambda}_m \) plugged into the continuity relation (66) and following the steps used in Section 2.1, one gets the equations
\[ m^\kappa = 0, \quad m^\kappa = 0, \quad \frac{dm}{ds} = 0, \quad mc^2 \frac{\delta}{ds} \left( \frac{\dot{\xi}^\kappa}{\| \dot{\xi} \|} \right) = eg^{\kappa \mu} F_{\mu \nu} (\xi^\sigma (s)) \dot{\xi}^\nu. \]  
The final conclusion is that the current density of charged matter distributed along a world line \( \gamma \) is represented by (70) and the energy–momentum density is represented by (15). The constants \( e \) and \( m \) are interpreted as charge and mass respectively. The world line \( \gamma \) satisfies Eq. (71), which reduces to the equation of motion of a neutral particle (14) when \( e = 0 \).

It easy to see that the current density (70) and the energy–momentum density (15) derive from the action
\[ S_m = -\int_{-\infty}^{\infty} \int_\Omega \left( mc \sqrt{g_{\kappa \lambda} (\chi^\sigma) \dot{\xi}^\kappa (s) \dot{\xi}^\lambda (s) + \frac{e}{c} A_\kappa (\chi^\sigma) \dot{\xi}^\kappa \right) \delta (\chi^\kappa - \xi^\kappa (s)) \, d^4 x \, ds. \]  
The integration of (72) over the domain \( \Omega \) produces
\[ S_m = -\int_{-\infty}^{\infty} \left( mc \sqrt{g_{\kappa \lambda} (\xi^\sigma (s)) \dot{\xi}^\kappa (s) \dot{\xi}^\lambda (s) + \frac{e}{c} A_\kappa (\xi^\sigma (s)) \dot{\xi}^\kappa \right) \, ds, \]  which is the mechanical action of a charged particle in the presence of electromagnetic and gravitational fields. The dynamics of the particle is derived from the mechanical Lagrangian
\[ L (\dot{\xi}^\kappa, \dot{\xi}^\lambda) = mc \sqrt{g_{\kappa \lambda} (\xi^\sigma) \dot{\xi}^\kappa \dot{\xi}^\lambda + \frac{e}{c} A_\kappa (\chi^\sigma) \dot{\xi}^\kappa \dot{\xi}^\sigma + g_{\kappa \lambda} (\chi^\sigma) \dot{\xi}^\kappa \dot{\xi}^\lambda} > 0. \]
The Lagrange equations
\[ p_k = \frac{\partial L}{\partial \dot{\xi}^k} = mcg_{\kappa\lambda} \frac{\dot{\xi}^\kappa}{\| \dot{\xi} \|} + e A_k, \quad \dot{p}_k = \frac{\partial L}{\partial \dot{\xi}^\kappa} = mcg_{\kappa\mu} \Gamma^\nu_{\kappa\mu} \frac{\dot{\xi}^\nu}{\| \dot{\xi} \|} + \frac{e}{c} \partial_\kappa A_\lambda \dot{\xi}^\lambda \] (75)

can be rewritten as
\[ p_k = mcg_{\kappa\lambda} \frac{\dot{\xi}^\kappa}{\| \dot{\xi} \|} + \frac{e}{c} A_k, \quad \frac{\delta p_k}{ds} = \dot{p}_k - \Gamma^\nu_{\kappa\mu} p_\nu \dot{\xi}^\mu = \frac{e}{c} \nabla_\kappa A_\lambda \dot{\xi}^\lambda. \] (76)

Note that space–time trajectories are solutions of the second order differential equation
\[ mc \frac{\delta}{ds} \left( \frac{\dot{\xi}^k}{\| \dot{\xi} \|} \right) = e g^{k\lambda} F_{\kappa\mu} \dot{\xi}^\kappa. \] (77)

which are the same equations that derive from the field Eq. (63).

Also in this case the dynamics is governed by a singular Lagrangian. By applying the generalized Legendre transformation on gets the Hamiltonian Morse family
\[ H(\lambda, \dot{\xi}^k, p_\lambda) = \lambda \left( \sqrt{g^{\kappa\sigma} (\dot{\xi}^\kappa)} \left( p_\kappa - \frac{e}{c} A_\kappa (\dot{\xi}^\sigma) \right) \left( p_\sigma - \frac{e}{c} A_\sigma (\dot{\xi}^\sigma) \right) - mc \right), \] (78)

defined for \( \lambda > 0 \). One can show that the generalized Hamilton equations
\[ \frac{\partial H}{\partial \lambda} = \left( g^{\kappa\lambda} (\dot{\xi}^\kappa) \right) \left( p_\kappa - \frac{e}{c} A_\kappa \right) \left( p_\lambda - \frac{e}{c} A_\lambda \right) - mc = 0, \]
\[ \dot{\xi}^k = \frac{\partial H}{\partial p_k} = \lambda g^{k\lambda} p_\lambda - \frac{e}{c} A_\lambda \sqrt{g^{\kappa\sigma} (\dot{\xi}^\kappa)} \left( p_\kappa - \frac{e}{c} A_\kappa \right) \left( p_\sigma - \frac{e}{c} A_\sigma \right), \] (79)
\[ \dot{p}_k = - \frac{\partial H}{\partial \dot{\xi}^k} = \frac{\lambda}{c} \delta_\kappa A_\lambda g^{\lambda\mu} \left( p_\mu - \frac{e}{c} A_\mu \right) \sqrt{g^{\rho\sigma} \left( p_\rho - \frac{e}{c} A_\rho \right) \left( p_\sigma - \frac{e}{c} A_\sigma \right)} + \lambda \Gamma^\lambda_{\kappa\mu} g^{\lambda\mu} \left( p_\mu - \frac{e}{c} A_\mu \right) \left( p_\rho - \frac{e}{c} A_\rho \right) \left( p_\sigma - \frac{e}{c} A_\sigma \right) \] are equivalent to the Lagrange equations (75). Also, the dynamics of a charged relativistic particle does not represent an example of a Dirac constrained Hamiltonian system (24).

References