Kolmogorov-Arnold-Moser Stability for Conserved Quantities in Finite-Dimensional Quantum Systems

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We show that for any finite-dimensional quantum systems the conserved quantities can be characterized by their robustness to small perturbations: for fragile symmetries, small perturbations can lead to large deviations over long times, while for robust symmetries, their expectation values remain close to their initial values for all times. This is in analogy with the celebrated Kolmogorov-Arnold-Moser theorem in classical mechanics. To prove this result, we introduce a resummation of a perturbation series, which generalizes the Hamiltonian of the quantum Zeno dynamics.

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Symmetries and conserved quantities are the cornerstones of modern theoretical physics [1]. In quantum mechanics, it is well known that conserved quantities are characterized by observables that commute with the system Hamiltonian. Here, we show that this characterization is incomplete, because some symmetries in quantum mechanics are more conserved than others.

More precisely, we can consider the *robustness* of symmetries. Some fundamental symmetries (such as those related to superselection rules [2]) are considered almost unbreakable in nonrelativistic quantum mechanics, while other, accidental [3], symmetries are easily perturbed.

We now introduce such distinction into fundamental, robust symmetries and accidental, fragile ones in an analogous, but much more applied context, namely, one provided by a time-independent Hamiltonian on a finite-dimensional quantum system. This Hamiltonian H acts as a reference for its symmetries S, characterized by $[H,S]=0$, and with respect to which we define their robust component S_{robust} as the part almost conserved [up to a term $O(\varepsilon)$] for *all* times and for *any* small time-independent perturbation εV, while we define their *fragile* component S_{fragile} as the part for which there are perturbations that will accumulate large amounts of change over time. As an alternative view, for any robust symmetry there is a slightly modified observable that is conserved in the perturbed system $S_{\text{robust}}\to S_{\text{robust}}^{\varepsilon}$, while for fragile symmetries there is not. Such conserved quantities were constructed recently in many-body systems for specific perturbations [4], while we provide a general construction and characterization, and show a natural decomposition of any symmetry S

$$S=S_{\text{robust}}+S_{\text{fragile}}. \tag{1}$$

The importance of robust observables is exemplified by analog quantum simulations [5], where the aim is to run a complex Hamiltonian long enough such that observable quantities are no longer easily computable by classical computers. The problem is, however, that small perturbations in the lab are not under control and can destroy the reliability of the simulation [6,7]. On the other hand, as we show below, the expectation values of robust observables remain reliable even in the long term.

More fundamentally, our result is in close analogy to the Kolmogorov-Arnold-Moser (KAM) perturbation theory in classical mechanics [8,9], which proved the long-time stability of planetary orbits, despite accumulating perturbations. Quantum mechanical versions of KAM perturbation have been considered previously by Scherer [10] to mimic a superconvergent series. In the context of many-body systems, Nekhoroshev estimates were used to show a robustness of certain observables for intermediate times [11–15]. Our focus, instead, is an algebraic approach based on the adiabatic theorem, enabling us to provide nonperturbative bounds valid for arbitrarily long times, with no structural assumptions on the generators and observables, and generalizations to open systems (Lindbladians). This way, we prove a result analogous to the KAM stability in finite-dimensional quantum systems.

How can we characterize which observables are fragile and which are robust? Under which conditions are there robust ones, and just *how* robust are they? In the unperturbed system, the conserved quantities are the observables commuting with H, given by all Hermitian matrices that are block diagonal with respect to the eigenspaces of H. They may share the degenerate eigenspaces of H or they may lift their degeneracy. In this Letter, we will show that this precisely distinguishes robust and fragile symmetries. Moreover, unless H is the identity, there always exist nontrivial robust symmetries.

Fragile symmetries.—First, consider a symmetry M that breaks degeneracy in an eigenspace of H. We show that such a conserved quantity is not robust against perturbation. For instance, take two simultaneous eigenstates of H and M, say $|e_1\rangle$ and $|e_2\rangle$, belonging to the same eigenspace of H but belonging to different eigenspaces of M; i.e., $H|e_1\rangle = e|e_1\rangle$ and $H|e_2\rangle = e|e_2\rangle$, while $M|e_1\rangle = m_1|e_1\rangle$ and $M|e_2\rangle = m_2|e_2\rangle$, with $\Delta = m_1 - m_2 > 0$. Let us take $V = |e_1\rangle\langle e_2| + |e_2\rangle\langle e_1|$ as a perturbation and consider $H + \varepsilon V$. If we focus on initial states $|\psi\rangle$ in the subspace spanned by $\{|e_1\rangle, |e_2\rangle\}$, the problem is reduced to a two-dimensional problem. Take, for instance, $|e_1\rangle$ as an initial state. We find

$$\langle M\rangle_t^\varepsilon - \langle M\rangle_t = \langle M\rangle_t^\varepsilon - \langle M\rangle_0 = -\Delta \sin^2 \varepsilon t, \qquad (2)$$

where the expectations $\langle\cdot\rangle_t$ and $\langle\cdot\rangle_t^\varepsilon$ are taken with respect to states evolved under the free and the perturbed evolution, $e^{-itH}|\psi\rangle$ and $e^{-it(H+\varepsilon V)}|\psi\rangle$, respectively. At time $t = \pi/(2\varepsilon)$ the error is Δ, which is independent of ε. This kind of example can be constructed for any M that is nondegenerate within a subspace of H, and we conclude that such conserved observables are fragile.

Robust symmetries.—Second, consider a conserved observable that acts uniformly within each eigenspace of H. We may write $M = \sum m_k P_k$, where $\{P_k\}$ are the spectral projections of $H = \sum_k e_k P_k$ (with $e_k \neq e_\ell$ for $k \neq \ell$ and $P_k P_\ell = \delta_{k\ell} P_\ell$). Using results on the quantum Zeno dynamics [16–19], one can show that such observables are endowed with some intrinsic robustness with respect to small perturbations εV, with $\|V\| = 1$. Indeed, we have a bound [20]

$$\delta_Z(t) = \|e^{it(H+\varepsilon V)} - e^{it(H+\varepsilon V_Z)}\| \leq 2\sqrt{d}\varepsilon(1+\varepsilon t)/\eta, \qquad (3)$$

where d is the number of distinct eigenvalues of the Hamiltonian H, $\eta = \min_{k\neq\ell} |e_k - e_\ell|$ is the spectral gap of H (strictly positive for any nontrivial H), and $V_Z = \sum_k P_k V P_k$ is the "Zeno Hamiltonian" [16,17]. By construction, $[M, V_Z] = 0$, and we obtain [19,20]

$$\|M_t^\varepsilon - M\| \leq 2\|M\|\delta_Z(t) \leq 4\sqrt{d}\|M\|\varepsilon(1+\varepsilon t)/\eta, \qquad (4)$$

where $M_t^\varepsilon = e^{it(H+\varepsilon V)} M e^{-it(H+\varepsilon V)}$ is the perturbed evolution of observable M. This bound, however, is informational as far as it is less than the trivial bound $2\|M\|$, which is not for sufficiently large times t.

This is, anyway, just an upper bound, and it might be a loose bound. Let us look more carefully at a two-dimensional example again and show that there are indeed perturbations V such that $\delta_Z(t)$ in (3) saturates the trivial bound 2, for every ε, however small. Consider $H = \sigma_z$ and $V = \sigma_x$, the third and first Pauli matrices, respectively. In this case, we have $V_Z = 0$, and $\delta_Z(t) = \|e^{it(\sigma_z+\varepsilon\sigma_x)} - e^{it\sigma_z}\|$. This is a complicated quasiperiodic function, with $\sup_t \delta_Z(t) = 2$ [31], proving that, in general, the Zeno Hamiltonian V_Z is not a good approximation for long times.

However, notwithstanding the negative result about the smallness of the distance $\delta_Z(t)$ in (3), the conserved quantity $M = \sum m_k P_k$ considered above is actually stable for all times, *eternally*. The key idea behind the above phenomenon is to choose an (ε-dependent) approximation of V that has the same block structure as H and is therefore commutative with M. The Zeno Hamiltonian V_Z is not a good choice. To make the point, consider again the above two-dimensional example with $H = \sigma_z$ and $V = \sigma_x$, and now choose, in place of V_Z, the operator $V_H(\varepsilon) = \varepsilon^{-1}(\sqrt{1+\varepsilon^2} - 1)\sigma_z$ as an approximation of V. Obviously, $[V_H, H] = 0$. Moreover, $V_H(\varepsilon) = V_Z + O(\varepsilon)$. With this choice, we get

$$\delta(t) = \|e^{it(\sigma_z+\varepsilon\sigma_x)} - e^{it[\sigma_z+\varepsilon V_H(\varepsilon)]}\|$$
$$= \sqrt{2\left(1 - \frac{1}{\sqrt{1+\varepsilon^2}}\right)}|\sin(t\sqrt{1/\varepsilon^2+1})| \leq \varepsilon. \qquad (5)$$

This bound is independent of time t and implies that [20]

$$\|M_t^\varepsilon - M\| \leq 2\|M\|\delta(t) \leq 2\|M\|\varepsilon, \qquad (6)$$

for all times and for any observable of the form $M = \mathrm{diag}(m_1, m_2)$. Such observables are robust.

General result.—Of course, we did not just guess $V_H(\varepsilon)$ arbitrarily. We discovered a way of constructing such eternal block-diagonal approximations for any finite-dimensional quantum systems, including noisy systems with Lindbladians. They can be seen as resummation of a perturbative series, whose zeroth-order term is the Zeno Hamiltonian $V_H(0) = V_Z$. Its theory, proof, and generalizations are discussed in great detail in Ref. [32].

The crucial ingredient is that the block-diagonal approximation $H + \varepsilon V_H(\varepsilon)$, unlike $H + \varepsilon V_Z$, can be chosen to have the *same* spectrum of $H + \varepsilon V$ and thus to be unitarily equivalent to it: $H + \varepsilon V_H(\varepsilon) = W_\varepsilon^\dagger(H + \varepsilon V)W_\varepsilon$, with a unitary $W_\varepsilon = \mathbb{1} + O(\varepsilon)$ [20]. This is a necessary condition, since geometrically the evolution of a Hamiltonian with d distinct eigenvalues yields a (quasi-)periodic motion of a point on a torus. Two motions with different frequencies, however small the differences may be, will eventually accumulate a divergence of $O(1)$. The only way to avoid this slow drift is that the two motions be isochronous, that is, the two Hamiltonians be isospectral. In such a case, we get

$$\begin{aligned}\delta_\infty &= \sup_t \| e^{it(H+\varepsilon V)} - e^{it[H+\varepsilon V_H(\varepsilon)]} \| \\ &= \sup_t \| e^{it(H+\varepsilon V)} - W_\varepsilon^\dagger e^{it(H+\varepsilon V)} W_\varepsilon \| < 7\sqrt{d}\varepsilon/\eta \quad (7)\end{aligned}$$

(see the Supplemental Material [20] for a perturbative proof and Ref. [32] for explicit bounds). It follows that any quantum system has robust conserved quantities $S = S_{\text{robust}}$, with $[S, H] = 0$, and hence $S = S_t$, such that for every perturbation εV,

$$\| S_t^\varepsilon - S \| \leq 2\|S\|\delta_\infty = O(\varepsilon), \quad (8)$$

for all times [20], where $S_t^\varepsilon = e^{it(H+\varepsilon V)} S e^{-it(H+\varepsilon V)}$, and they are precisely of the form $S = S_{\text{robust}}$, where

$$S_{\text{robust}} = \sum_k s_k P_k, \quad (9)$$

with $H = \sum_k e_k P_k$. All other conserved quantities are fragile, as the distance (8) becomes $O(1)$.

While this is a complete characterization of robust conserved quantities, the representation in terms of spectral projections requires diagonalization of the Hamiltonian H and is impractical for high-dimensional systems. However, given $H^n = \sum_k e_k^n P_k$, one can invoke the invertibility of the Vandermonde matrix (e_k^{j-1}) to see that $S = \sum_{k=0}^{d-1} c_k H^k$. This means that any primary matrix function $f(H)$ of the Hamiltonian H is robust. If the original Hamiltonian is sparse, for instance, low-order polynomials can be constructed efficiently. In particular, we obtain that for any state the energy expectation value and the variance

$$\langle H \rangle_t^\varepsilon - \langle H \rangle_0 = O(\varepsilon), \qquad \langle \Delta H^2 \rangle_t^\varepsilon - \langle \Delta H^2 \rangle_0 = O(\varepsilon) \quad (10)$$

remain close to their unperturbed values forever. This is easily generalized to higher moments.

We can also rephrase the fact that any robust observable is a polynomial function of H in terms of the symmetries of H. That is, S is robust if and only if it shares all symmetries of H: for any C such that $[H, C] = 0$, we also have $[S, C] = 0$ [33]. For more details on the algebraic structure, see the Supplemental Material [20].

Finally, let us emphasize that the above characterization provides a natural decomposition of any observable M into three parts: one dynamical part that is not conserved by H, one that is conserved but is fragile to perturbations, and one that is robust. The nonconserved part is off diagonal with respect to the spectral projections of H,

$$M_{\text{noncons}} = M - M_{\text{cons}} = M - \sum_k P_k M P_k, \quad (11)$$

the robust component of the conserved part M_{cons} acts trivially within the eigenspaces of H,

$$M_{\text{robust}} = \sum_k d_k^{-1} \text{tr}(P_k M P_k) P_k, \quad (12)$$

with d_k being the dimension of the kth eigenspace, and the fragile part is the remaining symmetry

$$M_{\text{fragile}} = \sum_k P_k [M - d_k^{-1} \text{tr}(P_k M P_k)] P_k. \quad (13)$$

Integrable example.—While an unambiguous and universal definition of integrability for quantum systems is still lacking [34,35], we take the Heisenberg chain as a typical example of a system which we think of as integrable. The Hamiltonian acts on N qubits and is given by $H = -J \sum_{n=1}^N \boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_{n+1}$, where $\boldsymbol{\sigma}_n = (\sigma_{n,x}, \sigma_{n,y}, \sigma_{n,z})$ is the vector of Pauli matrices acting on the nth qubit, and we impose the periodic boundary conditions $\boldsymbol{\sigma}_{N+1} = \boldsymbol{\sigma}_1$. The Heisenberg chain can be solved analytically by the algebraic Bethe ansatz. The corresponding conserved charges $Q_2, \ldots, Q_N$ can be generated using the boost operator $B = \frac{1}{2} \sum_{n=1}^N n \boldsymbol{\sigma}_n \cdot \boldsymbol{\sigma}_{n+1}$ as $Q_{n+1} = -i[B, Q_n]$ with $Q_2 = H$, and Q_n acts nontrivially on sets of n neighbors on the chain only [36]. Combined with the total magnetization $Q_1 = \sum_{n=1}^N \sigma_{n,z}$, they provide a maximal Abelian algebra. These conserved charges are the pinnacle of integrability. However, they are fragile: because the charges are algebraically independent, except for $Q_2 = H$, none are robust. Incidentally, this shows that the findings in Ref. [4] are restricted to specific perturbation classes. A simple example is given by the total magnetization Q_1: due to the rotational invariance of H, we could have equally chosen the magnetization in another direction, say $\tilde{Q}_1 = \sum_{n=1}^N \sigma_{n,x}$. As a perturbation, however, $\tilde{Q}_1$ causes the expectation value of Q_1 to oscillate and deviate vastly from its original value. For instance, if we start with a z-polarized state, we obtain $\langle Q_1 \rangle_t^\varepsilon = \cos(\omega t) N$ for some ω depending on the perturbation strength ε. We show numerical examples of the evolution of a randomly chosen observable (Fig. 1) as well as physical ones (Fig. 2).

Our bound (7) scales with the number d of distinct eigenvalues and the inverse of the spectral gap η of the Hamiltonian H. Therefore, in the context of many-body physics, it is useful only for particular systems as system size grows. In many-body setups, weaker types of robustness of observables were shown, assuming locality of the Hamiltonian, observable, and perturbation [11–15]. In the context of KAM, such bounds are analogous to Nekhoroshev estimates, showing stability for an

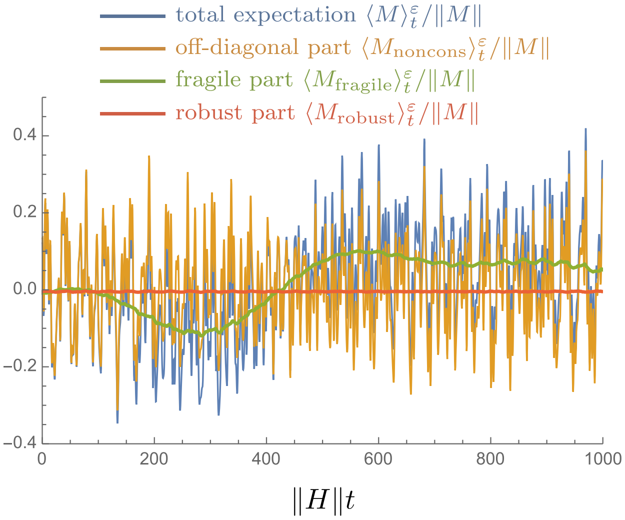

FIG. 1. Dynamics of the expectation of a randomly picked observable M in a Heisenberg chain with $N=4$ as a function of time. We show its decomposition into nonconserved, robust, and fragile parts [Eqs. (11)–(13)]. The initial state and the perturbation V are chosen randomly with strength $\varepsilon\|V\| = 0.02\|H\|$. Shown is one realization.

exponentially long time. While the spectral gap is important for our bound, our result is valid for arbitrarily long times, requiring no structural assumptions on the Hamiltonians, observables, and perturbations.

Thermalization.—It is one of the most celebrated results in mathematical quantum statistical mechanics that the Kubo-Martin-Schwinger state [the Gibbs state $\propto \exp(-\beta H)$] is the unique state that maximizes entropy, stationary under the time evolution of the Hamiltonian H, and robust under perturbations [37]. However, to date, this was only considered for short times. Our characterization of robust observables implies that

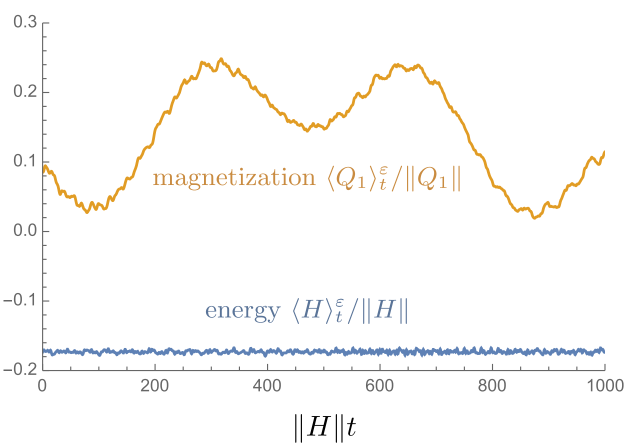

FIG. 2. Same setup and realization as in Fig. 1, but now showing the dynamics of the expectation of $Q_2 = H$ (robust) and $Q_1 = \sum_{n=1}^{N}\sigma_{n,z}$ (fragile).

$$e^{-it(H+\varepsilon V)}\exp(-\beta H)e^{it(H+\varepsilon V)} = \exp(-\beta H) + O(\varepsilon), \quad (14)$$

uniformly in time for any finite-dimensional system. Note, on the other hand, that generalized Gibbs ensembles [38–44] such as $\exp(-\sum_j \beta_j Q_j)$ for integrable charges are *not* robust.

Open systems.—How do we generalize this to Lindbladian dynamics? For a Lindbladian $\mathcal{L}$, it would be natural to consider $\mathcal{M} = \mathcal{M}_{\mathrm{robust}} = \sum_k m_k \mathcal{P}_k$, with $\{\mathcal{P}_k\}$ the spectral projections of $\mathcal{L}$, as a candidate for a robust symmetry. However, it is easy to see that the trace preservation of $\mathcal{L}$ implies that $\mathrm{tr}\mathcal{M}(\rho) = m_0 \mathrm{tr}\rho = m_0$, where $\mathcal{P}_0$ is the projection for the zero eigenvalue of $\mathcal{L}$. Therefore, this quantity is trivial. This is related to the fact that Noether's theorem breaks down for Lindbladian systems [45] and to the fact that we are talking about a superoperator structure on top of the usual observable space. Very recently, however, Styliaris and Zanardi showed [46] that for each conserved superoperator $\mathcal{M}$ satisfying $[\mathcal{M}, \mathcal{L}] = 0$ one can define a monotone function

$$f_{\mathcal{M}}(\rho) = \mathrm{tr}[\mathcal{M}(\rho)^\dagger(\mathbf{L}_\rho + \lambda\mathbf{R}_\rho)^{-1}(\mathcal{M}(\rho))], \quad (15)$$

with $\lambda \geq 0$, where $\mathbf{L}_\rho(X) = \rho X$ and $\mathbf{R}_\rho(X) = X\rho$ are the superoperators of left and right multiplication by ρ, respectively, and the inverse is well defined for strictly positive ρ. They showed that such a monotone, as complicated as it might look at first glance, is well motivated from entropic distances and is decreasing under the evolution $e^{t\mathcal{L}}$,

$$f_{\mathcal{M}}(\rho_t) \leq f_{\mathcal{M}}(\rho), \quad \text{for all } t \geq 0, \quad (16)$$

where $\rho_t = e^{t\mathcal{L}}\rho$. Using our generalized eternal block-diagonal approximation $\mathcal{V}_{\mathcal{L}}(\varepsilon)$ to a perturbation $\mathcal{V}$ for open systems [32], we can write the perturbed dynamics

$$e^{t(\mathcal{L}+\varepsilon\mathcal{V})} = e^{t[\mathcal{L}+\varepsilon\mathcal{V}_{\mathcal{L}}(\varepsilon)]} + O(\varepsilon) \quad (17)$$

for all times and see that, for any robust symmetry $\mathcal{M} = \mathcal{M}_{\mathrm{robust}}$ and for any perturbation $\varepsilon\mathcal{V}$, the monotone $f_{\mathcal{M}}(\rho)$ remains approximately monotonic [20],

$$f_{\mathcal{M}}(\rho_t^\varepsilon) \leq f_{\mathcal{M}}(\rho) + O(\varepsilon), \quad (18)$$

under the perturbed evolution $\rho_t^\varepsilon = e^{t(\mathcal{L}+\varepsilon\mathcal{V})}\rho$. In this sense, the monotone is robust against perturbation. See Fig. 3 for a couple of examples for a qubit dephasing evolution. The monotone defined with $\mathcal{M}_{\mathrm{robust}}$ is robust against perturbation: it is perturbed and becomes nonmonotonic, but the nonmonotonicity is small. On the other hand, the monotone defined with $\mathcal{M}_{\mathrm{fragile}}$ that lifts the degeneracy in $\mathcal{L}$ is fragile.

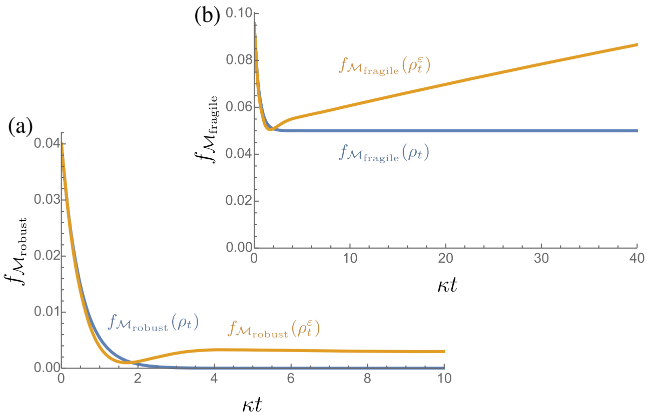

FIG. 3. The free and perturbed evolutions for a qubit of (a) the robust monotone $f_{\mathcal{M}_{\rm robust}}(\rho)=|\langle 0|\rho|1\rangle|^2$, with $\mathcal{M}_{\rm robust}=-(i/\sqrt{2})[\sigma_z,\cdot]$ and $\lambda=1$, and of (b) the fragile monotone $f_{\mathcal{M}_{\rm fragile}}(\rho)$, with $\mathcal{M}_{\rm fragile}=-(i/\sqrt{2})[\sigma_z,\cdot]+|0\rangle\langle 0|\cdot|0\rangle\langle 0|$ and $\lambda=1$, where $\mathcal{L}=-(i/2)\omega[\sigma_z,\cdot]-\frac{1}{2}\kappa(1-\sigma_z\cdot\sigma_z)$ and $\varepsilon\mathcal{V}=-(i/2)\varepsilon g[\sigma_x,\cdot]$. The perturbation creates coherence between the eigenstates of σ_z, $|0\rangle$ and $|1\rangle$. The initial state of the qubit is given by the coherence vector $(r_x,r_y,r_z)=(0.2,0,0.8)$, and the parameters are set at $g=\kappa$ and $\varepsilon=0.1$ (ω is irrelevant).

Conclusions.—While our results in spirit reproduce a lot of features one would hope a quantum KAM theory to feature—long-term stability of certain observables with respect to perturbations, in analogy with the KAM theory in classical mechanics [8,9]—there are also some perhaps surprising aspects. Conserved charges and generalized Gibbs states from quantum integrable models are not robust, while randomly chosen Hamiltonians (thus without degeneracies) have the property that all conserved quantities are robust.

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[1] D. J. Gross, The role of symmetry in fundamental physics, Proc. Natl. Acad. Sci. U.S.A. **93**, 14256 (1996).

[2] V. Moretti, *Fundamental Mathematical Structures of Quantum Theory* (Springer, Cham, 2019).

[3] M. Al-Hashimi, Accidental symmetry in quantum physics, Ph. D. thesis, Institute for Theoretical Physics, Bern University, 2008.

[4] G. P. Brandino, J.-S. Caux, and R. M. Konik, Glimmers of a Quantum KAM Theorem: Insights from Quantum Quenches in One-Dimensional Bose Gases, Phys. Rev. X **5**, 041043 (2015).

[5] I. M. Georgescu, S. Ashhab, and F. Nori, Quantum simulation, Rev. Mod. Phys. **86**, 153 (2014).

[6] M. Sarovar, J. Zhang, and L. Zeng, Reliability of analog quantum simulation, Eur. Phys. J. Quantum Technol. **4**, 1 (2017).

[7] I. Schwenk, J.-M. Reiner, S. Zanker, L. Tian, J. Leppäkangas, and M. Marthaler, Reconstructing the ideal results of a perturbed analog quantum simulator, Phys. Rev. A **97**, 042310 (2018).

[8] W. Thirring, *Classical Mathematical Physics: Dynamical Systems and Field Theories*, 3rd ed. (Springer, New York, 1997).

[9] V. I. Arnold, V. V. Kozlov, and A. I. Neishtadt, *Mathematical Aspects of Classical and Celestial Mechanics*, 3rd ed. (Springer, Berlin, 2006).

[10] W. Scherer, Superconvergent Perturbation Method in Quantum Mechanics, Phys. Rev. Lett. **74**, 1495 (1995).

[11] M. Moeckel and S. Kehrein, Interaction Quench in the Hubbard Model, Phys. Rev. Lett. **100**, 175702 (2008).

[12] M. Moeckel and S. Kehrein, Real-time evolution for weak interaction quenches in quantum systems, Ann. Phys. (Amsterdam) **324**, 2146 (2009).

[13] B. Bertini, F. H. L. Essler, S. Groha, and N. J. Robinson, Prethermalization and Thermalization in Models with Weak Integrability Breaking, Phys. Rev. Lett. **115**, 180601 (2015).

[14] K. Mallayya, M. Rigol, and W. De Roeck, Prethermalization and Thermalization in Isolated Quantum Systems, Phys. Rev. X **9**, 021027 (2019).

[15] W. De Roeck and V. Verreet, Very slow heating for weakly driven quantum many-body systems, arXiv:1911.01998.

[16] P. Facchi and S. Pascazio, Quantum Zeno Subspaces, Phys. Rev. Lett. **89**, 080401 (2002).

[17] P. Facchi and S. Pascazio, Quantum Zeno dynamics: Mathematical and physical aspects, J. Phys. A **41**, 493001 (2008).

[18] D. Burgarth, P. Facchi, H. Nakazato, S. Pascazio, and K. Yuasa, Generalized adiabatic theorem and strong-coupling limits, Quantum **3**, 152 (2019).

[19] Z. Gong, N. Yoshioka, N. Shibata, and R. Hamazaki, Universal Error Bound for Constrained Quantum Dynamics, Phys. Rev. Lett. **124**, 210606 (2020).

[20] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.126.150401 for additional details and explicit calculations, which includes Refs. [21–30].

[21] T. Kato, On the adiabatic theorem of quantum mechanics, J. Phys. Soc. Jpn. **5**, 435 (1950).

[22] J. des Cloizeaux, Extension d'une Formule de Lagrange à des Problèmes de Valeurs Propres, Nucl. Phys. **20**, 321 (1960).
[23] D. J. Klein, Degenerate perturbation theory, J. Chem. Phys. **61**, 786 (1974).
[24] T. Kato, *Perturbation Theory for Linear Operators*, 2nd ed. (Springer, Berlin, 1976).
[25] E. I. Dinaburg and Y. G. Sinai, The one-dimensional Schrödinger equation with a quasiperiodic potential, Funct. Anal. Appl. **9**, 279 (1975).
[26] H. Rüssmann, On the one-dimensional Schrödinger equation with a quasi-periodic potential, Ann. N.Y. Acad. Sci. **357**, 90 (1980).
[27] W. Craig, Pure point spectrum for discrete almost periodic Schrödinger operators, Commun. Math. Phys. **88**, 113 (1983).
[28] J. Pöschel, Examples of discrete Schrödinger operators with pure point spectrum, Commun. Math. Phys. **88**, 447 (1983).
[29] J. Bellissard, R. Lima, and E. Scoppola, Localization in ν-dimensional incommensurate structures, Commun. Math. Phys. **88**, 465 (1983).
[30] W. Scherer, Quantum averaging II: Kolmogorov's algorithm, J. Phys. A **30**, 2825 (1997).
[31] One can consider the specific sequence $\varepsilon_n = (\sqrt{4n+3}/2n+1) \to 0$ such that at times $t_n = (\pi/\sqrt{1+\varepsilon_n^2} - 1)$ the norm distance saturates, $\delta_Z(t_n) = 2$.
[32] D. Burgarth, P. Facchi, H. Nakazato, S. Pascazio, and K. Yuasa, Eternal adiabaticity in quantum evolution, Phys. Rev. A **103**, 032214 (2021).
[33] N. J. Higham, *Functions of Matrices: Theory and Computation* (SIAM, Philadelphia, 2008).
[34] J.-S. Caux and J. Mossel, Remarks on the notion of quantum integrability, J. Stat. Mech. (2011) P02023.
[35] D. Braak, Integrability of the Rabi Model, Phys. Rev. Lett. **107**, 100401 (2011).
[36] M. P. Grabowski and P. Mathieu, Quantum integrals of motion for the Heisenberg spin chain, Mod. Phys. Lett. A **09**, 2197 (1994).
[37] R. Haag, D. Kastler, and E. B. Trych-Pohlmeyer, Stability and equilibrium states, Commun. Math. Phys. **38**, 173 (1974).
[38] A. Polkovnikov, K. Sengupta, A. Silva, and M. Vengalattore, Nonequilibrium dynamics of closed interacting quantum systems, Rev. Mod. Phys. **83**, 863 (2011).
[39] J. Eisert, M. Friesdorf, and C. Gogolin, Quantum many-body systems out of equilibrium, Nat. Phys. **11**, 124 (2015).
[40] J. Goold, M. Huber, A. Riera, L. del Rio, and P. Skrzypczyk, The role of quantum information in thermodynamics: A topical review, J. Phys. A **49**, 143001 (2016).
[41] C. Gogolin and J. Eisert, Equilibration, thermalisation, and the emergence of statistical mechanics in closed quantum systems, Rep. Prog. Phys. **79**, 056001 (2016).
[42] L. Vidmar and M. Rigol, Generalized Gibbs ensemble in integrable lattice models, J. Stat. Mech. (2016) 064007.
[43] P. Calabrese, Entanglement and thermodynamics in non-equilibrium isolated quantum systems, Physica (Amsterdam) **504A**, 31 (2018).
[44] T. Mori, T. N. Ikeda, E. Kaminishi, and M. Ueda, Thermalization and prethermalization in isolated quantum systems: A theoretical overview, J. Phys. B **51**, 112001 (2018).
[45] V. V. Albert and L. Jiang, Symmetries and conserved quantities in Lindblad master equations, Phys. Rev. A **89**, 022118 (2014).
[46] G. Styliaris and P. Zanardi, Symmetries and monotones in Markovian quantum dynamics, Quantum **4**, 261 (2020).

KAM-Stability for Conserved Quantities in Finite-Dimensional Quantum Systems
Supplemental Material

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In this Supplemental Material, we discuss the mathematical details.

Zeno dynamics. Error bound

Consider the spectral resolution of $H = H^\dagger$:

$$H = \sum_{k=1}^{d} h_k P_k, \tag{1}$$

where

$$P_k P_\ell = \delta_{k\ell} P_\ell = \delta_{k\ell} P_\ell^\dagger, \qquad \sum_k P_k = \mathbb{1}, \tag{2}$$

$d \le \dim \mathcal{H}$ is the number of distinct eigenvalues of H, and $h_k \in \mathbb{R}$, with $h_k \neq h_\ell$ for $k \neq \ell$. Given a perturbation $V = V^\dagger$ its diagonal part (Zeno Hamiltonian) is given by

$$V_\mathrm{Z} = \sum_k P_k V P_k. \tag{3}$$

We want to bound the divergence

$$\delta_\mathrm{Z}(t) = \| \mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} - \mathrm{e}^{\mathrm{i}t(H+\varepsilon V_\mathrm{Z})} \| \tag{4}$$

between the dynamics generated by $H + \varepsilon V$ and the dynamics generated by its block-diagonal part $H + \varepsilon V_\mathrm{Z}$. We will use a trick elaborated in Ref. [1], which is based on Kato's seminal proof of the adiabatic theorem [2].

Fix a spectral projection P_ℓ and consider the reduced resolvent at h_ℓ, $\lim_{z\to h_\ell}(H - z\mathbb{1})^{-1}(\mathbb{1} - P_\ell)$, that is

$$S_\ell = \sum_{k\,:\,k\neq\ell} \frac{1}{h_k - h_\ell} P_k. \tag{5}$$

In the following, we will use 1 for the identity operator $\mathbb{1}$ and simply write $H - z$ instead of $H - z\mathbb{1}$. We get $P_\ell S_\ell = S_\ell P_\ell = 0$ and

$$(H - h_\ell) S_\ell = S_\ell (H - h_\ell) = \sum_{k\,:\,k\neq\ell} \frac{h_k - h_\ell}{h_k - h_\ell} P_k = \sum_{k\,:\,k\neq\ell} P_k = 1 - P_\ell, \tag{6}$$

that is S_ℓ is the inverse of $H - h_\ell$ on the subspace range of $1 - P_\ell$. We get

$$\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} - \mathrm{e}^{\mathrm{i}t(H+\varepsilon V_\mathrm{Z})} = -\int_0^t \mathrm{d}s\, \frac{\partial}{\partial s}\bigl(\mathrm{e}^{\mathrm{i}(t-s)(H+\varepsilon V)} \mathrm{e}^{\mathrm{i}s(H+\varepsilon V_\mathrm{Z})}\bigr) = \mathrm{i}\varepsilon \int_0^t \mathrm{d}s\, \mathrm{e}^{\mathrm{i}(t-s)(H+\varepsilon V)} (V - V_\mathrm{Z}) \mathrm{e}^{\mathrm{i}s(H+\varepsilon V_\mathrm{Z})}, \tag{7}$$

whence

$$\bigl(\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} - \mathrm{e}^{\mathrm{i}t(H+\varepsilon V_\mathrm{Z})}\bigr) P_\ell = \mathrm{i}\varepsilon \int_0^t \mathrm{d}s\, \mathrm{e}^{\mathrm{i}(t-s)(H+\varepsilon V)} (1 - P_\ell) V P_\ell \mathrm{e}^{\mathrm{i}s(h_\ell+\varepsilon V_\mathrm{Z})}. \tag{8}$$

By (6) we have

$$\big(\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)}-\mathrm{e}^{\mathrm{i}t(H+\varepsilon V_{\mathrm{Z}})}\big)P_\ell=\mathrm{i}\varepsilon\int_0^t \mathrm{d}s\,\mathrm{e}^{\mathrm{i}(t-s)(H+\varepsilon V)}(H-h_\ell)S_\ell V P_\ell \mathrm{e}^{\mathrm{i}s(h_\ell+\varepsilon V_{\mathrm{Z}})}. \tag{9}$$

Now notice that

$$\mathrm{i}\mathrm{e}^{\mathrm{i}(t-s)(H+\varepsilon V)}(H-h_\ell)=-\frac{\partial}{\partial s}\big(\mathrm{e}^{\mathrm{i}(t-s)(H+\varepsilon V)}\mathrm{e}^{\mathrm{i}s(h_\ell+\varepsilon V)}\big)\mathrm{e}^{-\mathrm{i}s(h_\ell+\varepsilon V)}, \tag{10}$$

and thus

$$\big(\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)}-\mathrm{e}^{\mathrm{i}t(H+\varepsilon V_{\mathrm{Z}})}\big)P_\ell=-\varepsilon\int_0^t \mathrm{d}s\,\frac{\partial}{\partial s}\big(\mathrm{e}^{\mathrm{i}(t-s)(H+\varepsilon V)}\mathrm{e}^{\mathrm{i}s(h_\ell+\varepsilon V)}\big)\mathrm{e}^{-\mathrm{i}s\varepsilon V}S_\ell V P_\ell \mathrm{e}^{\mathrm{i}s\varepsilon V_{\mathrm{Z}}}. \tag{11}$$

By integrating by parts

$$\begin{aligned}\big(\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)}-\mathrm{e}^{\mathrm{i}t(H+\varepsilon V_{\mathrm{Z}})}\big)P_\ell&=-\varepsilon\int_0^t \mathrm{d}s\,\frac{\partial}{\partial s}\big(\mathrm{e}^{\mathrm{i}(t-s)(H+\varepsilon V)}\mathrm{e}^{\mathrm{i}s(h_\ell+\varepsilon V)}\mathrm{e}^{-\mathrm{i}s\varepsilon V}S_\ell V P_\ell \mathrm{e}^{\mathrm{i}s\varepsilon V_{\mathrm{Z}}}\big)\\&\quad+\varepsilon\int_0^t \mathrm{d}s\,\mathrm{e}^{\mathrm{i}(t-s)(H+\varepsilon V)}\mathrm{e}^{\mathrm{i}s(h_\ell+\varepsilon V)}\frac{\partial}{\partial s}\big(\mathrm{e}^{-\mathrm{i}s\varepsilon V}S_\ell V P_\ell \mathrm{e}^{\mathrm{i}s\varepsilon V_{\mathrm{Z}}}\big)\\&=\varepsilon\big(\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)}S_\ell V P_\ell-S_\ell V P_\ell \mathrm{e}^{\mathrm{i}t(H+\varepsilon V_{\mathrm{Z}})}\big)\\&\quad-\mathrm{i}\varepsilon^2\int_0^t \mathrm{d}s\,\mathrm{e}^{\mathrm{i}(t-s)(H+\varepsilon V)}(VS_\ell V P_\ell-S_\ell V P_\ell V_{\mathrm{Z}})\mathrm{e}^{\mathrm{i}s(H+\varepsilon V_{\mathrm{Z}})}.\end{aligned} \tag{12}$$

Finally, by summing over ℓ we have

$$\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)}-\mathrm{e}^{\mathrm{i}t(H+\varepsilon V_{\mathrm{Z}})}=\varepsilon\big(\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)}X-X\mathrm{e}^{\mathrm{i}t(H+\varepsilon V_{\mathrm{Z}})}\big)-\mathrm{i}\varepsilon^2\int_0^t \mathrm{d}s\,\mathrm{e}^{\mathrm{i}(t-s)(H+\varepsilon V)}(VX-XV_{\mathrm{Z}})\mathrm{e}^{\mathrm{i}s(H+\varepsilon V_{\mathrm{Z}})}, \tag{13}$$

where

$$X=\sum_\ell S_\ell V P_\ell. \tag{14}$$

By taking the operator norm, one gets

$$\delta_{\mathrm{Z}}(t)=\|\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)}-\mathrm{e}^{\mathrm{i}t(H+\varepsilon V_{\mathrm{Z}})}\|\le 2\varepsilon\|X\|+\varepsilon^2\int_0^t \mathrm{d}s\,(\|V\|\|X\|+\|X\|\|V_{\mathrm{Z}}\|)=2\varepsilon\|X\|+\varepsilon^2\|X\|(\|V\|+\|V_{\mathrm{Z}}\|)t. \tag{15}$$

Now, we get

$$\|X\|^2=\|XX^\dagger\|=\Big\|\sum_\ell S_\ell V P_\ell V S_\ell\Big\|\le\sum_\ell\|S_\ell V P_\ell V S_\ell\|\le\sum_\ell\|S_\ell\|^2\|V\|^2, \tag{16}$$

while

$$\|S_\ell\|=\Big\|\sum_{k\,:\,k\neq\ell}\frac{P_k}{h_k-h_\ell}\Big\|=\max_{k\,:\,k\neq\ell}\Big|\frac{1}{h_k-h_\ell}\Big|\le\frac{1}{\eta}, \tag{17}$$

where

$$\eta=\min_{k,\ell\,:\,k\neq\ell}|h_k-h_\ell| \tag{18}$$

is the minimum spectral gap of H, and thus

$$\|X\|\le\frac{\sqrt{d}}{\eta}\|V\|. \tag{19}$$

Moreover, in the operator norm,

$$\|V_{\rm Z}\| = \left\| \sum_k P_k V P_k \right\| = \max_k \|P_k V P_k\| \leq \|V\|. \tag{20}$$

Therefore, by plugging (19) and (20) into (15), we finally get

$$\delta_{\rm Z}(t) = \|\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} - \mathrm{e}^{\mathrm{i}t(H+\varepsilon V_{\rm Z})}\| \leq \frac{2\sqrt{d}}{\eta} \varepsilon \|V\| (1 + \varepsilon \|V\| t), \tag{21}$$

which for $\|V\| = 1$ reduces to Eq. (3) of the Letter.

Robust symmetries

Consider now a robust symmetry

$$M = \sum_k m_k P_k, \tag{22}$$

with $m_k \in \mathbb{R}$. This is a conserved observable, $M = M^\dagger$, $[M, H] = 0$, that acts uniformly within each eigenspace of H. We have $M_t = \mathrm{e}^{\mathrm{i}tH} M \mathrm{e}^{-\mathrm{i}tH} = M$, and for every perturbation εV,

$$\begin{aligned} \|M_t^\varepsilon - M\| &= \|\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} M \mathrm{e}^{-\mathrm{i}t(H+\varepsilon V)} - M\| \\ &= \|\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} M - M \mathrm{e}^{\mathrm{i}t(H+\varepsilon V)}\| \\ &= \left\| \left(\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} - \mathrm{e}^{\mathrm{i}t(H+\varepsilon V_{\rm Z})} \right) M + \mathrm{e}^{\mathrm{i}t(H+\varepsilon V_{\rm Z})} M - M \mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} \right\|. \end{aligned} \tag{23}$$

By making use of the commutativity $[M, V_{\rm Z}] = 0$, one gets

$$\|M_t^\varepsilon - M\| = \left\| \left(\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} - \mathrm{e}^{\mathrm{i}t(H+\varepsilon V_{\rm Z})} \right) M - M \left(\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} - \mathrm{e}^{\mathrm{i}t(H+\varepsilon V_{\rm Z})} \right) \right\| \leq 2\|M\| \|\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} - \mathrm{e}^{\mathrm{i}t(H+\varepsilon V_{\rm Z})}\|, \tag{24}$$

that is

$$\|M_t^\varepsilon - M\| \leq 2\|M\| \delta_{\rm Z}(t), \tag{25}$$

which is the inequality (4) of the Letter.

Analogously, by substituting in the previous derivation $V_{\rm Z}$ with $V_H(\varepsilon)$, which still commutes with the robust conserved observable M, i.e. $[M, V_H(\varepsilon)] = 0$, one has the bound

$$\|M_t^\varepsilon - M\| = \left\| \left(\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} - \mathrm{e}^{\mathrm{i}t[H+\varepsilon V_H(\varepsilon)]} \right) M - M \left(\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} - \mathrm{e}^{\mathrm{i}t[H+\varepsilon V_H(\varepsilon)]} \right) \right\| \leq 2\|M\| \delta_\infty, \tag{26}$$

where

$$\delta_\infty = \sup_t \|\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} - \mathrm{e}^{\mathrm{i}t[H+\varepsilon V_H(\varepsilon)]}\| \tag{27}$$

is the uniform bound on the divergence of the two dynamics. This is the first inequality in Eq. (9) of the Letter.

The block-diagonal perturbation $V_H(\varepsilon)$ can be chosen such that $\delta_\infty = O(\varepsilon)$. The crucial ingredient is to choose a block-diagonal perturbation $H + \varepsilon V_H(\varepsilon)$ which is *isospectral* with $H + \varepsilon V$, and thus is unitarily equivalent to it:

$$H + \varepsilon V_H(\varepsilon) = W_\varepsilon^\dagger (H + \varepsilon V) W_\varepsilon, \tag{28}$$

with a unitary $W_\varepsilon = 1 + O(\varepsilon)$. Such a block-diagonal $V_H(\varepsilon)$ and a unitary W_ε actually exist [3–5]. By plugging (28) into (27), we get

$$\begin{aligned} \delta_\infty &= \sup_t \|\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} - W_\varepsilon^\dagger \mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} W_\varepsilon\| \\ &= \sup_t \|(1 - W_\varepsilon^\dagger) \mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} + W_\varepsilon^\dagger \mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} (1 - W_\varepsilon)\| \\ &\leq \sup_t \|1 - W_\varepsilon^\dagger\| \|\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)}\| + \|W_\varepsilon^\dagger \mathrm{e}^{\mathrm{i}t(H+\varepsilon V)}\| \|1 - W_\varepsilon\| \\ &= \|1 - W_\varepsilon^\dagger\| + \|1 - W_\varepsilon\| \\ &= O(\varepsilon). \end{aligned} \tag{29}$$

The existence, and the explicit construction, of a unitary W_ε, that carries the perturbed Hamiltonian into a block-diagonal form, is proved and discussed in detail in Ref. [5]. Here, in the next subsections, we will show the *necessity* of an isospectral perturbation, and then discuss its construction and prove that

$$V_H(\varepsilon) = V_{\mathrm{Z}} + O(\varepsilon), \tag{30}$$

by exploiting the connection with quantum KAM theory.

Isospectral perturbations

Consider a Hamiltonian $H = H^\dagger$ and a perturbation $\tilde{H} = H + O(\varepsilon)$, with small ε. We want to compare the two dynamics by looking at their divergence:

$$\delta_{H,\tilde{H}}(t) = \|\mathrm{e}^{\mathrm{i}t\tilde{H}} - \mathrm{e}^{\mathrm{i}tH}\|. \tag{31}$$

Consider the spectral decompositions

$$H = \sum_{k=1}^{d} h_k P_k, \qquad \tilde{H} = \sum_{k=1}^{d} \tilde{h}_k \tilde{P}_k, \tag{32}$$

where d is the number of distinct eigenvalues of $\tilde{H}$, i.e. $\tilde{h}_k \neq \tilde{h}_\ell$ for $k \neq \ell$. It may happen that $h_k = h_\ell$ for some $k \neq \ell$, if the degeneracy is lifted by the perturbation. However in such a case we choose the orthogonal projections P_k and P_ℓ such that they are adapted to the perturbation, that is $\tilde{P}_k = P_k + O(\varepsilon)$ and $\tilde{P}_\ell = P_\ell + O(\varepsilon)$ [6]. As for the eigenvalues, $\tilde{h}_k = h_k + O(\varepsilon)$.

We get

$$\mathrm{e}^{\mathrm{i}t\tilde{H}} - \mathrm{e}^{\mathrm{i}tH} = \sum_k \left(\mathrm{e}^{\mathrm{i}t\tilde{h}_k}\tilde{P}_k - \mathrm{e}^{\mathrm{i}th_k}P_k\right) = \sum_k \mathrm{e}^{\mathrm{i}t\tilde{h}_k}(\tilde{P}_k - P_k) - \sum_k (\mathrm{e}^{\mathrm{i}t\tilde{h}_k} - \mathrm{e}^{\mathrm{i}th_k})P_k. \tag{33}$$

The first sum on the right-hand side is $O(\varepsilon)$ uniformly in time, as

$$\left\|\sum_k \mathrm{e}^{\mathrm{i}t\tilde{h}_k}(\tilde{P}_k - P_k)\right\| \leq \sum_k \|\tilde{P}_k - P_k\| = O(\varepsilon). \tag{34}$$

On the other hand, the last term reads

$$\left\|\sum_k (\mathrm{e}^{\mathrm{i}t\tilde{h}_k} - \mathrm{e}^{\mathrm{i}th_k})P_k\right\| = \max_k |\mathrm{e}^{\mathrm{i}t\tilde{h}_k} - \mathrm{e}^{\mathrm{i}th_k}| = 2\max_k \left|\sin\left(t\frac{\tilde{h}_k - h_k}{2}\right)\right|, \tag{35}$$

so that

$$\delta_{H,\tilde{H}}(t) = 2\max_k \left|\sin\left(t\frac{\tilde{h}_k - h_k}{2}\right)\right| + O(\varepsilon). \tag{36}$$

Therefore, since $\tilde{h}_k - h_k = O(\varepsilon)$, we get

$$\delta_{H,\tilde{H}}(t) = O(\varepsilon), \qquad \text{for} \quad t = O(1). \tag{37}$$

However, the divergence has a slow drift (secular term) and becomes $O(1)$ for sufficiently large times $O(1/\varepsilon)$. Indeed,

$$\delta_{H,\tilde{H}}(t) = 2 + O(\varepsilon), \qquad \text{for} \quad t = \frac{\pi}{\tilde{h}_k - h_k} = O(1/\varepsilon), \tag{38}$$

that is, the maximal divergence

$$\delta_\infty = \sup_t \delta_{H,\tilde{H}}(t) = 2 + O(\varepsilon). \tag{39}$$

Geometrically, the evolution of a Hamiltonian with d distinct eigenvalues yields a (quasi-)periodic motion of a point on a torus. Two motions with different frequencies, however small the differences may be, will eventually accumulate a divergence of $O(1)$. The only way to avoid this slow drift is that the two motions be isochronous, that is the first term in (36) should be identically zero. This means that

$$\delta_\infty = O(\varepsilon) \qquad \text{iff} \qquad \tilde{h}_k = h_k, \qquad \text{for all } k, \tag{40}$$

i.e., the Hamiltonian H and its perturbation $\tilde{H}$ must be isospectral.

Quantum KAM iteration. Homological equation

We are looking for a unitary transformation W_ε close to the identity, such that the transformed total Hamiltonian is isospectral to $H+\varepsilon V$,

$$H+\varepsilon V_H(\varepsilon) = W_\varepsilon^\dagger (H+\varepsilon V) W_\varepsilon, \tag{41}$$

with the constraint that $V_H(\varepsilon)$ be block-diagonal,

$$V_H = \langle V_H \rangle := \sum_k P_k V_H P_k. \tag{42}$$

By writing

$$W_\varepsilon = \mathrm{e}^{\mathrm{i}K(\varepsilon)}, \qquad K(\varepsilon) = \varepsilon K_1 + O(\varepsilon^2), \tag{43}$$

with $K_1 = K_1^\dagger$, and

$$V_H(\varepsilon) = V_0 + O(\varepsilon), \tag{44}$$

with $V_0 = V_0^\dagger$, Eq. (41) reads

$$H+\varepsilon V_H(\varepsilon) = (1-\mathrm{i}\varepsilon K_1)(H+\varepsilon V)(1+\mathrm{i}\varepsilon K_1) + O(\varepsilon^2), \tag{45}$$

whence

$$V_0 = \mathrm{i}[H, K_1] + V. \tag{46}$$

Notice that

$$\langle [H, K_1] \rangle = \sum_k P_k (H K_1 - K_1 H) P_k = \sum_k P_k (h_k K_1 - K_1 h_k) P_k = 0. \tag{47}$$

Therefore, the constraint (42), which implies $V_0 = \langle V_0 \rangle$, gives

$$V_0 = \langle V \rangle = \sum_k P_k V P_k = V_{\mathrm{Z}}, \tag{48}$$

and

$$\mathrm{i}[H, K_1] = -\{V\}, \tag{49}$$

where

$$\{V\} := V - \langle V \rangle = \sum_{k,\ell\,:\,k\neq\ell} P_k V P_\ell = \sum_k P_k V (1-P_k) = \frac{1}{2}\sum_k [P_k, [P_k, V]] \tag{50}$$

is the off-diagonal part of V.

The expression (49) should be understood as an equation for K_1, the first-order term of the generator $K(\varepsilon)$ of the unitary W_ε. It is known as the *homological* equation and is the fundamental block of quantum KAM theory [7–11]. It is the quantum analog of the homological equation of KAM theory in classical mechanics, where the commutator is replaced by ($-\mathrm{i}$ times) the Poisson bracket, while $\langle\,\cdot\,\rangle$ and $\{\,\cdot\,\}$ are replaced by the averaged and the oscillating part of the perturbation, respectively [12, 13].

One can prove that the homological equation (49) has a unique solution with $\langle K_1 \rangle = 0$, for every H and V. Indeed, by sandwiching (49) between P_k and P_ℓ with $k \neq \ell$ we get

$$(h_k - h_\ell) P_k K_1 P_\ell = \mathrm{i} P_k V P_\ell, \tag{51}$$

that is

$$\{K_1\} = \mathrm{i} \sum_{k,\ell\,:\,k\neq\ell} \frac{P_k V P_\ell}{h_k - h_\ell} = \mathrm{i} \sum_\ell S_\ell V P_\ell. \tag{52}$$

Notice that $\{K_1\} = \{K_1\}^\dagger$, as it should be, and in fact one has

$$\{K_1\} = \mathrm{i}\sum_\ell S_\ell V P_\ell = -\mathrm{i}\sum_\ell P_\ell V S_\ell = \frac{\mathrm{i}}{2}\sum_\ell (S_\ell V P_\ell - P_\ell V S_\ell). \tag{53}$$

Moreover, notice that we have a complete freedom in the choice of the block-diagonal part $\langle K_1 \rangle$ of K_1, since it commutes with H and thus is immaterial in equation (49), so that

$$K_1 = \mathrm{i}\sum_\ell S_\ell V P_\ell + \sum_\ell P_\ell Z P_\ell, \tag{54}$$

with an arbitrary $Z = Z^\dagger$. In the following, for simplicity, we will *fix the gauge* $Z = 0$, i.e. $\langle K_1 \rangle = 0$, and thus will make the solution of (49) unique.

From the explicit expression of the generator K_1, we can now easily evaluate a uniform bound on the divergence (27). From the inequality (29), we get

$$\delta_\infty = \sup_t \|\mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} - \mathrm{e}^{\mathrm{i}t[H+\varepsilon V_H(\varepsilon)]}\| \le 2\|1 - W_\varepsilon\| \le 2\varepsilon\|K_1\| + O(\varepsilon^2) \le \frac{2\sqrt{d}}{\eta}\varepsilon\|V\| + O(\varepsilon^2), \tag{55}$$

where the last inequality is a consequence of the bound (19), since $K_1 = \mathrm{i}X$.

In fact, an explicit bound on the divergence δ_∞ is obtained in Ref. [5, Appendix E] as

$$\delta_\infty \le \hat{\delta}_\infty, \qquad \text{where} \qquad \hat{\delta}_\infty = 2\sqrt{d}\left(\frac{1}{\sqrt[4]{1 - 4\varepsilon/\eta}} - 1\right) = \frac{2\sqrt{d}}{\eta}\varepsilon + O(\varepsilon^2), \tag{56}$$

for $\|V\| = 1$, which is easily seen to be always larger than the first order term in (55), $\hat{\delta}_\infty \ge 2\sqrt{d}\,\varepsilon/\eta$.

This bound becomes trivial once it exceeds $\delta_\infty = 2$ as ε increases. Since $d \ge 2$, let us care only about the values of ε where $2\sqrt{2}\,(1/\sqrt[4]{1 - 4\varepsilon/\eta} - 1) \le 2$, namely, for $4\varepsilon/\eta \le (13 + 12\sqrt{2})/(17 + 12\sqrt{2}) = x_0$. Within this range, one gets the linear bound $2(1/\sqrt[4]{1 - 4\varepsilon/\eta} - 1) \le (\sqrt{2}/x_0)4\varepsilon/\eta < 7\varepsilon/\eta$. Therefore, we have

$$\delta_\infty < \frac{7\sqrt{d}}{\eta}\varepsilon. \tag{57}$$

This yields Eq. (7) of the Letter.

Higher-order terms

One can also show that all the following steps of the KAM iteration, giving higher-order terms V_n in $V_H(\varepsilon)$ and K_{n+1} in $K(\varepsilon)$, with $n \ge 1$, have the same structure as the first step and involve homological equations. For example, by considering the next-order terms,

$$K(\varepsilon) = \varepsilon K_1 + \varepsilon^2 K_2 + O(\varepsilon^3), \qquad V_H(\varepsilon) = V_0 + \varepsilon V_1 + O(\varepsilon^2), \tag{58}$$

one gets

$$H + \varepsilon V_0 + \varepsilon^2 V_1 = (H + \varepsilon V) + \mathrm{i}\varepsilon[H + \varepsilon V, K_1] - \frac{1}{2}\varepsilon^2[[H, K_1], K_1] + \mathrm{i}\varepsilon^2[H, K_2] + O(\varepsilon^3). \tag{59}$$

The second-order terms give

$$V_1 = \mathrm{i}[H, K_2] - \frac{1}{2}[[H, K_1], K_1] + \mathrm{i}[V, K_1], \tag{60}$$

that is

$$V_1 = \mathrm{i}[H, K_2] + \mathrm{i}\left[V - \frac{1}{2}\{V\}, K_1\right]. \tag{61}$$

This has the same structure as (46), and gives

$$V_1 = \langle V_1 \rangle = \left\langle \mathrm{i} \left[V - \frac{1}{2}\{V\}, K_1 \right] \right\rangle = -\sum_{\ell} P_\ell V S_\ell V P_\ell, \tag{62}$$

and a homological equation for K_2:

$$\mathrm{i}[H, K_2] = -\left\{ \mathrm{i} \left[V - \frac{1}{2}\{V\}, K_1 \right] \right\}. \tag{63}$$

In general, at order ε^{n+1} one gets an equation of the form

$$V_n = \mathrm{i}[H, K_{n+1}] + P_n(\mathcal{K}_1, \ldots, \mathcal{K}_n)(H) + Q_n(\mathcal{K}_1, \ldots, \mathcal{K}_n)(V), \tag{64}$$

where P_n and Q_n are polynomials of order n and $\mathcal{K}_j$ are the superoperators $\mathcal{K}_j(Y) = \mathrm{i}[Y, K_j]$. This has the same structure as (46) or (60). V_n will be given by the block-diagonal part of the right-hand side, while K_{n+1} will be the solution of the homological equation given by the off-diagonal part.

This is the algebraic structure of the KAM iteration scheme. And for our purposes this is enough. See for example [14, 15]. However, most difficulties and the hardest part of this scheme arises for infinite-dimensional systems with a vanishing minimal spectral gap η because of an accumulation point of the discrete spectrum. Interesting cases are systems with dense point spectrum [7–11]. In such a situation, at each iteration step, the solution of the homological equation (52) suffers from the plague of *small denominators*, the same problem that besets celestial mechanics. The reduced resolvent S_ℓ becomes unbounded, and the formal expression (52) is a bounded operator only for a particular class of perturbations V which are adapted to the Hamiltonian H: the closer are the eigenvalues h_k and h_ℓ of H at the denominator of (52), the smaller must be the numerator $P_k V P_\ell$. In such a case, the proof of the existence and the convergence of the series makes use of classical techniques of KAM perturbation theory with a careful control of small denominators through a Diophantine condition, and a super-convergent iteration scheme [12, 13].

Algebraic framework

In this section we want to set our analysis in an algebraic framework and gather our definitions and results in a more mathematical language.

Let $\mathcal{H} \simeq \mathbb{C}^n$ be an n-dimensional Hilbert space. The C*-algebra of observables is $B(\mathcal{H}) \simeq M_n(\mathbb{C}) \simeq \mathbb{C}^{n^2}$, where $M_n(\mathbb{C})$ is the set of $n \times n$ complex matrices. Consider a Hamiltonian $H = H^\dagger \in B(\mathcal{H})$. Its spectral decomposition reads

$$H = \sum_{k=1}^{d} e_k P_k, \tag{65}$$

where $\{e_k \,:\, k = 1, \ldots, d\} \subset \mathbb{R}$ is its spectrum consisting of $d \le n$ distinct eigenvalues, and $P_k = P_k^2 = P_k^\dagger$, with $P_k P_\ell = \delta_{k\ell} P_k$ and $\sum_{k=1}^d P_k = 1$, are its spectral projections of rank $d_k = \operatorname{tr} P_k$.

In our work two algebras associated with the Hamiltonian H play a fundamental role: the *commutant* $\{H\}'$ and the *bicommutant* $\{H\}''$ of H. Let us recall their definitions, together with some properties.

The commutant is the set of observables commuting with H, namely,

$$\begin{aligned} \{H\}' &= \{A \in B(\mathcal{H}) \,:\, [A, H] = 0\} \\ &= \left\{ A \in B(\mathcal{H}) \,:\, A = \sum_{k=1}^{d} P_k A P_k \right\}, \end{aligned} \tag{66}$$

that is the set of observables with the same block-diagonal structure of H. One has $\{H\}' \simeq \mathbb{C}^{\sum_{k=1}^d d_k^2}$, and obviously $H \in \{H\}'$.

The bicommutant is the set of observables commuting with all the elements of the commutant $\{H\}'$, namely,

$$\begin{aligned}\{H\}'' &= \{A \in B(\mathcal{H}) \,:\, [A,B]=0, \text{ for all } B \in \{H\}'\} \\ &= \{A \in B(\mathcal{H}) \,:\, A = f(H), \text{ for all } f:\mathbb{R}\to\mathbb{C}\} \\ &= \{A \in B(\mathcal{H}) \,:\, A = p_{d-1}(H), \text{ for all } p_{d-1} = \text{polynomial of degree} \le d-1\} \\ &= \left\{A \in B(\mathcal{H}) \,:\, A = \sum_{k=1}^{d} a_k P_k \right\} \simeq \mathbb{C}^d. \end{aligned} \tag{67}$$

The second equality (von Neumann theorem) identifies the bicommutant of H with the (Abelian) algebra generated by H, that is the set of all functions of H, which coincides with the set of polynomials of degree $\le d-1$, since the spectrum of H has d points [or H satisfies $\prod_{k=1}^{d}(H-e_k)=0$] (third equality). The last equality gives the explicit structure of the elements of the bicommutant as the observables which act as constants in each eigenspace of H.

We get

$$\{H\}'' \subseteq \{H\}' \subseteq B(\mathcal{H}), \tag{68}$$

the first inclusion being an equality iff $d=n$, that is if the spectrum of H is simple; the second inclusion being an equality if $d=1$, that is if the Hamiltonian is fully degenerate (proportional to the identity). Both $\{H\}''$ and $\{H\}'$ are C*-algebras.

Now let us go back to our object of analysis by starting with the definition of a symmetry.

Definition 1. An observable $A \in B(\mathcal{H})$ is a *symmetry* of H or a *conserved quantity* if

$$A_t = \mathrm{e}^{\mathrm{i}tH} A \mathrm{e}^{-\mathrm{i}tH} = A, \qquad \text{for all } t \in \mathbb{R}.$$

It is easy to prove the characterization of the symmetries as the elements of the commutant:

Theorem 1. *A is a symmetry of H iff $A \in \{H\}'$.*

We are seeking for a finer classification of symmetries in terms of their robustness with respect to a perturbation of H. By definition we have that if A is a symmetry, the distance $\|A_t - A\| = 0$ for all times t. What happens if we instead consider a perturbed evolution $A_t^{\varepsilon} = \mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} A \mathrm{e}^{-\mathrm{i}t(H+\varepsilon V)}$? Does it remain close to the unperturbed one?

For small times and small ε we get that the distance $\|A_t^{\varepsilon} - A_t\| = \|A_t^{\varepsilon} - A\|$ is of order ε. We will consider a symmetry robust under perturbations if the distance remains small for all times t for every small perturbation, that is if the observable is an approximate symmetry of the perturbed dynamics. Notice that this is the case for $A = cI$, so the question is whether there exist nontrivial robust symmetries.

At the other extreme there might be symmetries such that, however small a particular perturbation may be, the distance $\|A_t^{\varepsilon} - A\|$ accumulates over large times and eventually becomes $O(1)$. We will consider such a symmetry fragile. Notice that by the triangle inequality and the unitarity of evolution,

$$\|A_t^{\varepsilon} - A\| \le \|A_t^{\varepsilon}\| + \|A\| = 2\|A\|, \tag{69}$$

so the maximum divergence is $2\|A\|$ and is obtained if $A_t^{\varepsilon} = \mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} A \mathrm{e}^{-\mathrm{i}t(H+\varepsilon V)} = -A$. In the following definition we will consider a fragile symmetry whose divergence gets to reach its upper bound $2\|A\|$ over time, that is the worst situation.

Definition 2. A symmetry $A \in \{H\}'$ is *fragile* if there exists a perturbation $V = V^{\dagger} \in B(\mathcal{H})$ such that

$$\sup_{t\in\mathbb{R}} \left\| \mathrm{e}^{\mathrm{i}t(H+\varepsilon V)} A \mathrm{e}^{-\mathrm{i}t(H+\varepsilon V)} - A \right\| = 2\|A\|, \qquad \text{for all } \varepsilon > 0. \tag{70}$$

A symmetry which is not fragile is called *robust*.

We gather our main results in the following theorem.

Theorem 2. *Let $A \in \{H\}'$ be a symmetry of H. Then the following assertions are equivalent:*

1. *A is robust;*
2. *$A \in \{H\}''$;*

3. There exists a constant $c = c_H > 0$ such that for all $V = V^\dagger \in B(\mathcal{H})$ and for all $t \in \mathbb{R}$ one has

$$\left\|\mathrm{e}^{\mathrm{i}t(H+V)} A \mathrm{e}^{-\mathrm{i}t(H+V)} - A\right\| \le c\|V\|\|A\|. \tag{71}$$

Item 2 gives an algebraic characterization of the set of robust symmetries as the bicommutant of H, and thus a characterization of fragile symmetries as elements of the complement $\{H\}' \setminus \{H\}''$. Item 3 is of dynamical nature, and characterizes robust symmetries as those symmetries whose divergence remains forever small for small perturbations. Moreover, the divergence is linearly controlled by the perturbation strength $\|V\|$ times a universal constant which depends only on H. We have shown above that $c \le 2\delta_\infty < 14\sqrt{d}/\eta$. Its bound depends on the spectral gap η and the number d of eigenvalues of H.

A striking consequence of the above theorem is that in fact only the two possible extreme scenarios discussed above are possible: either a symmetry is an approximate symmetry for all perturbations (robust symmetry), or it maximally diverges over time from its unperturbed value for a particular perturbation however small the latter may be (fragile symmetry).

The proof of Theorem 2 goes as follows in the Letter. We first prove that if $M \in \{H\}' \setminus \{H\}''$ then we get the saturation (70) for some perturbation V, and thus M is fragile according to Definition 2; this is done by reducing the analysis to the two-dimensional case. Then we prove, by isospectral deformations, that if the symmetry is instead in the complementary set $M \in \{H\}''$, then it satisfies (71) for all perturbations V (see the above analysis). Therefore, M cannot saturate the divergence (70) for any V and thus it is robust according to Definition 2.

Robustness of monotones

In Ref. [16], it is shown that for a symmetry $\mathcal{M}$ of a Lindbladian $\mathcal{L}$ satisfying $[\mathcal{M}, \mathcal{L}] = 0$ one can define a monotone

$$f_{\mathcal{M}}(\rho) = \mathrm{tr}[\mathcal{M}(\rho)^\dagger (\mathbf{L}_\rho + \lambda \mathbf{R}_\rho)^{-1}(\mathcal{M}(\rho))], \tag{72}$$

which decreases under the evolution $\rho_t = \mathrm{e}^{t\mathcal{L}}\rho$,

$$f_{\mathcal{M}}(\rho_t) \le f_{\mathcal{M}}(\rho), \quad \text{for all } t \ge 0, \tag{73}$$

where $\mathbf{L}_\rho(X) = \rho X$ and $\mathbf{R}_\rho(X) = X\rho$ are the superoperators of left and right multiplication by ρ, respectively, and the inverse with $\lambda \ge 0$ is well defined for strictly positive ρ. Here, we prove that a monotone defined with respect to a symmetry of the form

$$\mathcal{M} = \sum_k m_k \mathcal{P}_k, \tag{74}$$

where $\{\mathcal{P}_k\}$ are the spectral projections of the Lindbladian $\mathcal{L}$, remains a monotone up to an error $O(\varepsilon)$ eternally even in the presence of a perturbation $\varepsilon\mathcal{V}$, namely,

$$f_{\mathcal{M}}(\rho_t^\varepsilon) \le f_{\mathcal{M}}(\rho) + O(\varepsilon), \quad \text{for all } t \ge 0, \tag{75}$$

where $\rho_t^\varepsilon = \mathrm{e}^{t(\mathcal{L}+\varepsilon\mathcal{V})}\rho$. In this sense, $\mathcal{M}$ in (74) is a robust symmetry of the evolution $\mathcal{L}$.

To show this, we first note that even in the case of open-system evolution one can find a block-diagonal approximation $\mathcal{V}_{\mathcal{L}}(\varepsilon)$ of the perturbation $\mathcal{V}$ such that $\mathcal{L} + \varepsilon\mathcal{V}_{\mathcal{L}}(\varepsilon)$ is similar to $\mathcal{L} + \varepsilon\mathcal{V}$ [5],

$$\mathcal{L} + \varepsilon\mathcal{V}_{\mathcal{L}}(\varepsilon) = \mathcal{W}_\varepsilon^{-1}(\mathcal{L} + \varepsilon\mathcal{V})\mathcal{W}_\varepsilon. \tag{76}$$

Then, let us consider

$$\tilde{\mathcal{M}} = \mathcal{W}_\varepsilon \mathcal{M} \mathcal{W}_\varepsilon^{-1}. \tag{77}$$

This is a symmetry of the perturbed system $\mathcal{L} + \varepsilon\mathcal{V}$, corresponding to the symmetry $\mathcal{M}$ of the unperturbed system $\mathcal{L}$, since $[\mathcal{M}, \mathcal{V}_{\mathcal{L}}(\varepsilon)] = 0$. Since this similarity transformation is small, $\mathcal{W}_\varepsilon = 1 + O(\varepsilon)$, we have

$$\tilde{\mathcal{M}} = \mathcal{M} + O(\varepsilon). \tag{78}$$

Notice that the monotone $f_{\tilde{\mathcal{M}}}(\rho)$ defined with respect to $\tilde{\mathcal{M}}$ is decreasing under the perturbed evolution $\rho_t^\varepsilon = \mathrm{e}^{t(\mathcal{L}+\varepsilon\mathcal{V})}\rho$. Therefore,

$$f_{\mathcal{M}}(\rho_t^\varepsilon) = f_{\tilde{\mathcal{M}}}(\rho_t^\varepsilon) + O(\varepsilon) \le f_{\tilde{\mathcal{M}}}(\rho) + O(\varepsilon) = f_{\mathcal{M}}(\rho) + O(\varepsilon), \quad \text{for all } t \ge 0. \tag{79}$$

This proves the approximate monotonicity (75).

[1] D. Burgarth, P. Facchi, H. Nakazato, S. Pascazio, and K. Yuasa, Generalized Adiabatic Theorem and Strong-Coupling Limits, Quantum **3**, 152 (2019).
[2] T. Kato, On the Adiabatic Theorem of Quantum Mechanics, J. Phys. Soc. Jpn. **5**, 435 (1950).
[3] J. des Cloizeaux, Extension d'une Formule de Lagrange à des Problèmes de Valeurs Propres, Nucl. Phys. **20**, 321 (1960).
[4] D. J. Klein, Degenerate Perturbation Theory, J. Chem. Phys. **61**, 786 (1974).
[5] D. Burgarth, P. Facchi, H. Nakazato, S. Pascazio, and K. Yuasa, Eternal Adiabaticity in Quantum Evolution, Phys. Rev. A **103**, 032214 (2021).
[6] T. Kato, *Perturbation Theory for Linear Operators*, 2nd ed. (Springer, Berlin, 1976).
[7] E. I. Dinaburg and Y. G. Sinai, The One-Dimensional Schrödinger Equation with a Quasiperiodic Potential, Funct. Anal. Its Appl. **9**, 279 (1975).
[8] H. Rüssmann, On the One-Dimensional Schrödinger Equation with a Quasi-Periodic Potential, Ann. NY Acad. Sci. **357**, 90 (1980).
[9] W. Craig, Pure Point Spectrum for Discrete Almost Periodic Schrödinger Operators, Commun. Math. Phys. **88**, 113 (1983).
[10] J. Pöschel, Examples of Discrete Schrödinger Operators with Pure Point Spectrum, Commun. Math. Phys. **88**, 447 (1983).
[11] J. Bellissard, R. Lima, and E. Scoppola, Localization in ν-Dimensional Incommensurate Structures, Commun. Math. Phys. **88**, 465 (1983).
[12] W. Thirring, *Classical Mathematical Physics: Dynamical Systems and Field Theories*, 3rd ed. (Springer, New York, 1997).
[13] V. I. Arnold, V. V. Kozlov, and A. I. Neishtadt, *Mathematical Aspects of Classical and Celestial Mechanics*, 3rd ed. (Springer, Berlin, 2006).
[14] W. Scherer, Superconvergent Perturbation Method in Quantum Mechanics, Phys. Rev. Lett. **74**, 1495 (1995).
[15] W. Scherer, Quantum Averaging II: Kolmogorov's Algorithm, J. Phys. A: Math. Gen. **30**, 2825 (1997).
[16] G. Styliaris and P. Zanardi, Symmetries and Monotones in Markovian Quantum Dynamics, Quantum **4**, 261 (2020).