Constraints on the opacity profile of the sun from helioseismic and solar neutrino data

F. L. Villante – Università dell’Aquila and LNGS-INFN

Work done in collaboration with B. Ricci
Outline

- Solar abundances: the solar composition problem and the SSM
- Linear Solar Models: a tool to investigate the solar interior
- Application: what we know about opacity (and metals) in the sun
- Summary and conclusions
The solar composition problem

The latest solar photospheric abundances leads to SSMs which do not correctly reproduce helioseismic observables

\textit{squared isothermal sound speed}

\[ u = \frac{P}{\rho} \]

Note that: \[ c^2 = \gamma u \]

\[ \begin{align*}
    c^2 &= \left. \frac{\partial P}{\partial \rho} \right|_{\text{ad}} \\
    \gamma &= \left. \frac{\partial \ln P}{\partial \ln \rho} \right|_{\text{ad}}
\end{align*} \]

\begin{tabular}{|l|c|c|c|}
\hline
 & FRANEC\(^(*)\)-GS98 & FRANEC-AGS05 & FRANEC\(^(*)\)-AGSS09 \\
\hline
\textbf{Photospheric abund.} & & & \\
\((Z/X)_b\) & 0.0231 & 0.0165 & 0.0181 \\
\(Y_b\) & 0.245 & 0.229 & 0.232 \\
\textbf{Convective Zone} & & & \\
\(R_b/R_o\) & 0.716 & 0.730 & 0.725 \\
\hline
\end{tabular}

\begin{tabular}{|c|}
\hline
\textbf{Helioseismic Values} & \\
\hline
0.2485 \pm 0.0034 & \\
0.715 \pm 0.001 & \\
\hline
\end{tabular}

\(^(*)\) Estimated by LSM approach. See Later
Standard Solar Models

Stellar structure equations are solved, starting from a ZAMS model to present solar age (we neglect rotation, magnetic fields, etc.):

\[
\begin{align*}
\frac{\partial m}{\partial r} &= 4\pi r^2 \rho \\
\frac{\partial P}{\partial r} &= -\frac{G_N m}{r^2} \rho \\
P &= P(\rho, T, X_i) \\
\frac{\partial l}{\partial r} &= 4\pi r^2 \rho \epsilon(\rho, T, X_i) \\
\frac{\partial T}{\partial r} &= -\frac{G_N m T \rho}{r^2 P} \nabla \\
\nabla &= \text{Min}(\nabla_{\text{rad}}, \nabla_{\text{ad}}) \\
\nabla_{\text{rad}} &= \frac{3}{16\pi ac G_N} \frac{\kappa(\rho, T, X_i) l P}{m T^4} \\
\nabla_{\text{ad}} &= (d \ln T / d \ln P)_s \simeq 0.4
\end{align*}
\]

Chemical evolution driven by nuclear reaction, diffusion and gravitational settling, convection

Standard input physics for equation of states, nuclear reaction rates, opacity, etc.

Free-parameters (mixing length, $Y_{\text{ini}}$, $Z_{\text{ini}}$) adjusted to match the observed properties of the Sun (radius, luminosity, $Z/X$).

Note that equations are non-linear $\rightarrow$ Iterative method to determine mixing length, $Y_{\text{ini}}$, $Z_{\text{ini}}$. 
Metals in the Sun

- $Z_{\text{CNO}}$ control the efficiency of CNO cycle

- Metals give a substantial contribution to opacity:

  Energy producing region ($R < 0.3 \, R_\odot$)

  $$\kappa_Z \approx \frac{1}{2} \kappa_{\text{tot}}$$

  Fe gives the largest contribution.

  Outer radiative region ($0.3 < R < 0.73 \, R_\odot$)

  $$\kappa_Z \sim 0.8 \, \kappa_{\text{tot}}$$

  Relevant contributions from several diff. elements (O, Fe, Si, Ne,...)

To understand metals: What we know about opacity in the sun?
Linear Solar Models

Linear Solar Models: the basic idea

- **The starting point:**
  SSMs provide a good approximation of the real sun. Small modifications are likely to explain disagreement with helioseismology.
Linear Solar Models: the basic idea

- **The starting point:**
  SSMs provide a good approximation of the real sun. Small modifications are likely to explain disagreement with helioseismology.

- **The method:**
  We write:
  \[
  h(r) = \overline{h}(r)[1 + \delta h(r)] \\
  X_i(r) = \overline{X}_i(r)[1 + \delta X_i(r)] \\
  Y(r) = \overline{Y}(r) + \Delta Y(r)
  \]
  where \( \overline{h}(r) \), \( \overline{X}_i(r) \) are the SSMs predicted values, and we expand linearly in \( \begin{bmatrix} \delta h(r) \\ \delta X_i(r) \\ \Delta Y(r) \end{bmatrix} \)

  **Assumption:** the variation of the present solar composition (i.e. the \( \delta X_i(r) \), \( \Delta Y(r) \)) can be deduced with sufficient accuracy from the variation of the nuclear reaction efficiency and diffusion velocities in the present sun (i.e. the \( \delta h(r) \))
Linear Solar Models: the basic idea

• **The starting point:**
SSMs provide a good approximation of the real sun. Small modifications are likely to explain disagreement with helioseismology.

• **The method:**
We write:
\[
\begin{align*}
  h(r) & = \overline{h}(r)[1 + \delta h(r)] \\
  X_i(r) & = \overline{X}_i(r)[1 + \delta X_i(r)] \\
  Y(r) & = \overline{Y}(r) + \Delta Y(r)
\end{align*}
\]

where \(\overline{h}(r), \overline{X}_i(r)\) are the SSMs predicted values, and we expand linearly in \(\begin{bmatrix} \delta h(r) \\ \delta X_i(r) \\ \Delta Y(r) \end{bmatrix}\)

*Assumption:* the variation of the *present* solar composition (i.e. the \(\delta X_i(r), \Delta Y(r)\)) can be deduced with sufficient accuracy from the variation of the nuclear reaction efficiency and diffusion velocities in the *present* sun (i.e. the \(\delta h(r)\))

• **The result:**
A linear system of ordinary differential equations that can be used to study the response of the sun to an arbitrary modification input parameters.
Linear Solar Models – Final set of equations equations

\[
\frac{d\delta m}{dr} = \frac{1}{l_m} \left[ \gamma_P \delta P + \gamma_T \delta T - \delta m + \gamma_Y \Delta Y_{ini} + \gamma_\epsilon \delta \epsilon \right]
\]

\[
\frac{d\delta P}{dr} = \frac{1}{l_P} \left[ (\gamma_P - 1) \delta P + \gamma_T \delta T + \delta m + \gamma_Y \Delta Y_{ini} + \gamma_\epsilon \delta \epsilon \right]
\]

\[
\frac{d\delta l}{dr} = \frac{1}{l_l} \left[ \beta'_P \delta P + \beta'_T \delta T - \delta l + \beta'_Y \Delta Y_{ini} + \beta'_C \delta C + \beta'_\epsilon \delta \epsilon \right]
\]

\[
\frac{d\delta T}{dr} = \frac{1}{l_T} \left[ \alpha'_P \delta P + \alpha'_T \delta T + \delta l + \alpha'_Y \Delta Y_{ini} + \alpha'_C \delta C + \delta \kappa + \alpha'_\epsilon \delta \epsilon \right]
\]

To be solved between with the boundary conditions

**At the center of the sun (r = 0)**

\[
\delta m = \gamma_{P,0} \delta P_0 + \gamma_{T,0} \delta T_0 + \gamma_{Y,0} \Delta Y_{ini} + \gamma_{\epsilon,0} \delta \epsilon_0
\]

\[
\delta P = \delta P_0
\]

\[
\delta T = \delta T_0
\]

\[
\delta l = \beta'_{P,0} \delta P_0 + \beta'_{T,0} \delta T_0 + \beta'_{Y,0} \Delta Y_{ini} + \beta'_{C,0} \delta C + \beta'_{\epsilon,0} \delta \epsilon_0
\]

**Univocally determine the parameters**

\(\delta P_0, \delta T_0, \Delta Y_{ini}, \delta C\)

**At the convective boundary (r = \(R_b\))**

\[
\delta m = -m_{conv} \delta C'
\]

\[
\delta P = \delta C
\]

\[
\delta T = A'_Y \Delta Y_{ini} + A'_C \delta C
\]

\[
\delta l = 0
\]
Solid – Linear solar models
Dotted - SSMs
Linear Solar Models – Validation

\[ \delta u(r) = \delta P(r) - \delta \rho(r) \]

**Solid** - Linear solar models
**Dotted** - SSMs

**OPA1**
Const. Opa. rescaling

**OPA2**
Opa. rescaled by a smooth fnct.

**OPA3**
Opa. rescaled in an interval

**Spp**
\( \delta S_{pp} = 0.1 \)
### Linear Solar Models – Validation

#### Table 1:
Comparison between the predictions of LSM and "standard" non-linear SM for the initial and surface chemical abundances, the convective radius and the solar neutrino fluxes. Note that the absolute variations are reported for Helium, whereas the relative variations are shown for all the other quantities.

<table>
<thead>
<tr>
<th></th>
<th>OPA1 SM</th>
<th>OPA1 LSM</th>
<th>OPA2 SM</th>
<th>OPA2 LSM</th>
<th>OPA3 SM</th>
<th>OPA3 LSM</th>
<th>Spp SM</th>
<th>Spp LSM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta Y_b$</td>
<td>0.014</td>
<td>0.014</td>
<td>-0.0037</td>
<td>-0.0036</td>
<td>0.0038</td>
<td>0.0038</td>
<td>0.0031</td>
<td>0.0034</td>
</tr>
<tr>
<td>$\delta R_b$</td>
<td>-0.0020</td>
<td>-0.0020</td>
<td>-0.0067</td>
<td>-0.0070</td>
<td>-0.014</td>
<td>-0.015</td>
<td>-0.0058</td>
<td>-0.0064</td>
</tr>
<tr>
<td>$\delta \Phi_{pp}$</td>
<td>-0.011</td>
<td>-0.010</td>
<td>0.0045</td>
<td>0.0052</td>
<td>-0.0020</td>
<td>-0.0011</td>
<td>0.0090</td>
<td>0.0092</td>
</tr>
<tr>
<td>$\delta \Phi_{Be}$</td>
<td>0.13</td>
<td>0.13</td>
<td>-0.067</td>
<td>-0.064</td>
<td>0.017</td>
<td>0.016</td>
<td>-0.11</td>
<td>-0.11</td>
</tr>
<tr>
<td>$\delta \Phi_B$</td>
<td>0.27</td>
<td>0.27</td>
<td>-0.17</td>
<td>-0.17</td>
<td>0.029</td>
<td>0.028</td>
<td>-0.27</td>
<td>-0.28</td>
</tr>
<tr>
<td>$\delta \Phi_N$</td>
<td>0.14</td>
<td>0.14</td>
<td>-0.10</td>
<td>-0.094</td>
<td>0.003</td>
<td>0.004</td>
<td>-0.21</td>
<td>-0.22</td>
</tr>
<tr>
<td>$\delta \Phi_O$</td>
<td>0.21</td>
<td>0.22</td>
<td>-0.14</td>
<td>-0.14</td>
<td>0.012</td>
<td>0.012</td>
<td>-0.29</td>
<td>-0.31</td>
</tr>
</tbody>
</table>

Surface helium:

$$\Delta Y_b = A_Y \Delta Y_{ini} + A_C \delta C$$

Convective radius:

$$\delta R_b = \Gamma_Y \Delta Y_{ini} + \Gamma_C \delta C + \Gamma_\kappa \delta \kappa_b$$

Neutrino fluxes:

$$\delta \Phi_\nu = \int dr \left[ \phi_{\nu,\rho}(r) \delta \rho(r) + \phi_{\nu,T}(r) \delta T(r) + \phi_{\nu,Y}(r) \Delta Y(r) + \phi_{\nu,Z}(r) \delta Z(r) + \phi_{\nu,Spp}(r) \delta S_{pp} \right]$$

where:

$$\phi_{\nu,j}(r) = \frac{r^2 \bar{\rho}(r) \bar{n}_\nu(r) n_{\nu,j}(r)}{\int dr \ r^2 \bar{\rho}(r) \bar{n}_\nu(r)}$$
Opacity (and metals) in the sun

The relation between opacity and metals

\[
\frac{d\delta m}{dr} = \frac{1}{l_m} \left[ \gamma_P \delta P + \gamma_T \delta T - \delta m + \gamma_Y \Delta Y_{\text{ini}} \right]
\]

\[
\frac{d\delta P}{dr} = \frac{1}{l_P} \left[ (\gamma_P - 1) \delta P + \gamma_T \delta T + \delta m + \gamma_Y \Delta Y_{\text{ini}} \right]
\]

\[
\frac{d\delta l}{dr} = \frac{1}{l_l} \left[ \beta'_P \delta P + \beta'_T \delta T - \delta l + \beta'_Y \Delta Y_{\text{ini}} + \beta'_C \delta C \right]
\]

\[
\frac{d\delta T}{dr} = \frac{1}{l_T} \left[ \alpha'_P \delta P + \alpha'_T \delta T + \delta l + \alpha'_Y \Delta Y_{\text{ini}} + \alpha'_C \delta C + \delta \kappa \right]
\]

The source term that is responsible for the modification of the sun (and that can be bounded from obs. data) is:

\[
\delta \kappa(r) = \delta \kappa_1(r) + \delta \kappa_Z(r)
\]

**Intrinsic opacity change**

\[
\delta \kappa_1(r) = \frac{\kappa(\bar{p}(r), \bar{T}(r), \bar{Y}(r), \bar{Z}_i(r))}{\kappa(\bar{p}(r), \bar{T}(r), \bar{Y}(r), \bar{Z}_i(r))} - 1
\]

**Composition opacity change**

\[
\delta \kappa_Z(r) = \frac{\kappa(\bar{p}(r), \bar{T}(r), \bar{Y}(r), Z_i(r))}{\kappa(\bar{p}(r), \bar{T}(r), \bar{Y}(r), \bar{Z}_i(r))} - 1
\]
The relation between opacity and metals

\[
\delta \kappa_Z(r) = \frac{\kappa(\bar{\rho}(r), \bar{T}(r), \bar{Y}(r), Z_i(r))}{\kappa(\bar{\rho}(r), \bar{T}(r), \bar{Y}(r), \bar{Z}_i(r))} - 1 \approx \sum_i \frac{\partial \ln \kappa}{\partial \ln Z_i} \delta z_i
\]

\[
\delta z_i = \frac{(Z_{i,b}/X_b) - (\bar{Z}_{i,b}/\bar{X}_b)}{\bar{Z}_{i,b}/\bar{X}_b}
\]
The opacity kernels

We study the response of the sun to arbitrary opacity variations:

\[ \delta \kappa(r) = \delta \kappa_1(r) + \delta \kappa_Z(r) \]

If we consider a small variation of the opacity, the sun respond linearly. The variation of a generic quantity \( Q \) is then given by:

\[ \delta Q = \int dr \: K_Q(r) \delta \kappa(r) \]
The opacity kernels

We study the response of the sun to arbitrary opacity variations:

\[ \delta \kappa(r) = \delta \kappa_1(r) + \delta \kappa_2(r) \]

If we consider a small variation of the opacity, the sun respond linearly. The variation of a generic quantity \( Q \) is then given by:

\[ \delta Q = \int dr \, K_Q(r) \delta \kappa(r) \]

We calculate numerically the kernel \( K_Q(r) \)

by considering localised increase of opacity in LSM:

\[ \delta \kappa(r) = G(r - r_0) = \frac{1}{\sqrt{2\pi \delta r}} \exp \left[ -\frac{(r - r_0)^2}{2\delta r^2} \right] \]

\( (\delta r = 0.01 R_\odot) \)

The obtained results are adequate to describe all the situations in which opacity varies on scales larger the \( \delta r = 0.01 R_\odot \)

\[ \delta Q(r_0) = \int dr \, K_Q(r) \, G(r - r_0) \approx K_Q(r_0) \]
The sound speed kernels

\[ \delta u(r) = \int dr' \ K_u(r, r') \ \delta \kappa(r') \]
The sound speed kernels

\[ \delta u(r) = \int dr' K_u(r, r') \delta \kappa(r') \]

The kernels are not positive definite \( \Rightarrow \) compensating effects can occur ...

\[ \delta u_0(r) = \int dr' K_u(r, r') \approx 0 \]

The sound speed is \textit{insensitive to a global rescaling of opacity}
Useful parameterizations for $\delta \kappa(r)$

![Graph showing parameterizations]

Two zones:

$$\delta \kappa(r) = A_{\text{in}} \delta \kappa_{\text{in}}(r) + A_{\text{out}} \delta \kappa_{\text{out}}(r)$$

Linear tilt:

$$\delta \kappa(r) = A_0 \delta \kappa_0(r) + A_1 \delta \kappa_1(r) = A_0 + A_1 \left( r/R_b \right)$$
The sound speed

\[
\delta \kappa(r) = A_{\text{in}} \delta \kappa_{\text{in}}(r) + A_{\text{out}} \delta \kappa_{\text{out}}(r)
\]

\[
\delta \kappa(r) = A_0 \delta \kappa_0(r) + A_1 \delta \kappa_1(r)
\]
The sound speed

\[ \delta \kappa(r) = A_{\text{in}} \delta \kappa_{\text{in}}(r) + A_{\text{out}} \delta \kappa_{\text{out}}(r) \]

\[ \delta \kappa(r) = A_0 \delta \kappa_0(r) + A_1 \delta \kappa_1(r) \]

The sound speed provide a bound on the differential opacity increase \((A_{\text{out}} - A_{\text{in}})\) or on the tilt \((A_1)\):

\[ A_{\text{out}} - A_{\text{in}} \approx 0.15 \]

\[ A_1 \approx 0.2 \]

No relevant bound on the opacity scale \(A_0\) (or \(A_{\text{in}} + A_{\text{out}}\)).
The convective radius and the surface helium abundance

Convective radius:

\[ \delta R_b = \int dr \ K_R(r) \ \delta \kappa(r) \]

\[ \delta R_b = 0.12 \ A_{in} - 0.14 \ A_{out} \]

\[ \approx 0.13 \ (A_{in} - A_{out}) \]

\[ \delta R_b = -0.02 \ A_0 - 0.10 \ A_1 \]
The convective radius and the surface helium abundance

Convective radius:

\[ \delta R_b = \int dr \, K_R(r) \, \delta \kappa(r) \]
\[ \delta R_b = 0.12 \, A_{\text{in}} - 0.14 \, A_{\text{out}} \]
\[ \simeq 0.13 \, (A_{\text{in}} - A_{\text{out}}) \]
\[ \delta R_b = -0.02 \, A_0 - 0.10 \, A_1 \]

Surface helium:

\[ \Delta Y_b = \int dr \, K_Y(r) \, \delta \kappa(r) \]
\[ \Delta Y_b = 0.073 \, A_{\text{in}} + 0.069 \, A_{\text{out}} \]
\[ \simeq 0.07 \, (A_{\text{in}} + A_{\text{out}}) \]
\[ \Delta Y_b = 0.142 \, A_0 + 0.062 \, A_1 \]

To reproduce helioseismic results:

\[ A_{\text{in}} = 0.07 \pm 0.04 \quad A_{\text{out}} = 0.21 \pm 0.04 \]
A model independent relation for $\delta \kappa_b$

We have that:

$$\Delta Y_b = A_Y \Delta Y_{\text{ini}} + A_C \delta C$$

$$\delta R_b = \Gamma_Y \Delta Y_{\text{ini}} + \Gamma_C \delta C + \Gamma_\kappa \delta \kappa_b$$

Remember that: $\delta C = \delta P_b = \delta \rho_b$

We eliminate $\Delta Y_{\text{ini}}$ from equations and obtain:

$$\delta \kappa_b = C_Y \Delta Y_b + C_R \delta R_b + C_\rho \delta \rho_b$$

*Model independent:* no parametrization for $\delta \kappa(r)$ is needed

By using $\delta R_b = -0.0205 \pm 0.0015$, $\Delta Y_b = 0.0195 \pm 0.0034$ and $\delta \rho_b = 0.08$:

$$\delta \kappa_b \approx 0.24 \pm 0.03$$
Neutrino Fluxes

\[ \delta \Phi_\nu = \int dr \, K_\nu(r) \, \delta \kappa(r) \]

\[ \delta \Phi_\nu = A_{\text{in}} \, \delta \Phi_{\nu,\text{in}} + A_{\text{out}} \, \delta \Phi_{\nu,\text{out}} \]
\[ \approx (\xi \, A_{\text{in}} + A_{\text{out}}) \, \delta \Phi_{\nu,\text{in}} \quad (\xi \sim 2 \rightarrow 3) \]

\[ \delta \Phi_\nu = A_0 \, \delta \Phi_{\nu,0} + A_1 \, \delta \Phi_{\nu,1} \]

\[
\begin{array}{|c|c|c|c|c|}
\hline
\delta \Phi_\nu & \delta \Phi_{\nu,\text{in}} & \delta \Phi_{\nu,\text{out}} & \delta \Phi_{\nu,0} & \delta \Phi_{\nu,1} \\
\hline
\delta \Phi_{\text{pp}} & -0.069 & -0.031 & -0.100 & -0.030 \\
\delta \Phi_{\text{Be}} & 0.85 & 0.41 & 1.26 & 0.38 \\
\delta \Phi_{\text{B}} & 1.93 & 0.75 & 2.68 & 0.68 \\
\delta \Phi_{\text{O}} & 1.65 & 0.50 & 2.15 & 0.48 \\
\delta \Phi_{\text{N}} & 1.14 & 0.28 & 1.43 & 0.30 \\
\hline
\end{array}
\]

See also Tripathy and Christensen-Dalsgaard 98
A final look ...

helioseismology
A final look ...

\[ \delta \kappa(r) = A_0 \delta \kappa_0(r) + A_1 \delta \kappa_1(r) = A_0 + A_1 \left( \frac{r}{R_b} \right) \]

\[ Y_b = 0.2485 \pm 0.0034 \]

\[ R_b = (0.715 \pm 0.001) R_\odot \]

AGS05 predictions

\[ \overline{Y}_b = 0.229 \]

\[ \overline{R}_b = 0.730 R_\odot \]
A final look ...

\[ \delta \kappa(r) = A_0 \delta \kappa_0(r) + A_1 \delta \kappa_1(r) = A_0 + A_1 \left( \frac{r}{R_b} \right) \]

\( Y_b = 0.2485 \pm 0.0034 \)

\( R_b = (0.715 \pm 0.001) R_\odot \)

\( \Phi_B = (5.18 \pm 0.29) \times 10^6 \text{ cm}^{-2} \text{ s}^{-1} \)


\( \Phi_{Be} = (5.18 \pm 0.51) \times 10^9 \text{ cm}^{-2} \text{ s}^{-1} \)

Borexino 2008

\[ \chi^2_{\text{min}}/\text{d.o.f.} = 1.7/2 \]

AGS05 predictions

\( \overline{Y}_b = 0.229 \)

\( \overline{R}_b = 0.730 R_\odot \)

\( \overline{\Phi}_B = (4.66 \pm 0.42) \times 10^6 \text{ cm}^{-2} \text{ s}^{-1} \)

\( \overline{\Phi}_{Be} = (4.54 \pm 0.23) \times 10^9 \text{ cm}^{-2} \text{ s}^{-1} \)

v-fluxes from Serenelli 09 – (error estimate sdo not include opa and metals)
Summary and conclusions

• **Linear Solar Models**: a simple and accurate tool to investigate the sun interior
• **Application**: analysis of the role of opacity (and metals) in the sun
• **Sound speed and convective radius**: sensitive to differential opacity variations
• **Helium and neutrinos**: sensitive to overall opacity rescaling and, in particular, to opacity of the inner radiative region.
• **Nice complementarity** between different observational data.
• The opacity changes required to fit helioseismic and solar neutrino data seem large with respect to current opacity (and composition) uncertainties.
• Hopefully, **future neutrino flux** measurements (Be, CNO, pep) will provide additional clues for the solar composition puzzle.
Additional Slides
**Linear Solar Models**

- **SSMs** provide a good approximation of the real sun. Small modifications are likely to explain disagreement with helioseismology.

- We can expand linearly the solar models around the SSM and calculate:

\[
\delta(\text{output}) = L \left[ \delta(\text{input}) \right]
\]

\[
\begin{align*}
\delta(\text{input}) &= \delta \kappa(r), \, \delta \epsilon(r), \, \delta(\text{composition}), \, \ldots \\
\delta(\text{output}) &= \delta u(r), \, \delta R_b, \, \Delta Y_b, \, \delta \Phi_\nu, \, \ldots
\end{align*}
\]

- A tool to investigate the solar interior (accessible also to non SSM builders)

- It can help to understand the origin of discrepancy with helioseismic data or to describe new effects.
Linear Solar Models – Structure equations

We write:

\[ h(r) = \bar{h}(r)[1 + \delta h(r)] \]
\[ X_i(r) = \bar{X}_i(r)[1 + \delta X_i(r)] \]
\[ Y(r) = \bar{Y}(r) + \Delta Y(r) \]

where \( \bar{h}(r), \bar{X}_i(r) \) are the SSMs predicted values.

We then expand structure eqs. to first order in perturbations:

\[
\frac{\partial \delta m}{\partial r} = \frac{1}{l_m} [\delta \rho - \delta m] \\
\frac{\partial \delta P}{\partial r} = \frac{1}{l_P} [\delta m + \delta \rho - \delta P] \\
\delta P = [P_\rho \delta \rho + P_T \delta T + P_Y \Delta Y + \sum_i P_i \delta X_i]
\]

\[
\frac{\partial \delta l}{\partial r} = \frac{1}{l_l} [(1 + \epsilon_\rho) \delta \rho + \epsilon_T \delta T + \epsilon_Y \Delta Y + \sum_i \epsilon_i \delta X_i - \delta l + \delta \epsilon]
\]

\[
\begin{aligned}
\frac{\partial \delta T}{\partial r} &= \frac{1}{l_T} [\delta l + (\kappa_T - 4) \delta T + (\kappa_\rho + 1) \delta \rho + \kappa_Y \Delta Y + \sum_i \kappa_i \delta X_i + \delta \kappa] \quad \text{Rad.} \\
\frac{\partial \delta T}{\partial r} &= \frac{1}{l_T} [\delta m + \delta \rho - \delta P] \quad \text{Conv.}
\end{aligned}
\]

where \( l_h = \left[ \frac{d \ln(h)}{dr} \right]^{-1} \) and \( P_h = \left[ \frac{d \ln(P)}{d \ln(h)} \right], \epsilon_h = \left[ \frac{d \ln(\epsilon)}{d \ln(h)} \right], \kappa_h = \left[ \frac{d \ln(\kappa)}{d \ln(h)} \right] \)
Linear Solar Models – Structure equations

\[ \frac{\partial \delta m}{\partial r} = \frac{1}{l_m} [\delta \rho - \delta m] \]

\[ \frac{\partial \delta P}{\partial r} = \frac{1}{l_P} [\delta m + \delta \rho - \delta P] \]

\[ \delta P = [P_\rho \delta \rho + P_T \delta T + P_Y \Delta Y + \sum_i P_i \delta X_i] \]

\[ \frac{\partial \delta l}{\partial r} = \frac{1}{l_l} [(1 + \epsilon_\rho)\delta \rho + \epsilon_T \delta T + \epsilon_Y \Delta Y + \sum_i \epsilon_i \delta X_i - \delta l + \delta \epsilon] \]

\[ \begin{align*}
\frac{\partial \delta T}{\partial r} &= \frac{1}{l_T} [\delta l + (\kappa_T - 4)\delta T + (\kappa_\rho + 1)\delta \rho + \kappa_Y \Delta Y + \sum_i \kappa_i \delta X_i + \delta \kappa] & \text{Rad.} \\
\frac{\partial \delta T}{\partial r} &= \frac{1}{l_T} [\delta m + \delta \rho - \delta P] & \text{Conv.}
\end{align*} \]
**Linear Solar Models – Simplifications**

EOS: we assume perfect gas scaling and neglect the role of metals

\[
\delta \rho(r) = \delta P(r) - \delta T(r) - P_Y \Delta Y(r)
\]

\[
P_Y(r) = -\frac{\partial \ln \mu}{\partial Y} = -\frac{5}{8 - 5Y(r) - 6Z(r)}
\]

We eliminate density from equations, obtaining:

\[
\frac{d\delta m}{dr} = \frac{1}{l_m} \left[ \delta P - \delta T - \delta m - P_Y \Delta Y \right]
\]

\[
\frac{d\delta P}{dr} = \frac{1}{l_P} \left[ -\delta T + \delta m - P_Y \Delta Y \right]
\]

\[
\frac{d\delta \ell}{dr} = \frac{1}{l_\ell} \left[ \beta_P \delta P + \beta_T \delta T - \delta \ell + \beta_Y \Delta Y + \sum_i \beta_i \delta Z_i + \delta \epsilon \right]
\]

\[
\frac{d\delta T}{dr} = \frac{1}{l_T} \left[ \alpha_P \delta P + \alpha_T \delta T + \delta \ell + \alpha_Y \Delta Y + \sum_i \alpha_i \delta Z_i + \delta \kappa \right]
\]

Rad.

\[
\frac{d\delta T}{dr} = \frac{1}{l_T} \left[ -\delta T + \delta m - P_Y \Delta Y \right]
\]

Conv.

with:

\[
\begin{align*}
\alpha_P &= \kappa_\rho + 1 & \alpha_T &= \kappa_T - \kappa_\rho - 5 & \alpha_Y &= - (\kappa_\rho + 1) P_Y + \kappa_Y & \alpha_i &= \partial \ln \kappa / \partial \ln Z_i \\
\beta_P &= \epsilon_\rho + 1 & \beta_T &= \epsilon_T - \epsilon_\rho - 1 & \beta_Y &= - (\epsilon_\rho + 1) P_Y + \epsilon_Y & \beta_i &= \partial \ln \epsilon / \partial \ln Z_i
\end{align*}
\]
Linear Solar Models – Boundary conditions

Central conditions (r=0):

\[
\begin{align*}
\delta m(0) & = \delta P_0 - \delta T_0 - P_{Y,0} \Delta Y_0 \\
\delta P(0) & = \delta P_0 \\
\delta T(0) & = \delta T_0 \\
\delta l(0) & = \beta P_0 + \beta T_0 \delta T_0 + \beta Y_0 \Delta Y_0 + \sum_i \beta_{i,0} \delta Z_{i,0} + \delta \epsilon_0
\end{align*}
\]

Surface conditions implemented at the convective boundary:

\[
\begin{align*}
m(\bar{R}_b) & \simeq M_\odot \\
l(\bar{R}_b) & = L_\odot \\
Y(\bar{R}_b) & = \text{const}
\end{align*}
\]

\[
\begin{align*}
\partial \frac{\delta u}{\partial r} & = -\frac{\delta u}{l_T} \\
\delta m(\bar{R}_b) & = -m_{\text{conv}} \delta C \\
\delta P(\bar{R}_b) & = \delta C = \delta \rho(\bar{R}_b) \\
\delta T(\bar{R}_b) & = \delta \mu_b = -P_{Y,b} \Delta Y_b \\
\delta l(\bar{R}_b) & = 0
\end{align*}
\]

\[\delta u(\bar{R}_b) = 0\]

Remind that:

\[\delta u(r) = \delta P(r) - \delta \rho(r) = \delta T(r) - \delta \mu(r)\]
Linear Solar Models – Chemical composition

The elemental abundances in the present Sun are determined by:

- initial abundances;  \[ Y(r) = Y_{\text{ini}} [1 + D_Y(r)] + Y_{\text{nuc}}(r) \]
- nuclear burnings;  \[ Z_i(r) = Z_{\text{ini},i} [1 + D_Z(r)] \]
- diffusion and gravitational settling.

We assume that:

- helium nuclear production \( Y_{\text{nuc}} \) scales proportionally to the nuclear reaction efficiency in the present sun

\[ \Delta Y_{\text{nuc}}(r) = \overline{Y}_{\text{nuc}}(r) \delta e^{\text{tot}}(r) \]

- Diffusion effects in the convective envelope scales proportionally to diffusion velocity of the present sun

\[ \Delta Y_b = A_Y \Delta Y_{\text{ini}} + A_C \delta C \]
\[ \Delta Z_{b,i} = B_Y \Delta Y_{\text{ini}} + B_C \delta C + \delta z_i \]

\[ \delta z_i = \frac{(Z_{i,b}/X_b) - (\overline{Z}_{i,b}/\overline{X}_b)}{(\overline{Z}_{i,b}/\overline{X}_b)} \]

- Variations of diffusion efficiency in the radiative core can be neglected:

\[ \Delta Y(r) = \xi_Y(r) \Delta Y_{\text{ini}} + \xi_T(r) \delta T(r) + \xi_P(r) \delta P(r) \]
\[ \delta Z_i(r) = \delta Z_{i,\text{ini}} = Q_Y \Delta Y_{\text{ini}} + Q_C \delta C + \delta z_i \]
Linear Solar Models – Final set of equations equations

\[
\begin{align*}
\frac{d\delta m}{dr} &= \frac{1}{l_m} \left[ \gamma_P \delta P + \gamma_T \delta T - \delta m + \gamma_Y \Delta Y_{ini} + \gamma_\epsilon \delta \epsilon \right] \\
\frac{d\delta P}{dr} &= \frac{1}{l_P} \left[ (\gamma_P - 1) \delta P + \gamma_T \delta T + \delta m + \gamma_Y \Delta Y_{ini} + \gamma_\epsilon \delta \epsilon \right] \\
\frac{d\delta l}{dr} &= \frac{1}{l_l} \left[ \beta'_P \delta P + \beta'_T \delta T - \delta l + \beta'_Y \Delta Y_{ini} + \beta'_C \delta C + \beta'_\epsilon \delta \epsilon \right] \\
\frac{d\delta T}{dr} &= \frac{1}{l_T} \left[ \alpha'_P \delta P + \alpha'_T \delta T + \delta l + \alpha'_Y \Delta Y_{ini} + \alpha'_C \delta C + \delta \kappa + \alpha'_\epsilon \delta \epsilon \right]
\end{align*}
\]

to be solved between with the boundary conditions

\( r = 0 \)

\[
\begin{align*}
\delta m &= \gamma_{P,0} \delta P_0 + \gamma_{T,0} \delta T_0 + \gamma_{Y,0} \Delta Y_{ini} + \gamma_{\epsilon,0} \delta \epsilon_0 \\
\delta P &= \delta P_0 \\
\delta T &= \delta T_0 \\
\delta l &= \beta'_{P,0} \delta P_0 + \beta'_{T,0} \delta T_0 + \beta'_{Y,0} \Delta Y_{ini} + \beta'_{C,0} \delta C + \beta'_{\epsilon,0} \delta \epsilon_0
\end{align*}
\]

\( r = \bar{R}_b \)

\[
\begin{align*}
\delta m &= -m_{\text{conv}} \delta C \\
\delta P &= \delta C \\
\delta T &= A'_Y \Delta Y_{ini} + A'_C \delta C \\
\delta l &= 0
\end{align*}
\]

univocally determine the parameters \( \delta P_0, \delta T_0, \Delta Y_{ini}, \delta C \)
The “stability” of sound speed ...

Schematically, we can note that:

\[
\frac{G M m_u}{R} \sim \frac{k_B T}{\mu} = \frac{P}{\rho} = u
\]

This quantity is fixed for the Sun.

In a “normal star”, opacity determine luminosity:

\[
L \sim \frac{E_\gamma}{t_{\text{diff}}} = \frac{M^3 \mu^4}{\kappa}
\]

In the sun:
To keep L constant, we have to vary helium abundance.
An increase of Y implies a decrease of \(\kappa\) and an increase of \(\mu\).
The sound speed

$$\delta \kappa(r) = A_{in} \delta \kappa_{in}(r) + A_{out} \delta \kappa_{out}(r)$$

$$\delta \kappa(r) = \Lambda_0 \delta \kappa_0(r) + \Lambda_1 \delta \kappa_1(r)$$

$$\delta u(r) = A_{in} \delta u_{in}(r) + A_{out} \delta u_{out}(r)$$

$$\delta u(r) = \Lambda_0 \delta u_0(r) + \Lambda_1 \delta u_1(r)$$

$$\delta u_{in}(r) \approx -\delta u_{out}(r)$$

$$\delta u_0(r) \ll \delta u_1(r)$$