Multipartite entangled states in particle mixing

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- Flavor mixing and entanglement;
- Entanglement in neutrino oscillations:
  - Flavor entanglement;
  - Decoherence;
- Neutrino oscillations as a resource for quantum information.
- Particle mixing and entanglement in Quantum Field Theory.
Motivations

- Evidence of neutrino mixing and oscillations;
- Importance of entanglement both at a fundamental level and for quantum information;
- Necessity for a treatment of entanglement in the context of Quantum Field Theory*;
- Entanglement in particle physics: entanglement, decoherence, Bell inequalities for the $K^0\bar{K}^0$ (or $B^0\bar{B}^0$) system†;
- Entanglement in neutrino oscillations: entanglement between neutrino and recoil particle‡.

Entanglement in particle mixing

- Flavor mixing (neutrinos)

\[ |\nu_e\rangle = \cos \theta |\nu_1\rangle + \sin \theta |\nu_2\rangle \]
\[ |\nu_\mu\rangle = -\sin \theta |\nu_1\rangle + \cos \theta |\nu_2\rangle \]

- Correspondence with two-qubit states:

\[ |\nu_1\rangle \equiv |1\rangle_1 |0\rangle_2 \equiv |10\rangle, \quad |\nu_2\rangle \equiv |0\rangle_1 |1\rangle_2 \equiv |01\rangle, \]

where \( |\rangle_i \) denotes states in the Hilbert space for neutrinos with mass \( m_i \).

\[ \Rightarrow \text{flavor states are entangled superpositions of the mass eigenstates:} \]
\[ |\nu_e\rangle = \cos \theta |10\rangle + \sin \theta |01\rangle. \]
Single-particle entanglement*

– A state like \( |\psi\rangle_{A,B} = |0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B \) is entangled;

– entanglement among field modes, rather than particles;

– entanglement is a property of composite systems, rather than of many-particle systems;

– entanglement and non-locality are not synonyms;

– single-particle entanglement is as good as two-particle entanglement for applications (quantum cryptography, teleportation, violation of Bell inequalities, etc.).


One photon is split, creating an entangled one-photon state.

Each photon mode interacts with a two-level atom. Resonance is tuned to give a $\pi$ pulse, if a photon is present. The excitation is transferred to the atomic pair.

One excitation is distributed between two atoms. A Bell state of excited-ground states is created.

One atom is split between two traps, creating an entangled one-atom state.

Each atomic trap interacts with an attenuated atomic beam. Resonance is tuned to create a molecule if one atom is found in the trap. The traps are left empty, and the atom is transferred to the beams.

The (dark grey) trapped atom is distributed between two (light grey) atomic beams. A Bell state of molecule–atom states is created.

Protocols for extraction of single-particle entanglement
Multipartite entanglement

– Characterization of entanglement for multipartite systems is a non-trivial task. Several approaches have been developed: global entanglement, tangle, geometric measures*, etc...

In the 3-qubit case, the two fundamental classes† of states are those of the GHZ state and of the $W$ state:

$$|\text{GHZ}\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle),$$

$$|W\rangle = \frac{1}{\sqrt{3}} (|100\rangle + |010\rangle + |001\rangle).$$

*T.C.Wei and P.M.Goldbart Phys. Rev. A (2003);
Multipartite entanglement measures: pure states

Let $\rho = |\psi\rangle\langle\psi|$ be the density operator corresponding to a pure state $|\psi\rangle$, describing the system $S$ partitioned into $N$ parties.

Bipartition of the $N$-partite system $S = \{S_1, S_2, \ldots, S_N\}$ in two subsystems:

$S_{A_n} = \{S_{i_1}, S_{i_2}, \ldots, S_{i_n}\}, \quad 1 \leq i_1 < i_2 < \ldots < i_n \leq N; \quad (1 \leq n < N)$

and

$S_{B_{N-n}} = \{S_{j_1}, S_{j_2}, \ldots, S_{j_{N-n}}\}, \quad 1 \leq j_1 < j_2 < \ldots < j_{N-n} \leq N; \quad i_q \neq j_p$

- Reduced density matrix of $S_{A_n}$ after tracing over $S_{B_{N-n}}$:

$\rho_{A_n} \equiv \rho_{i_1, i_2, \ldots, i_n} = Tr_{B_{N-n}}[\rho] = Tr_{j_1, j_2, \ldots, j_{N-n}}[\rho]$

• von Neumann entropy associated with the above bipartition:

\[ E_{vN}^{(A_n;B_{N-n})} = -Tr_{A_n}[\rho_{A_n} \log_2 \rho_{A_n}] . \]

• Average von Neumann entropy (global entanglement)

\[ \langle E_{vN}^{(n:N-n)} \rangle = \binom{N}{n}^{-1} \sum_{A_n} E_{vN}^{(A_n;B_{N-n})} , \]

where the sum is intended over all the possible bipartitions of the system in two subsystems each with \( n \) and \( N - n \) elements \( (1 \leq n < N) \).
Multipartite entanglement measures: mixed states

- Entropic measures cannot be used to quantify the entanglement of mixed states ⇒ logarithmic negativity.

We denote by

\[ \tilde{\rho}_{An} \equiv \rho^{PT} B_{N-n} \equiv \rho^{PT j_1, j_2, \ldots, j_{N-n}} \]

the *bona fide* density matrix, obtained by the partial transposition of \( \rho \) with respect to the parties belonging to the subsystem \( S_{B_{N-n}} \).

- Logarithmic negativity associated with the above bipartition

\[ E_{N}^{(A_n; B_{N-n})} = \log_2 \| \tilde{\rho}_{A_n} \|_1 \]

- Average logarithmic negativity (global entanglement)

\[ \langle E_{N}^{(n:N-n)} \rangle = \binom{N}{n}^{-1} \sum_{A_n} E_{N}^{(A_n; B_{N-n})}, \]

where the sum is intended over all the possible bipartitions of the system.
Multipartite entanglement in neutrino mixing

– Neutrino mixing (three flavors):

$$|\nu_f\rangle = U(\tilde{\theta}, \delta) |\nu_m\rangle$$

with

$$|\nu_f\rangle = (|\nu_e\rangle, |\nu_\mu\rangle, |\nu_\tau\rangle)^T$$

and

$$|\nu_m\rangle = (|\nu_1\rangle, |\nu_2\rangle, |\nu_3\rangle)^T.$$ 

– Mixing matrix (MNSP)

$$U(\tilde{\theta}, \delta) = \begin{pmatrix}
  c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
  -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
  s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix},$$

where

$$(\tilde{\theta}, \delta) \equiv (\theta_{12}, \theta_{13}, \theta_{23}; \delta),$$

$c_{ij} \equiv \cos \theta_{ij}$ and $s_{ij} \equiv \sin \theta_{ij}.$

• Correspondence with three-qubit states:

$$|\nu_1\rangle \equiv |1\rangle_1|0\rangle_2|0\rangle_3 \equiv |100\rangle,$$

$$|\nu_2\rangle \equiv |0\rangle_1|1\rangle_2|0\rangle_3 \equiv |010\rangle,$$

$$|\nu_3\rangle \equiv |0\rangle_1|0\rangle_2|1\rangle_3 \equiv |001\rangle.$$ 

Flavor states as generalized W states

- Define the generalized class of three-qubit W states as

\[
|W^{(3)}(\tilde{\theta}; \delta)\rangle \equiv U^{(3f)}(\tilde{\theta}, \delta) |\nu^{(3)}\rangle
\]

\[
U^{(3f)}(\tilde{\theta}, \delta) = U(\tilde{\theta}, \delta) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix},
\]

where \( |W^{(3)}(\tilde{\theta}; \delta)\rangle = (|W_e^{(3)}(\tilde{\theta}, \delta)\rangle, |W_\mu^{(3)}(\tilde{\theta}, \delta)\rangle, |W_\tau^{(3)}(\tilde{\theta}, \delta)\rangle)^T \) and \( |\nu^{(3)}\rangle = (|\nu_1^{(3)}\rangle, |\nu_2^{(3)}\rangle, |\nu_3^{(3)}\rangle)^T \).

- The entanglement properties of the states associated with matrices \( U(\tilde{\theta}, \delta) \) and \( U^{(3f)}(\tilde{\theta}, \delta) \) are identical.

⇔ we regard flavor neutrino states as generalized W states.
Entanglement properties of states with maximal mixing

- Flavor mixing is maximal for

$$\theta_{12}^{\text{max}} = \frac{\pi}{4}; \quad \theta_{23}^{\text{max}} = \frac{\pi}{4}; \quad \theta_{13}^{\text{max}} = \arccos \sqrt{\frac{2}{3}}; \quad \delta^{\text{max}} = \frac{\pi}{2}.$$ 

⇒ all elements of CKM matrix have modulus $= 1/3$:

$$U_{\text{max}}^{(3f)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ iy & iy^2 & i \\ iy^2 & iy & i \end{pmatrix} \quad \text{with} \quad y = \exp (2i\pi/3).$$

In this case, all the $|W_\alpha^{(3)}(\tilde{\theta}, \delta)\rangle$ states have the same entanglement of $|W^{(3)}\rangle$:

$$E_{vN}^{(A_2;B_1)}(|W^{(3)}(\tilde{\theta}^{\text{max}}, \delta^{\text{max}})\rangle) = \langle E_{vN}^{(2:1)}(|W^{(3)}(\tilde{\theta}^{\text{max}}, \delta^{\text{max}})\rangle) \rangle = E_{21}^{(3)}.$$
Entanglement properties of states $|W^{(3)}_{\alpha}(\tilde{\theta}_{\text{max}}; \delta)\rangle$ ($\alpha = e, \mu, \tau$)

– We study the dependence of entanglement on the phase $\delta$, with the rotation angles set at their maximal values $\theta_{ij}^{\text{max}}$.

The matrix $U^{(3f)}$ becomes

$$
U^{(3f)}(\delta) = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 1 & 1 \\
-\frac{1}{2}(\sqrt{3} + e^{i\delta}) & \frac{1}{2}(\sqrt{3} - e^{i\delta}) & e^{i\delta} \\
\frac{1}{2}(\sqrt{3} - e^{i\delta}) & -\frac{1}{2}(\sqrt{3} + e^{i\delta}) & e^{i\delta}
\end{pmatrix}.
$$

We get:

$$
E_{vN_{e}}^{(1,2;3)} = E_{vN_{e}}^{(1,3;2)} = E_{vN_{e}}^{(2,3;1)} = E_{vN_{\mu}}^{(1,2;3)} = E_{vN_{\tau}}^{(1,2;3)} = \log_2 3 - \frac{2}{3},
$$

$$
E_{vN_{\mu}}^{(1,3;2)} = E_{vN_{\tau}}^{(2,3;1)} = -\left(\frac{1}{3} - \frac{\cos \delta}{2\sqrt{3}}\right) \log_2 \left[\frac{1}{3} - \frac{\cos \delta}{2\sqrt{3}}\right] - \left(\frac{2}{3} + \frac{\cos \delta}{2\sqrt{3}}\right) \log_2 \left[\frac{2}{3} + \frac{\cos \delta}{2\sqrt{3}}\right],
$$

$$
E_{vN_{\mu}}^{(2,3;1)} = E_{vN_{\tau}}^{(1,3;2)} = -\left(\frac{1}{3} + \frac{\cos \delta}{2\sqrt{3}}\right) \log_2 \left[\frac{1}{3} + \frac{\cos \delta}{2\sqrt{3}}\right] - \left(\frac{2}{3} - \frac{\cos \delta}{2\sqrt{3}}\right) \log_2 \left[\frac{2}{3} - \frac{\cos \delta}{2\sqrt{3}}\right],
$$

where $E_{vN_{\alpha}}^{(i,j;k)} = E_{vN}^{(i,j;k)}(|W_{\alpha}^{(3)}(\delta)\rangle)$. 

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Entanglement of the state $|W^{(3)}_{\mu}(\tilde{\theta}_{max}; \delta)\rangle$

von Neumann entropy $E_{vN\mu}^{(i,j;k)}$ and average von Neumann entropy $\langle E_{vN\mu}^{(2;1)} \rangle$ (full line) as functions of the $CP$-violating phase $\delta$.

$E_{vN\mu}^{(i,j;k)}$ is plotted for the bipartitions $i = 1, j = 2, k = 3$ (dotted line); $i = 1, j = 3, k = 2$ (dashed line); $i = 2, j = 3, k = 1$ (dot-dashed line).

$E_{vN\mu}^{(1,2;3)}$ is constant and takes the reference value $E_{21}^{(3)} = 0.918296$. 
“Squeezing” of entanglement: $E_{vN \mu}^{(1,3;2)}$ and $E_{vN \mu}^{(2,3;1)}$ vary with $\delta$, attaining the absolute maximum 1 at the points $\delta_1 = \pm \arccos \left(-\frac{1}{\sqrt{3}}\right) \pm 2p\pi$ and $\delta_2 = \pm \arccos \left(\frac{1}{\sqrt{3}}\right) \pm 2p\pi$ (with $p$ integer), respectively, and exceeding the reference value $E_{21}^{(3)}$.

The average von Neumann entropy $\langle E_{vN \mu}^{(2:1)} \rangle$ stays below the reference value $E_{21}^{(3)}$, attaining it at the points $\delta = \frac{\pi}{2} \pm p\pi$.

The free parameter $\delta$ can be used to concentrate and squeeze the entanglement in a specific bipartition, allowing a sharply peaked distribution of entanglement, at the expense of the average von Neumann entropy.
(Flavor) Entanglement in neutrino oscillations*

- Two-flavor neutrino states

\[ |\nu^{(f)}\rangle = U(\tilde{\theta}, \delta) |\nu^{(m)}\rangle \]

where \( |\nu^{(f)}\rangle = (|\nu_e\rangle, |\nu_\mu\rangle)^T \) and \( |\nu^{(m)}\rangle = (|\nu_1\rangle, |\nu_2\rangle)^T \) and \( U(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \).

- Flavor states at time \( t \):

\[ |\nu^{(f)}(t)\rangle = U(\tilde{\theta}, \delta) U_0(t) U(\tilde{\theta}, \delta)^{-1} |\nu^{(f)}\rangle \equiv \widetilde{U}(t) |\nu^{(f)}\rangle, \]

with \( U_0(t) = \begin{pmatrix} e^{-iE_1 t} & 0 \\ 0 & e^{-iE_2 t} \end{pmatrix} \).

- Transition probability for $\nu_\alpha \to \nu_\beta$

$$P_{\nu_\alpha \to \nu_\beta}(t) = |\langle \nu_\beta|\nu_\alpha(t)\rangle|^2 = |\tilde{U}_{\alpha\beta}(t)|^2.$$ 

We now take the flavor states at initial time as our qubits:

$$|\nu_e\rangle \equiv |1\rangle_e|0\rangle_\mu \equiv |10\rangle_f, \quad |\nu_\mu\rangle \equiv |0\rangle_e|1\rangle_\mu \equiv |01\rangle_f,$$

- Starting from $|10\rangle_f$ or $|01\rangle_f$, time evolution generates the (entangled) Bell-like states:

$$|\nu_\alpha(t)\rangle = \tilde{U}_{\alpha e}(t)|1\rangle_e|0\rangle_\mu + \tilde{U}_{\alpha\mu}(t)|0\rangle_e|1\rangle_\mu, \quad \alpha = e, \mu.$$
Entanglement measure

- Let $\rho = |\psi\rangle\langle\psi|$ be the density operator for a pure state $|\psi\rangle$

Bipartition of the $N$-partite system $S = \{S_1, S_2, \ldots, S_N\}$ in two subsystems:

$S_{A_n} = \{S_{i_1}, S_{i_2}, \ldots, S_{i_n}\}, \quad 1 \leq i_1 < i_2 < \ldots < i_n \leq N; (1 \leq n < N)$

and

$S_{B_{N-n}} = \{S_{j_1}, S_{j_2}, \ldots, S_{j_{N-n}}\} \quad 1 \leq j_1 < j_2 < \ldots < j_{N-n} \leq N; i_q \neq j_p$

- Reduced density matrix of $S_{A_n}$ after tracing over $S_{B_{N-n}}$:

$\rho_{A_n} \equiv \rho_{i_1, i_2, \ldots, i_n} = Tr_{B_{N-n}}[\rho] = Tr_{j_1, j_2, \ldots, j_{N-n}}[\rho]$
• Linear entropy associated to such a bipartition:

\[ S_{L}^{(A_n;B_{N-n})}(\rho) = \frac{d}{d-1} \left( 1 - Tr_{A_n}[\rho_{A_n}^2] \right), \]

\( d \) is the Hilbert-space dimension: \( d = \min\{\dim S_{A_n}, \dim S_{B_{N-n}}\} = \min\{2^n, 2^{N-n}\}. \)

• Average linear entropy (global entanglement):

\[ \langle S_{L}^{(n:N-n)}(\rho) \rangle = \left( \begin{array}{c} N \\ n \end{array} \right)^{-1} \sum_{A_n} S_{L}^{(A_n;B_{N-n})}(\rho), \]

sum over all the possible bi-partitions of the system in two subsystems, respectively with \( n \) and \( N - n \) elements \( (1 \leq n < N) \).
Entanglement in neutrino oscillations: two-flavors

Consider the density matrix for the electron neutrino state \( \rho^{(e)} = |\nu_e(t)\rangle \langle \nu_e(t)| \), and trace over mode \( \mu \Rightarrow \rho^{(e)}_e \).

- The associated linear entropy is:
  \[
  S^{(e;\mu)}_L(\rho^{(e)}) = 4 |\bar{U}_{e\mu}(t)|^2 |\bar{U}_{ee}(t)|^2 = 4 P_{\nu_e\rightarrow\nu_e}(t) P_{\nu_e\rightarrow\nu_\mu}(t)
  \]

- The linear entropy for the state \( \rho^{(\alpha)} \) is:
  \[
  S^{(e;\mu)}_{L\alpha} = S^{(\mu;e)}_{L\alpha} = \langle S^{(1:1)}_{L\alpha} \rangle = 4 |\bar{U}_{\alpha\mu}(t)|^2 |\bar{U}_{\alpha e}(t)|^2
  = 4 |\bar{U}_{\alpha e}(t)|^2 (1 - |\bar{U}_{\alpha e}(t)|^2)
  = 4 |\bar{U}_{\alpha\mu}(t)|^2 (1 - |\bar{U}_{\alpha\mu}(t)|^2).
  \]

- Linear entropy given by product of transition probabilities!
Linear entropy $S_{Le}^{(e;\mu)}$ (full) as a function of the scaled time $T = \frac{2Et}{\Delta m^2_{12}}$, with $\sin^2 \theta = 0.314$. Transition probabilities $P_{\nu_e \rightarrow \nu_e}$ (dashed) and $P_{\nu_e \rightarrow \nu_\mu}$ (dot-dashed) are reported for comparison.
(Flavor) Entanglement in neutrino oscillations: three flavors

Three-flavor neutrino states

$$|\nu^{(f)}\rangle = \mathbf{U}(\tilde{\theta}, \delta) |\nu^{(m)}\rangle$$

where $$|\nu^{(f)}\rangle = (|\nu_e\rangle, |\nu_\mu\rangle, |\nu_\tau\rangle)^T$$ and $$|\nu^{(m)}\rangle = (|\nu_1\rangle, |\nu_2\rangle, |\nu_3\rangle)^T$$

$$\mathbf{U}(\tilde{\theta}, \delta) = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix},$$

where $$(\tilde{\theta}, \delta) \equiv (\theta_{12}, \theta_{13}, \theta_{23}; \delta), c_{ij} \equiv \cos \theta_{ij}$$ and $s_{ij} \equiv \sin \theta_{ij}.$$
– Flavor states at time $t$:

$$\left| \nu^{(f)}(t) \right\rangle = U(\tilde{\theta}, \delta) U_0(t) U(\tilde{\theta}, \delta)^{-1} \left| \nu^{(f)} \right\rangle \equiv \tilde{U}(t) \left| \nu^{(f)} \right\rangle,$$

with $\left| \nu^{(f)} \right\rangle$ flavor states at $t = 0$, $U_0(t) = \text{diag}(e^{-iE_1 t}, e^{-iE_2 t}, e^{-iE_3 t})$,

and $\tilde{U}(t) = U(\tilde{\theta}, \delta) U_0(t) U(\tilde{\theta}, \delta)^{-1}$, with $\tilde{U}(t = 0) = 1I$.

– Transition probability for $\nu_\alpha \rightarrow \nu_\beta$

$$P_{\nu_\alpha \rightarrow \nu_\beta}(t) = |\langle \nu_\beta | \nu_\alpha(t) \rangle|^2 = |\tilde{U}_{\alpha\beta}(t)|^2,$$

• Take the flavor states at time $t = 0$ as the qubits:

$$\left| \nu_e \right\rangle \equiv \left| 1 \right\rangle_e \left| 0 \right\rangle_\mu \left| 0 \right\rangle_\tau \equiv \left| 100 \right\rangle_f, \quad \left| \nu_\mu \right\rangle \equiv \left| 0 \right\rangle_e \left| 1 \right\rangle_\mu \left| 0 \right\rangle_\tau \equiv \left| 010 \right\rangle_f,$$

$$\left| \nu_\tau \right\rangle \equiv \left| 0 \right\rangle_e \left| 0 \right\rangle_\mu \left| 1 \right\rangle_\tau \equiv \left| 001 \right\rangle_f.$$
Entanglement in neutrino oscillations: three-flavors

– In the three-flavor case, we obtain

\[ S_{L\alpha}^{(e,\mu;\tau)} = 4|\tilde{U}_{\alpha\tau}(t)|^2 (|\tilde{U}_{\alpha e}(t)|^2 + |\tilde{U}_{\alpha\mu}(t)|^2) \]

\[ = 4|\tilde{U}_{\alpha\tau}(t)|^2 (1 - |\tilde{U}_{\alpha\tau}(t)|^2). \]

The linear entropies for the two remaining bi-partitions are easily obtained by permuting the indexes \( e, \mu, \tau \).

– The average linear entropy is

\[ \langle S_{L\alpha}^{(2:1)} \rangle = \frac{8}{3} (|\tilde{U}_{\alpha e}(t)|^2 |\tilde{U}_{\alpha\mu}(t)|^2 + |\tilde{U}_{\alpha e}(t)|^2 |\tilde{U}_{\alpha\tau}(t)|^2 + |\tilde{U}_{\alpha\mu}(t)|^2 |\tilde{U}_{\alpha\tau}(t)|^2). \]
Linear entropies $S_{Le}^{(\alpha, \beta; \gamma)}$ and $\langle S_{Le}^{(2;1)} \rangle$ as functions of the scaled time $T = \frac{2E_t}{\Delta m_{12}^2}$. Curves correspond to the partial linear entropies $S_{Le}^{(e, \mu; \tau)}$ (long-dashed), $S_{Le}^{(e, \tau; \mu)}$ (dashed), $S_{Le}^{(\mu, \tau; e)}$ (dot-dashed), and to the average linear entropy $\langle S_{Le}^{(2;1)} \rangle$ (full).

Parameters are fixed at central experimental values: $\sin^2 \theta_{12} = 0.314$, $\sin^2 \theta_{23} = 0.45$, $\sin^2 \theta_{12} = 0.008$, $\Delta m_{12}^2 = 7.92 \times 10^{-5} eV^2$, $\Delta m_{23}^2 = 2.6 \times 10^{-3} eV^2$. 
Because of \( CPT \) invariance, the \( CP \) asymmetry \( \Delta \alpha,\beta_{CP} \) is equal to the asymmetry under time reversal:

\[
\Delta \alpha,\beta_{CP} = \Delta \alpha,\beta_T = P_{\nu_\alpha \rightarrow \nu_\beta}(t) - P_{\nu_\beta \rightarrow \nu_\alpha}(t) = P_{\nu_\alpha \rightarrow \nu_\beta}(t) - P_{\nu_\alpha \rightarrow \nu_\beta}(-t).
\]

\( \Delta \alpha,\beta_{CP} \neq 0 \) for \( \delta \neq 0 \). Note that \( \sum_\beta \Delta \alpha,\beta_{CP} = 0 \) with \( \alpha, \beta = e, \mu, \tau \).

– Define the “imbalances”, i.e. the difference between the linear entropies and their time-reversed expressions:

\[
\Delta S_{L\lambda}^{(\alpha,\beta;\gamma)} = S_{L\lambda}^{(\alpha,\beta;\gamma)}(t) - S_{L\lambda}^{(\alpha,\beta;\gamma)}(-t),
\]

– We have for example:

\[
\Delta S_{Le}^{(e,\mu;\tau)} = 4 \Delta_{CP}^{e,\mu} (|\tilde{U}_{e\tau}(t)|^2 + |\tilde{U}_{\tau e}(t)|^2 - 1),
\]

where the last factor is \( CP \)-even.
The imbalances $\Delta S^{(\alpha,\beta;\gamma)}_{Le}$ as functions of the scaled time $T$. Curves correspond to $\Delta S^{(e,\mu;\tau)}_{Le}$ (long-dashed) and $\Delta S^{(e,\tau;\mu)}_{Le}$ (dot-dashed). The quantity $\Delta S^{(\mu,\tau;e)}_{Le}$ is vanishing.

The $CP$-violating phase is set at the value $\delta = \pi/2$. 
Neutrino oscillations as a resource for quantum information

- Single-particle entanglement encoded in flavor states $|\nu^{(f)}(t)\rangle$ is a real physical resource that can be used, at least in principle, for protocols of quantum information.

- Experimental scheme for the transfer of the flavor entanglement of a neutrino beam into a single-particle system with spatially separated modes.

Charged-current interaction between a neutrino $\nu_\alpha$ with flavor $\alpha$ and a nucleon $N$ gives a lepton $\alpha^-$ and a baryon $X$:

$$\nu_\alpha + N \rightarrow \alpha^- + X.$$
Generation of a single-particle entangled lepton state (two flavors):

In the target the charged-current interaction occurs: \( \nu_\alpha + n \rightarrow \alpha^- + p \) with \( \alpha = e, \mu \).

A spatially nonuniform magnetic field \( \mathbf{B}(r) \) constraints the momentum of the outgoing lepton within a solid angle \( \Omega_i \), and ensures spatial separation between lepton paths.

The reaction produces a superposition of electronic and muonic spatially separated states.
The initial Bell-like superposition is given by $|\nu_\alpha(t)\rangle$. The unitary process associated with the weak interaction leads to the superposition

$$|\alpha(t)\rangle = \Lambda_e |1\rangle_e |0\rangle_\mu + \Lambda_\mu |0\rangle_e |1\rangle_\mu,$$

where $|\Lambda_e|^2 + |\Lambda_\mu|^2 = 1$, and $|\kappa\rangle_\alpha$, with $\kappa = 0, 1$, represents the lepton qubit.

The coefficients $\Lambda_\alpha$ are proportional to $\tilde{U}_{\alpha\beta}(t)$ and to the cross sections associated with the creation of an electron or a muon.

Analogy with single-photon system: quantum uncertainty on the so-called "which path" of the photon at the output of an unbalanced beam splitter $\Leftrightarrow$ uncertainty on the "which flavor" of the produced lepton.

The coefficients $\Lambda_\alpha$ play the role of the transmissivity and of the reflectivity of the beam splitter.
Entanglement for mixed particles in QFT*

– Extension of the above analysis to QFT

– Non-trivial nature of mixing transformations in QFT

– Dynamical symmetry approach to entanglement

– Entropic measures in QFT

*M. Blasone, F. Dell’Anno, F. Illuminati and S. De Siena, work in progress.
Entanglement in relativistic systems

• Necessity* for a treatment of entanglement in the context of Quantum Field Theory.

• Lorentz invariance of entanglement†;

Neutrino mixing in QFT

- Mixing relations for two Dirac fields

\[ \nu_e(x) = \nu_1(x) \cos \theta + \nu_2(x) \sin \theta \]

\[ \nu_\mu(x) = -\nu_1(x) \sin \theta + \nu_2(x) \cos \theta \]

can be written as

\[ \nu^\alpha_e(x) = G_\theta^{-1}(t) \nu_1^\alpha(x) G_\theta(t) \]

\[ \nu^\alpha_\mu(x) = G_\theta^{-1}(t) \nu_2^\alpha(x) G_\theta(t) \]

- Mixing generator:

\[ G_\theta(t) = \exp \left[ \theta \int d^3x \left( \nu_1^\dagger(x)\nu_2(x) - \nu_2^\dagger(x)\nu_1(x) \right) \right] \]

• The vacuum $|0\rangle_{1,2}$ is not invariant under the action of the generator $G_\theta(t)$:

$$|0(t)\rangle_{e,\mu} \equiv G_\theta^{-1}(t) |0\rangle_{1,2} = e^{-\theta(S_+(t)-S_-(t))} |0\rangle_{1,2}$$

• Relation between $|0\rangle_{1,2}$ and $|0(t)\rangle_{e,\mu}$: orthogonality! (for $V \to \infty$)

$$\lim_{V \to \infty} \lim_{1,2} \langle 0 | 0(t) \rangle_{e,\mu} = \lim_{V \to \infty} e V \int \frac{d^3 k}{(2\pi)^3} \ln (1-\sin^2 \theta |V_k|^2)^2 = 0$$

with

$$|V_k|^2 \equiv \sum_{r,s} |v^{r\dagger}_{-k,1} u_{k,2}^s|^2$$

• The “flavor vacuum” $|0(t)\rangle_{e,\mu}$ is a $SU(2)$ generalized coherent state*:

$$|0\rangle_{e,\mu} = \prod_{k,r} \left[ (1 - \sin^2 \theta |V_k|^2) - \epsilon^r \sin \theta \cos \theta |V_k| \left( \alpha^{r\dagger}_{k,1} \beta^{r\dagger}_{-k,2} + \alpha^{r\dagger}_{k,2} \beta^{r\dagger}_{-k,1} \right) \right.$$  

$$+ \epsilon^r \sin^2 \theta |V_k||U_k| \left( \alpha^{r\dagger}_{k,1} \beta^{r\dagger}_{-k,1} - \alpha^{r\dagger}_{k,2} \beta^{r\dagger}_{-k,2} \right) + \sin^2 \theta |V_k|^2 \alpha^{r\dagger}_{k,1} \beta^{r\dagger}_{-k,2} \alpha^{r\dagger}_{k,2} \beta^{r\dagger}_{-k,1} \right] |0\rangle_{1,2}$$

*A. Perelomov, Generalized Coherent States and Their Applications, (Springer V., 1986)
• Structure of the annihilation operators for $|0(t)\rangle_{e,\mu}$:

$$\alpha_{k,e}^{r}(t) = \cos \theta \alpha_{k,1}^{r} + \sin \theta \left( U_{k}^{*}(t) \alpha_{k,2}^{r} + \epsilon^{r} V_{k}(t) \beta_{-k,2}^{r} \right)$$

$$\alpha_{k,\mu}^{r}(t) = \cos \theta \alpha_{k,2}^{r} - \sin \theta \left( U_{k}(t) \alpha_{k,1}^{r} - \epsilon^{r} V_{k}(t) \beta_{-k,1}^{r} \right)$$

$$\beta_{-k,e}^{r}(t) = \cos \theta \beta_{-k,1}^{r} + \sin \theta \left( U_{k}^{*}(t) \beta_{-k,2}^{r} - \epsilon^{r} V_{k}(t) \alpha_{k,2}^{r} \right)$$

$$\beta_{-k,\mu}^{r}(t) = \cos \theta \beta_{-k,2}^{r} - \sin \theta \left( U_{k}(t) \beta_{-k,1}^{r} + \epsilon^{r} V_{k}(t) \alpha_{k,1}^{r} \right)$$

• Mixing transformation = Rotation + Bogoliubov transformation .

- Bogoliubov coefficients:

$$U_{k}(t) = u_{k,2}^{r \dagger} u_{k,1}^{r} e^{i(\omega_{k,2} - \omega_{k,1})t} ; \quad V_{k}(t) = \epsilon^{r} u_{k,1}^{r \dagger} v_{-k,2}^{r} e^{i(\omega_{k,2} + \omega_{k,1})t}$$

$$|U_{k}|^{2} + |V_{k}|^{2} = 1$$
Currents and charges for mixed fermions *

- Lagrangian in the mass basis:

\[ \mathcal{L} = \bar{\Psi}_m (i \not{\partial} - M_d) \Psi_m \]

where \( \Psi^T_m = (\nu_1, \nu_2) \) and \( M_d = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \).

- Lagrangian in the flavor basis:

\[ \mathcal{L} = \bar{\Psi}_f (i \not{\partial} - M) \Psi_f \]

where \( \Psi^T_f = (\nu_e, \nu_\mu) \) and \( M = \begin{pmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{pmatrix} \).

– Two sets of charges:

\[
Q_{\nu_i} = \int d^3x \, \nu_i^\dagger(x) \, \nu_i(x) ; \quad i = 1, 2 \\
Q_{\nu_\sigma}(t) = \int d^3x \, \nu_\sigma^\dagger(x) \, \nu_\sigma(x) ; \quad \sigma = e, \mu
\]

with

\[
Q_{\nu_e}(t) = \cos^2 \theta \, Q_{\nu_1} + \sin^2 \theta \, Q_{\nu_2} + \sin \theta \, \cos \theta \int d^3x \left[ \nu_1^\dagger \nu_2 + \nu_2^\dagger \nu_1 \right]
\]

\[
Q_{\nu_\mu}(t) = \sin^2 \theta \, Q_{\nu_1} + \cos^2 \theta \, Q_{\nu_2} - \sin \theta \, \cos \theta \int d^3x \left[ \nu_1^\dagger \nu_2 + \nu_2^\dagger \nu_1 \right].
\]

• Problem: find the eigenstates of the above charges.
The flavor charge operators are diagonal in the flavor ladder operators:

\[
\begin{align*}
\mathcal{Q}_{\nu_\sigma}(t) & \equiv \int d^3x \, \nu_\sigma(x) \nu_\sigma(x) \\
& = \sum_r \int d^3k \left( \alpha_{k,\sigma}^r(t) \bar{\alpha}_{k,\sigma}^r(t) - \beta_{-k,\sigma}^r(t) \bar{\beta}_{-k,\sigma}^r(t) \right), \quad \sigma = e, \mu.
\end{align*}
\]

Here \(\mathcal{Q} \ldots \mathcal{Q}\) denotes normal ordering with respect to the flavor vacuum:

\[
\begin{align*}
\mathcal{A} & \equiv A - e,\mu \langle 0 | A | 0 \rangle_{e,\mu}
\end{align*}
\]

Define flavor neutrino states with definite momentum and helicity:

\[
| \nu_{k,\sigma}^r \rangle \equiv \alpha_{k,\sigma}^r(0) | 0 \rangle_{e,\mu}
\]

– Such states are eigenstates of the flavor charges (at \(t=0\)):

\[
\begin{align*}
\mathcal{Q}_{\nu_\sigma} | \nu_{k,\sigma}^r \rangle & = | \nu_{k,\sigma}^r \rangle
\end{align*}
\]
Lepton charge violation for Pontecorvo states

- Pontecorvo states:

\[ |\nu^r_{k,e}\rangle_P = \cos \theta |\nu^r_{k,1}\rangle + \sin \theta |\nu^r_{k,2}\rangle \]
\[ |\nu^r_{k,\mu}\rangle_P = -\sin \theta |\nu^r_{k,1}\rangle + \cos \theta |\nu^r_{k,2}\rangle , \]

are not eigenstates of the flavor charges.

\[ \Rightarrow \text{violation of lepton charge conservation} \text{ in the production/detection vertices, at tree level:} \]
\[ P\langle \nu^r_{k,e} : Q_e(0) : |\nu^r_{k,e}\rangle_P = \cos^4 \theta + \sin^4 \theta + 2|U_k| \sin^2 \theta \cos^2 \theta < 1 , \]

for any \( \theta \neq 0, k \neq 0 \) and for \( m_1 \neq m_2 \).

C. C. Nishi, Phys. Rev. D (2008);
- We have, for an electron neutrino state:

\[
Q_{k,\nu_\sigma}(t) \equiv \langle \nu_{k,e}^r \mid \bar{Q} \nu_\sigma(t) \mid \nu_{k,e}^r \rangle
\]

\[
= \left| \{ \alpha_{k,\sigma}^r(t), \alpha_{k,e}^{r\dagger}(0) \} \right|^2 + \left| \{ \beta_{-k,\sigma}^{r\dagger}(t), \alpha_{k,e}^{r\dagger}(0) \} \right|^2
\]

- Neutrino oscillation formula (exact result)*:

\[
Q_{k,\nu_e}(t) = 1 - |U_k|^2 \sin^2(2\theta) \sin^2 \left( \frac{\omega_{k,2} - \omega_{k,1}}{2} t \right) - |V_k|^2 \sin^2(2\theta) \sin^2 \left( \frac{\omega_{k,2} + \omega_{k,1}}{2} t \right)
\]

\[
Q_{k,\nu_\mu}(t) = |U_k|^2 \sin^2(2\theta) \sin^2 \left( \frac{\omega_{k,2} - \omega_{k,1}}{2} t \right) + |V_k|^2 \sin^2(2\theta) \sin^2 \left( \frac{\omega_{k,2} + \omega_{k,1}}{2} t \right)
\]

- For \( k \gg \sqrt{m_1 m_2} \), \( |U_k|^2 \to 1 \) and \( |V_k|^2 \to 0 \).

Dynamical symmetry approach to entanglement

- Entanglement can be characterized by total variance of the operators generating the dynamical algebra.*

- Consider the observables $X_i$ elements of the basis of a Lie algebra $\mathcal{L}$ such that the Lie group $G = \exp(i\mathcal{L})$ defines the dynamic symmetry of the system.

- Entanglement of a state $|\psi\rangle$ is given by the total amount of uncertainty:

$$\Delta(\psi) = \sum_i \left( \langle \psi | X_i^2 | \psi \rangle - \langle \psi | X_i | \psi \rangle^2 \right)$$

Consistency with above QM results:

Define flavor annihilation operators:

\[ \alpha_e(t) = \cos \theta \alpha_1(t) + \sin \theta \alpha_2(t) \]
\[ \alpha_\mu(t) = -\sin \theta \alpha_2(t) + \cos \theta \alpha_1(t) \]

Flavor states:

\[ |\nu_\sigma(t)\rangle \equiv \alpha_\sigma^\dagger(t)|0\rangle_m, \quad \sigma = e, \mu. \]

- Variance of the number operators \( N_i = \alpha_i^\dagger \alpha_i \) and \( N_\sigma(t) = \alpha_\sigma^\dagger(t) \alpha_\sigma(t) \):

- Variance of \( N_i \) \( \Rightarrow \) static entanglement:
  \[ \Delta N_i(\nu_e) = \cos^2 \theta \sin^2 \theta \]

- Variance of \( N_\sigma(t) \) \( \Rightarrow \) flavor entanglement:
  \[ \Delta N_\sigma(\nu_e)(t) = P_{\nu_e\to\nu_e}(t) P_{\nu_e\to\nu_\mu}(t) \]
Entanglement for flavor neutrino states in QFT I

- Entanglement for flavor neutrino states in QFT can be expressed by means of the variances of the neutrino charges: $Q_{\nu_i}$, $Q_{\nu_\sigma}(t)$

- Variance of $Q_{\nu_i}$ → static entanglement:

$$\Delta Q_{\nu_i}(\nu_e) = \langle \nu^r_{k,e} | Q^2_{\nu_i}(t) | \nu^r_{k,e} \rangle - \langle \nu^r_{k,e} | Q_{\nu_i} | \nu^r_{k,e} \rangle^2$$

$$= \cos^2 \theta \sin^2 \theta$$

- Variance of $Q_{\nu_\sigma}$ → flavor entanglement:

$$\Delta Q_{\nu_\sigma}(\nu_e)(t) = \langle \nu^r_{k,e} | Q^2_{\nu_\sigma}(t) | \nu^r_{k,e} \rangle - \langle \nu^r_{k,e} | Q_{\nu_\sigma} | \nu^r_{k,e} \rangle^2$$

$$= Q^k_{\nu_e \rightarrow \nu_e}(t) Q^k_{\nu_e \rightarrow \nu_\mu}(t)$$

in formal agreement with results obtained in QM.

QM vs. QFT flavor entanglement for $|\nu_e(t)\rangle$. 
Entanglement for flavor states in QFT II *

Rewrite QFT flavor neutrino state as:

\[ |\nu_e(t)\rangle = [U_{ee}(t) \alpha^\dagger_e + U_{e\mu}(t) \alpha^\dagger_\mu + U_{e\bar{e}}(t) \alpha^\dagger_e \alpha^\dagger_\mu \beta^\dagger_e + U_{e\bar{\mu}}(t) \alpha^\dagger_e \alpha^\dagger_\mu \beta^\dagger_\mu] |0\rangle_{e,\mu} \]

with

\[
U_{ee}(t) = e^{-i\omega_1 t} \left[ \cos^2 \theta + \sin^2 \theta (e^{-i(\omega_2-\omega_1)t}|U|^2 + e^{-i(\omega_2+\omega_1)t}|V|^2) \right] \\
U_{e\mu}(t) = e^{-i\omega_1 t} U \cos \theta \sin \theta (e^{-i(\omega_2-\omega_1)t} - 1) \\
U_{e\bar{e}}(t) = e^{-i\omega_1 t} V \cos \theta \sin \theta (1 - e^{-i(\omega_2+\omega_1)t}) \\
U_{e\bar{\mu}}(t) = e^{-i\omega_1 t} UV \sin^2 \theta (e^{-i(\omega_2+\omega_1)t} - e^{-i(\omega_2-\omega_1)t}),
\]

\[|U_{ee}(t)|^2 + |U_{e\mu}(t)|^2 + |U_{e\bar{e}}(t)|^2 + |U_{e\bar{\mu}}(t)|^2 = 1.\]

- In QFT, flavor neutrino states exhibit multiparticle components:

\[|\nu_e\rangle = U_{ee}|1000\rangle + U_{e\mu}|0100\rangle + U_{e\bar{e}}|1110\rangle + U_{e\bar{\mu}}|1101\rangle.\]

* M. Blasone, F. Dell’Anno, S. De Siena and F. Illuminati, work in progress
The linear entropies $S_L^{(\alpha;\beta,\gamma,\delta)}$ associated with $|\nu_e\rangle$ are:

$$S_L^{(\nu_e;\nu_{\mu},\bar{\nu}_e,\bar{\nu}_{\mu})} = 4|U_{e\mu}|^2(1 - |U_{e\mu}|^2),$$

(1)

$$S_L^{(\nu_{\mu};\nu_e,\bar{\nu}_e,\bar{\nu}_{\mu})} = 4|U_{e\mu}|^2(1 - |U_{e\mu}|^2),$$

(2)

$$S_L^{(\bar{\nu}_e;\nu_e,\nu_{\mu},\bar{\nu}_{\mu})} = 4|U_{e\mu}|^2(1 - |U_{e\mu}|^2),$$

(3)

$$S_L^{(\bar{\nu}_{\mu};\nu_e,\nu_{\mu},\bar{\nu}_e)} = 4|U_{e\mu}|^2(1 - |U_{e\mu}|^2).$$

(4)

In the quantum mechanical limit, Eqs. (1) and (2) reduce to the Pontecorvo analogs, while Eqs. (3) and (4) go to zero.
Linear entropies: $S^{(\nu_e;\nu_\mu,\bar{\nu}_e,\bar{\nu}_\mu)}_L$ (double-dot-dashed line), $S^{(\nu_\mu,\nu_e,\bar{\nu}_e,\bar{\nu}_\mu)}_L$ (dot-dashed line), $S^{(\bar{\nu}_e,\nu_e,\nu_\mu,\bar{\nu}_\mu)}_L$ (dotted line), $S^{(\bar{\nu}_\mu,\nu_e,\nu_\mu,\bar{\nu}_e)}_L$ (dashed line), and the average linear entropy $\langle S^{(1:3)}_L \rangle$ (full line) as functions of the scaled time $\tau = (\omega_2 - \omega_1)t$. 
• Alternatively, we can quantify entanglement between two single parties, by tracing over other degrees of freedom. The resulting state is a mixed state for which we calculate the concurrence.

\[ C(\alpha; \beta) \]

The concurrences \( C^{(\alpha; \beta)} \) associated with the reduced mixed states \( \rho_{\nu_e \nu_\mu} \), \( C^{(\nu_e; \nu_\mu)} \), full line), and \( \rho_{\nu_e \nu_\mu} \), \( C^{(\nu_e; \nu_\mu)} \), dashed line), as functions of the scaled time \( \tau = (\omega_2 - \omega_1)t \).

The concurrences \( C^{(\nu_e; \nu_e)} \), \( C^{(\nu_e; \nu_\mu)} \), \( C^{(\nu_\mu; \nu_e)} \), and \( C^{(\nu_\mu; \nu_\mu)} \) vanish for every \( \tau \).

– Connection among the above QFT results...
Conclusions

- Elementary particles are produced as entangled states in the SM;
- Quantification of multipartite entanglement for neutrinos and quarks;
- Neutrino oscillations as a resource for quantum information;
- Extension to QFT: entropic measures vs variances.
Condensation density for mixed fermions

- \( V_k = 0 \) when \( m_1 = m_2 \) and/or \( \theta = 0 \).

- Max. at \( k = \sqrt{m_1 m_2} \) with \( V_{\text{max}} \to \frac{1}{2} \) for \( \frac{(m_2 - m_1)^2}{m_1 m_2} \to \infty \).

- \( |V_k|^2 \approx \frac{(m_2 - m_1)^2}{4 k^2} \) for \( k \gg \sqrt{m_1 m_2} \).
Neutrino oscillations (wave packets)*

– Consider, in one dimension, a neutrino with definite flavor, propagating along the $x$ direction:

$$\left| \nu_\alpha(x, t) \right> = \sum_j U_{\alpha,j} \psi_j(x, t) \left| \nu_j \right>,$$

where $U_{\alpha,j}$ is an element of the mixing matrix, $\left| \nu_j \right>$ the mass eigenstate with mass $m_j$, and $\psi_j(x, t)$ its wave function.

– Assume Gaussian distribution $\psi_j(p)$ for the momentum of the massive neutrino $\left| \nu_j \right>$:

$$\psi_j(x, t) = \frac{1}{\sqrt{2\pi}} \int dp \psi_j(p) e^{ipx - iE_j(p)t}, \quad \psi_j(p) = \frac{1}{(2\pi\sigma_p^2)^{1/4}} e^{-\frac{1}{4\sigma_p^2}(p-p_j)^2},$$

where $E_j(p) = \sqrt{p^2 + m_j^2}$.

– The associated density matrix writes:

$$\rho_\alpha(x, t) = \left| \nu_\alpha(x, t) \right> \left< \nu_\alpha(x, t) \right|.$$

If $\sigma_p \ll E_j^2(p_j)/m_j$, one can write $E_j(p) \simeq E_j + v_j(p - p_j)$, with $E_j \equiv \sqrt{p_j^2 + m_j^2}$, and $v_j \equiv \frac{\partial E_j(p)}{\partial p} \bigg|_{p=p_j} = \frac{p_j}{E_j}$ is the group velocity of the wave packet for $\nu_j$.

– In this case, a Gaussian integration yields:

\[
\rho_\alpha(x, t) = \frac{1}{\sqrt{2\pi \sigma_x^2}} \sum_{j,k} U_{\alpha j} U_{\alpha k}^* e^{-i(E_j-E_k)t + i(p_j-p_k)x - \frac{1}{4\sigma_x^2}[(x-v_jt)^2+(x-v_kt)^2]} |\nu_j\rangle \langle \nu_k|,
\]

where \( \sigma_x = (2\sigma_p)^{-1} \). For extremely relativistic neutrinos, one has

\[
E_j \approx E + \xi \frac{m_j^2}{2E}, \quad p_j \approx E - (1 - \xi) \frac{m_j^2}{2E}, \quad v_j \approx 1 - \frac{m_j^2}{2E_j^2}
\]

where \( E \) is the neutrino energy in the limit of zero mass, and \( \xi \) a dimensionless constant depending on the characteristic of the production process.

– The density matrix \( \rho_\alpha(x, t) \) provides a space-time description of neutrino dynamics.

– In realistic situations, it is convenient to consider the time-independent density matrix \( \rho_\alpha(x) \) obtained by the time average of \( \rho_\alpha(x, t) \):

\[
\rho_\alpha(x) = \sum_{j,k} U_{\alpha j} U_{\alpha k}^* \exp \left[ -i \frac{\Delta m_{jk}^2 x}{2E} - \left( \frac{\Delta m_{jk}^2 x}{4\sqrt{2E^2\sigma_x}} \right)^2 - \left( \xi \frac{\Delta m_{jk}^2}{4\sqrt{2E\sigma_p}} \right)^2 \right] |\nu_j\rangle \langle \nu_k|,
\]

with \( \Delta m_{jk}^2 = m_j^2 - m_k^2 \).
Flavor oscillations in space.

\[ P_{\nu_e \rightarrow \nu_e}(x) \simeq 1 - \frac{1}{2} \sin^2(2\theta) \left\{ 1 - \cos \left(2\pi \frac{x}{L^{\text{osc}}} \right) \exp \left[ - \left( \frac{x}{L^{\text{coh}}} \right)^2 - 2\pi^2 \left( \frac{\sigma_x}{L^{\text{osc}}} \right)^2 \right] \right\} \]

- Oscillation length: \( L^{\text{osc}} = \frac{4\pi p}{\Delta m^2} \)
- Coherence length: \( L^{\text{coh}} = \frac{L^{\text{osc}} p}{\sqrt{2\pi} \sigma_p} \).
Decoherence in neutrino oscillations

- We analyze the coherence of the quantum superposition of the neutrino mass eigenstates, by looking at the spatial behavior of the multipartite entanglement of the above state.

By means of the identification $|\nu_i\rangle = |\delta_{i,1}\rangle_1|\delta_{i,2}\rangle_2|\delta_{i,3}\rangle_3$, with $i = 1, 2, 3$, we construct the matrix with elements

$$\langle l m n | \rho_\alpha(x) | i j k \rangle,$$

where $i, j, k, l, m, n = 0, 1$

- We analytically compute logarithmic negativities $E_{N_\alpha}^{(i,j;k)}$, for $i, j, k = 1, 2, 3$ and $i \neq j \neq k$, and average logarithmic negativity $\langle E_{N_\alpha}^{(2:1)} \rangle$, for the neutrino states with flavor $\alpha = e, \mu, \tau$.

We assume for the mixing angles the experimental values

\[
\sin^2 \theta_{12}^{MNSP} = 0.314(1^{+0.18}_{-0.15}), \quad \sin^2 \theta_{13}^{MNSP} = (0.8^{+2.3}_{-0.8}) \times 10^{-2}, \quad \sin^2 \theta_{23}^{MNSP} = 0.45(1^{+0.35}_{-0.20})
\]

The squared mass differences are fixed at the experimental values*

\[
\Delta m_{21}^2 = \delta m^2, \quad \Delta m_{31}^2 = \Delta m^2 + \frac{\delta m^2}{2}, \quad \Delta m_{32}^2 = \Delta m^2 - \frac{\delta m^2}{2},
\]

\[
\delta m^2 = 7.92 \times 10^{-5} \text{eV}^2, \quad \delta m^2 = 2.6 \times 10^{-3} \text{eV}^2.
\]

We take \( E = 10 \text{GeV} \) and \( \sigma_p = 1 \text{GeV} \). The parameter \( \xi \) is put to zero for simplicity.

Logarithmic negativities $E_{Ne}^{(i,j;k)}$ for all possible bipartitions and average logarithmic negativity $\langle E_{Ne}^{(2:1)} \rangle$ (solid line) as functions of the distance $x$ (meters).

In panel II we plot a zoom of $E_{Ne}^{(1,3;2)}$ and $E_{Ne}^{(2,3;1)}$

All plotted quantities are independent of the CP-violating phase $\delta$. 
Logarithmic negativities $E_{N\alpha}^{(i,j;k)}$ for all possible bipartitions and average logarithmic negativity $\langle E_{N\alpha}^{(2:1)} \rangle$ (solid line), with $\alpha = \mu, \tau$, as functions of the distance $x$ (meters).

The CP-violating phase $\delta$ is put to zero. The $x$ axis is in logarithmic scale, and the dimensions are meters.
Logarithmic negativities $E_{N\mu}^{(1,3;2)}$ (panel I) and $E_{N\mu}^{(2,3;1)}$ (panel II) as functions of the distance $x$ (meters) for different choices of the CP-violating phase $\delta$: (a) $\delta = 0$ (dotted line); (b) $\delta = \frac{\pi}{2}$ (dashed line); (b) $\delta = \pi$ (dot-dashed line). $E_{N\mu}^{(1,2;3)}$ is independent of $\delta$. 

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