Flavour symmetries in the symmetric limit
(and the neutrino normal hierarchy)

Andrea Romanino, SISSA

Reyimuaji, R 1801.10530, JHEP
Reyimuaji, R to appear
The content of this talk
**Q1**: can a flavour symmetry constraining light neutrino Majorana masses provide an approximate description of lepton flavour in the symmetric limit?

**A1**: yes, but only if neutrinos are inverted hierarchical or unconstrained (anarchical)

If NH is confirmed, symmetry breaking must play a leading role in the understanding of lepton flavour
Light neutrino majorana masses may originate from high-scale physics

**Q2:** is studying the flavour symmetry at low-scale equivalent to studying it at high-scale?

**A2:** not necessarily

Necessary and sufficient conditions for the equivalence in the case of see-saw I
Q3: can a flavour symmetry constraining a see-saw lagrangian provide an approximate description of lepton flavour in the symmetric limit?

A3: yes, and neutrinos can be normally hierarchical if the high-scale and low-scale actions of the flavour symmetry are not equivalent
Introduction: flavour symmetries
The flavour puzzle in the SM

- 3 families $\leftrightarrow U(3)^5$ symmetry of the gauge lagrangian

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l$</td>
<td>$l_1$</td>
<td>$l_2$</td>
<td>$l_3$</td>
</tr>
<tr>
<td>$e^c$</td>
<td>$(e^c)_1$</td>
<td>$(e^c)_2$</td>
<td>$(e^c)_3$</td>
</tr>
<tr>
<td>$q$</td>
<td>$q_1$</td>
<td>$q_2$</td>
<td>$q_3$</td>
</tr>
<tr>
<td>$u^c$</td>
<td>$(u^c)_1$</td>
<td>$(u^c)_2$</td>
<td>$(u^c)_3$</td>
</tr>
<tr>
<td>$d^c$</td>
<td>$(d^c)_1$</td>
<td>$(d^c)_2$</td>
<td>$(d^c)_3$</td>
</tr>
</tbody>
</table>

Gauge irreps (vertical) understood?  
Family number (horizontal) not understood
The flavour puzzle in the SM

- 3 families $\leftrightarrow U(3)^5$ symmetry of the gauge lagrangian

- Charged fermions: $m_1 \ll m_2 \ll m_3$
  Quarks: $V_{\text{CKM}} \sim 1$

- Neutrinos: lighter, milder hierarchy, $U_{\text{PMNS}} \neq 1$
Smallness of neutrino masses and high scales

\[ \mathcal{L}_{E \ll \Lambda}^{\text{eff}} = \mathcal{L}_{\text{SM}}^{\text{ren}} + \frac{h_{ij}}{2\Lambda} (l_i h) (l_j h) + \text{h.c.} \]

\[ m_{u,d,e} = \lambda_{u,d,e} v \quad m_\nu = h v \times \frac{v}{\Lambda} \]

\[ \Lambda \sim (0.5 \cdot 10^{15} \text{ GeV}) h \left( \frac{0.05 \text{ eV}}{m_\nu} \right) \]

This is the framework we consider
The results can be extended to other set-ups
Flavour symmetries

- $G_f$ flavour group acting on “i”, $\mathcal{L}$ invariant

\[ \mathcal{L}_f = \lambda_{ij} \psi_i \psi_j h + \frac{\lambda_{ijk}}{\Lambda_f} \phi_k \psi_i \psi_j h + \ldots \]

$\downarrow$

\[ m_{ij}^0 = \lambda_{ij} v \]

$\downarrow$

\[ m_{ij}^1 = \lambda_{ijk} \frac{\langle \phi_k \rangle}{\Lambda} v + \ldots \]

\[ m_{ij} = m_{ij}^0 + m_{ij}^1 \]

$\Phi_k$ scalar “flavon”
(SM invariant)
spontaneously breaks $G_f$

vanishes for $\phi_k = 0$
“symmetric limit”
Q1: can a flavour symmetry constraining light neutrino Majorana masses provide an approximate description of lepton flavour in the symmetric limit?
Flavour group

- $G_f$ flavour group

- any: discrete/continuous, abelian/non-abelian, global/gauge, etc

- includes all “hidden” factors

- unitary representation, commuting with Poincaré and $G_{SM}$

Flavour representation

$$g \in G_f : \begin{cases} 
  l_i & \rightarrow & U_l(g)_{ij} l_j \\
  e_i^c & \rightarrow & U_{e^c}(g)_{ij} e_j^c 
\end{cases} \quad e^c \leftrightarrow e_R$$
Invariant lagrangian, \( <\phi> = 0 \) (low-scale)

\[
\mathcal{L}_{\text{low-scale}}^{(0)} = \lambda^E_{ij} e_i^c l_j h^\ast + \frac{c_{ij}}{2\Lambda} l_i l_j h h \\
\rightarrow m^0_{ij} e_i^c e_j + \frac{m^0_{ij}}{2} \nu_i \nu_j
\]
Symmetric limit

\[
U_{EC}(g)^T m^0_E U_l(g) = m^0_E \\
U_l(g)^T m^0_\nu U_l(g) = m^0_\nu
\]

(from the invariance of the lagrangian)

Symmetry breaking

\[
m_E = m^0_E + m^1_E \\
m_\nu = m^0_\nu + m^1_\nu
\]

invariant under $G_f$  
not invariant under $G_f$ generated by $\phi$
The symmetric limit provides an approximate description of lepton flavour

- \( m_E \neq 0 \) and \( m_\nu \neq 0 \)

\[
\begin{align*}
  m_E & = m_0^E + m_1^E \\
  m_\nu & = m_0^\nu + m_1^\nu
\end{align*}
\]

approximate description of lepton observables

moderate correction necessary for an accurate description

\[
\begin{align*}
  m_D^0 & = \begin{pmatrix} 0 & 0 \\ 0 & m_b \end{pmatrix} \\
  m_U^0 & = \begin{pmatrix} 0 & 0 \\ 0 & m_t \end{pmatrix} \\
  m_E^0 & = \begin{pmatrix} 0 & 0 \\ 0 & m_\tau \end{pmatrix} \\
  V_{\text{CKM}}^0 & = \begin{pmatrix} 1 \\ \theta_C \text{ undetermined} \end{pmatrix}
\end{align*}
\]
The LO pattern of lepton flavour is determined by symmetry breaking

- e.g. if $m_E = 0$ or $m_\nu = 0$

\[
\begin{align*}
    m_E &= m^0_E + m^1_E \\
    m_\nu &= m^0_\nu + m^1_\nu
\end{align*}
\]

- e.g. $m_E = 0$ or $m_\nu = 0$ in the symmetric limit
- fully determine the PMNS matrix

\[
\begin{pmatrix}
    a & 0 & 0 \\
    0 & 0 & a \\
    0 & a & 0
\end{pmatrix}
\quad
\begin{pmatrix}
    0 & 0 & 0 \\
    0 & 0 & 0 \\
    0 & 0 & 0
\end{pmatrix}
\]

- e.g. $G = A_4$

- $m^0_\nu$: $H_1$ invariant
- $m^1_\nu$: $H_2$ invariant
- $m^0_E$: $H_2$ invariant
- $m^1_E$: $H_2$ invariant
The symmetric limit provides an approximate description of lepton flavour

- \( m_E \neq 0 \) and \( m_\nu \neq 0 \)

\[
\begin{align*}
    m_E &= m^0_E + m^1_E \\
    m_\nu &= m^0_\nu + m^1_\nu
\end{align*}
\]

approximate description of lepton observables

<table>
<thead>
<tr>
<th>Neutrino masses</th>
<th>Charged lepton masses</th>
<th>PMNS matrix</th>
</tr>
</thead>
</table>
| NH/IH | (a 0 0) (0 a a) | (A 0 0) | \[
\begin{pmatrix}
X & X & 0 \\
X & X & X \\
X & X & X
\end{pmatrix}
\]
| NH or IH | (a a a) (a b 0) | (A B 0) | or \[
\begin{pmatrix}
X & X & X \\
X & X & X \\
X & X & X
\end{pmatrix}
\]
| IH | (a b b) (a b c) | (A B C) | \((X \neq 0 \text{ generic})\)
$G_f U_1 U_e$ leading, in the symmetric limit, to lepton masses and mixings in the above form

- A complete and concise classification is possible, as the predictions in the symmetric limit only depend on the structure of the decomposition of the representations in irreducible components (irreps) and in particular on their
  - **Type** (real, pseudoreal, complex)
  - **Dimension**
  - **Equivalence**
  - **Notation**
    - “n”: dimension n complex or pseudoreal irrep
    - “n”: dimension n real irrep
    - “n, n’, n””: dimension n inequivalent irreps
$G_f U_l U_e$ leading, in the symmetric limit, to lepton masses and mixings in the above form

- Only 6 cases
- Only $d = 1$ (abelian) irreps and no pseudoreal irreps (except possibly if $m_e, \mu = 0$)
- Neutrinos are either unconstrained (anarchical) or inverted hierarchical
- If NH confirmed, lepton flavour at low-scale can only be accounted for by SB
$1+1+1$

- “1” = real one-dimensional: $f \rightarrow \pm f$
- $1+1+1$: $U(g)_{ij} = \pm 1_{ij}$
- any $m_\nu$ is trivially invariant
- neutrino masses and mixing completely unconstrained
- (anarchy)
SU(5) and SO(10)

- **SU(5)**: assume $U_5 = U_l$ and $U_{10} = U_{\text{ec}}$, require $(V_{\text{CKM}})_0 = 1$ or $V_{12}$
  - only unconstrained (anarchical) neutrinos are allowed

- **SO(10)**: assume $U_l = U_{\text{ec}} = U_{16}$
  - no solutions
Features of the proof

• in 2 steps: masses first, then mixings

• no need to write down any mass matrix, texture: the flavour pattern is directly determined by the irrep decomposition

• in particular, the form of the PMNS matrix is given by

$$U = H_E P_E V D^{-1} P_{\nu}^{-1} H_{\nu}^{-1}$$

• $V$ commutes with $U_I$

• $D$ maximal rotation, if $U_I$ contains conjugated complex irreps (Dirac substructure)

• $P$ permutations possibly needed to bring mass eigenvalues in standard ordering

• $H$ ambiguity in case of degeneracies (fixed by symmetry breaking effects)
Q2: is studying the flavour symmetry at low-scale equivalent to studying it at high-scale?

[to appear]
Origin of lepton masses (high-scale)

\[ \mathcal{L}_{\text{low-scale}} = \lambda_{ij}^E e_i^c l_j h^* + \frac{c_{ij}}{2\Lambda} l_i l_j h h \]

from

Flavour representation

\[
\begin{align*}
\text{high scale} & \quad \left\{ \begin{array}{c}
  l_i \rightarrow U_l(g)_{ij} l_j \\
  e_i^c \rightarrow U_{e^c}(g)_{ij} e_j^c \\
  \nu_i^c \rightarrow U_{\nu^c}(g)_{ij} \nu_j^c
\end{array} \right. \\
\text{low scale version}
\end{align*}
\]

Equivalent, at least in the symmetric limit?
Equivalence of high and low-scale representations (in the symmetric limit)

• By definition, when for each invariant \( m_v \) there exists invariant \( m_N \) and \( M \) such that \( m_v = - m_N^T M^{-1} m_N \) (converse is always true)

• Given a low-scale representation does an equivalent high-scale version always exists? **YES**

• Is the low-scale version of a representation always equivalent to the high-scale version? **NO**

• **Necessary and sufficient conditions** for LS to be equivalent to HS:
  1. \( U_{vc} \) vectorlike real, or pairs of complex conjugated, or pairs of equivalent pseudoreal
  2. The vectorlike part of \( U_l \) is contained in \( U_{vc} \)
1. $U_\nu^c$ is not vectorlike

- $U_\nu^c$ not vectorlike $\Leftrightarrow$ M forced to be *singular* in the symmetric limit: the see-saw formula does not apply

- Example:

<table>
<thead>
<tr>
<th>low-scale</th>
<th>high-scale</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_l = 1+1+1$</td>
<td>$m_{ei} = (A \ 0 \ 0)$</td>
</tr>
<tr>
<td>$U_{e^c} = \bar{1}+1+1$</td>
<td>$m_{vi} = (a \ 0 \ 0)$</td>
</tr>
<tr>
<td>$U_{\nu^c} = \bar{1}$ + real</td>
<td>$U = \begin{pmatrix} ? &amp; ? &amp; X \ X &amp; X &amp; ? \ X &amp; X &amp; 0 \end{pmatrix}$</td>
</tr>
<tr>
<td>$m_E = \begin{pmatrix} X &amp; X \ X &amp; X \end{pmatrix}$</td>
<td>$m_N = \begin{pmatrix} X &amp; X \ X &amp; X \end{pmatrix}$</td>
</tr>
<tr>
<td>$m_\nu = \begin{pmatrix} X \ X \end{pmatrix}$</td>
<td>$M = \begin{pmatrix} X &amp; X \ X &amp; X \end{pmatrix}$</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

*single RH neutrino dominance*
2. \( \mathbf{U}_{\nu c} \) is vectorlike but the vectorlike part of \( \mathbf{U}_l \) is not contained in \( \mathbf{U}_{\nu c} \)

- Example:

\[
\begin{align*}
\mathbf{U}_l &= 1+1+1 \\
\mathbf{U}_{\nu c} &= 1+\bar{1}+1
\end{align*}
\]

\[
\begin{align*}
\mathbf{m}_{\nu i} &= \begin{pmatrix} a & b & 0 \end{pmatrix} \\
\mathbf{m}_{\nu i} &= \begin{pmatrix} 1+1+1 \end{pmatrix} \\
\mathbf{m}_{\nu i} &= \begin{pmatrix} 1+\bar{1}+1 \end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\mathbf{m}_E &= \begin{pmatrix} X & X & 0 \\ X & X & 0 \\ X & X & 0 \\ X & X & 0 \end{pmatrix} \\
\mathbf{m}_N &= \begin{pmatrix} X & X & X \\ X & X & X \\ X & X & X \\ X & X & X \end{pmatrix} \\
\mathbf{m}_E &= \begin{pmatrix} X & X \\ X & X \end{pmatrix} \\
\mathbf{m}_N &= \begin{pmatrix} X & X \\ X & X \end{pmatrix} \\
\mathbf{M} &= \begin{pmatrix} X & X \\ X & X \end{pmatrix} \\
\mathbf{m}_N &= \begin{pmatrix} X & X \\ X & X \end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\mathbf{V} &= \begin{pmatrix} X & X & X \\ ? & X & X \\ 0 & X & X \end{pmatrix} \\
\mathbf{V} &= \begin{pmatrix} X & X & X \\ X & X & ? \\ X & X & ? \end{pmatrix}
\end{align*}
\]
Q3: can a flavour symmetry constraining a see-saw lagrangian provide an approximate description of lepton flavour in the symmetric limit?
• If $U_{\nu c}$ vectorlike and the vectorlike part of $U_l$ is contained in $U_{\nu c}$: yes, at the same conditions as in the low-scale analysis

• If instead the low- and high-scale analyses are not equivalent, predictive (non-unconstrained) cases corresponding to NH can be found

• The complete list of solutions can be again found based only on the structure of the irrep decompositions
Conclusions

• The complete set of lepton flavour predictions of any flavour group and representation in the symmetric limit has been found, both at low scale (Weinberg operator) and high scale (see-saw). The predictions only depend on the type, dimension, and equivalence of the irrep components.

• In the low-scale case: the symmetric limit is close to what observed only if neutrinos are unconstrained (anarchical) or inverted hierarchical.

• If the present hint for normal hierarchy was confirmed, we would conclude that symmetry breaking plays a leading order role in constraining lepton flavour observables at low scale.

• In the high-scale case: the results do not change, except when the low- and high-scale analyses are not equivalent. The conditions for equivalence have been found.

• The complete set of additional predictions in the symmetric limit that can obtained at high-scale has been found. A normal hierarchy for the neutrinos can be obtained.

• If the present hint for normal hierarchy was confirmed, a predictive symmetric limit could be close to what observed only because the low- and high-scale actions of the flavour symmetry are not equivalent. Otherwise, symmetry breaking effects necessary play a leading order role in determining lepton flavour observables.