Extended Neutrino Sphere effects on Supernova $\nu$ Oscillations

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Impact of neutrino oscillations

- Supernova explosion mechanism:

\[ E(\nu_{\mu/\tau}) > E(\nu_e) \]

- Modify the neutrino signal:

\[ \nu_e \rightarrow \nu_{\mu/\tau} \quad \nu_{\mu/\tau} \rightarrow \nu_e \]

- Nucleosynthesis depends on the neutron fraction:

\[ p + \bar{\nu}_e \rightarrow n + e^+ \quad n + \nu_e \rightarrow p + e^- \]
Matter effect

Electron background shifts the energy eigenvalues.

\[ H = \frac{\Delta m^2}{2E} \begin{pmatrix} -c_{2\theta} & s_{2\theta} \\ s_{2\theta} & c_{2\theta} \end{pmatrix} + \sqrt{2} G_F n_e \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \]

\[ m_{2,\text{eff.}} \sim n_e \]

\[ m_{1,\text{eff.}} \sim \nu_x \]

\[ m_{\text{vacuum}} \]

\[ m_{\text{solar centre}} \]
Density matrix formalism

In the mean field approximation:

\[
\rho = \begin{pmatrix} \rho_{ee} & \rho_{ex} \\ \rho_{xe} & \rho_{xx} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} P_0 + P_z & P_x - iP_y \\ P_x + iP_y & P_0 - P_z \end{pmatrix} = \frac{1}{2} (P_0 + \vec{P} \cdot \vec{\sigma}).
\]

Similarly for the Hamiltonian:

\[
H = \frac{1}{2} \begin{pmatrix} V_z & V_x - iV_y \\ V_x + iV_y & -V_z \end{pmatrix} = \frac{1}{2} \vec{V} \cdot \vec{\sigma}.
\]

Equation of motion (in absence of collisions):

\[
i \dot{\rho} = [H, \rho] \quad \Leftrightarrow \quad \dot{\vec{P}} = \vec{V} \times \vec{P}
\]
Collective oscillations

Normal oscillations:

\[ \text{prob}(\nu_e \rightarrow \nu_e) \propto \cos^2(\Delta m^2 L/4E). \]

Neutrino background:

\[ H_{\nu\nu} = \sqrt{2} G_F \int d\rho (\rho - \bar{\rho}). \]

Conversion independent of \( E \).

Non-linear problem \( \rightarrow \) hard to solve in a realistic setting.

Hannestad et al. 2006
Observation
Considering the individual neutrino, its oscillations in a SN is a linear problem.

Idea
How much of the neutrino-neutrino refraction can we describe using linear equations.

Ultimate goal
General conclusions about the behaviour of neutrino oscillations in presence of neutrino-neutrino refraction.

Methods
- Solve equations from first principles, analytic and numeric.
- Describe complicated systems using effective potentials.
General equations

Probe neutrino in arbitrary neutrino and matter background.

\[ H^{(p)} = \frac{1}{2} \left( -c_{2\theta} \omega_p + V_e + V_{\nu} \quad s_{2\theta} \omega_p + 2 \bar{V}_{\nu} e^{i\phi_B} \right) \left( s_{2\theta} \omega_p + 2 \bar{V}_{\nu} e^{-i\phi_B} \right) c_{2\theta} \omega_p - V_e - V_{\nu} , \quad (1) \]

\[ V_{\nu} = \int d\mathbf{k} V_{\nu}^0(\mathbf{k}) \left[ \rho_{ee}(\mathbf{k}) - \rho_{\tau\tau}(\mathbf{k}) \right] , \]

\[ \bar{V}_{\nu} e^{i\phi_B} = \int d\mathbf{k} V_{\nu}^0(\mathbf{k}) \rho_{e\tau}(\mathbf{k}) \]

\[ V_{\nu}^0(\mathbf{k}) = \sqrt{2} G_F n(\mathbf{k}) \left( 1 - \mathbf{v}_{bg} \cdot \mathbf{v}_p \right) \]
General equations - rotated

Rewrite the off-diagonal as $V'e^{i\phi'} = s_{2\theta}\omega_p + 2\tilde{V}_\nu e^{i\phi_B}$. Apply the transformation $U = \text{diag}\left(e^{i\phi'/2}, e^{-i\phi'/2}\right)$.

$$H^{(p)} = \frac{1}{2} \begin{pmatrix} V^r & V' \\ V' & -V^r \end{pmatrix},$$

where

$$V' = \sqrt{4\tilde{V}_\nu^2 + 4s_{2\theta}\omega_p \cos\phi_B \tilde{V}_\nu + s_{2\theta}^2\omega_p^2},$$

$$V^r = V_e + V_\nu + \dot{\phi}' - c_{2\theta}\omega_p.$$
Conditions for a large conversion

Our Ansatz is that every case where a large conversion happens can be described in terms of at least one of these frameworks:

1. Resonance
   Vanishing diagonal:
   \[ V_e + V_\nu + \dot{\phi}' - c_{2\theta}\omega_p = 0. \]

2. Adiabatic conversion
   Fast oscillations in \( V' \) and \( \phi' \) can be removed by going to a rotating frame. This can result in a Hamiltonian describing adiabatic evolution.

3. Parametric enhancement
   Present if the period of oscillation equals the period of change of mixing angle.
Energy spectrum

$P_{er}$

- $\omega_k = 0.5\omega_p$
- $\omega_k = \omega_p$
- $\omega_k = 1.5\omega_p$
- Box spectrum
- Analytic

RSLH and Smirnov, 2018
Neutrino emission

\( \nu_\mu/\tau \) decouple first, then \( \bar{\nu}_e \) and last \( \nu_e \).

\( \nu_\mu/\tau \) have:

Number sphere:

\[ e^+ e^- \rightarrow \nu_\mu/\tau \bar{\nu}_\mu/\tau \]

Energy sphere:

\[ e \nu_\mu/\tau \rightarrow e \nu_\mu/\tau \]

Transport sphere:

\[ N \nu_\mu/\tau \rightarrow N \nu_\mu/\tau \]
Neutrino emission

$\nu_{\mu/\tau}$ decouple first, then $\bar{\nu}_e$ and last $\nu_e$.

$\nu_{\mu/\tau}$ have:
Number sphere:
\[ e^+ e^- \rightarrow \nu_{\mu/\tau} \bar{\nu}_{\mu/\tau} \]

Energy sphere:
\[ e \nu_{\mu/\tau} \rightarrow e \nu_{\mu/\tau} \]

Transport sphere:
\[ N \nu_{\mu/\tau} \rightarrow N \nu_{\mu/\tau} \]

$\bar{\nu}_e$ and $\nu_e$ are dominated by absorption and emission from nucleons. ($n_n > n_p$)

Mean free path:
\[ \frac{1}{\lambda} \approx \frac{G_F^2}{\pi} (g_V^2 + 3g_A^2) E^2 n_N. \]

Emissivity:
\[ j = \frac{1}{\lambda} \exp(-E/T) \]
Extended source

Rough estimates:

- Neutrino sphere: $\sim 10\text{km}$.
- Width of neutrino sphere: $\sim 1\text{km}$.
- Oscillation length: $\sim \frac{1}{G_F n_e} \sim 10^{-8} – 10^{-7}\text{km}$.

Average over emission region suppresses oscillatory terms by $10^7 – 10^8$.
Parametric resonance is not removed as such.
Extended source - non-linear

\[ \rho_{ee} = \frac{1}{2}(1 + P_z) \]

\[ \Delta z = 0 \]
\[ \Delta z = \frac{3\pi \sin \beta}{\lambda} \]
\[ \Delta z = \frac{7\pi \sin \beta}{\lambda} \]
\[ \Delta z = \frac{21\pi \sin \beta}{\lambda} \]

\[ \text{emission region} \]

\[ |\rho_{ex} - H_{ex}| \]

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Extended source - non-linear
Changing background - only matter

Coordinate system with \( \vec{V} \) along \( z \)-axis:

\[
\rho_{12}(r) = \rho_{12,\text{initial}} \exp \left( i \int_{r_{e}}^{r} \omega_{m}(r') dr' \right).
\]

Average over emission point:

\[
\langle \rho_{12}(r) \rangle = \int_{0}^{r} p(r_{e}) \frac{1}{2} \sin 2\theta_{m}(r_{e}) \exp \left( i \int_{r_{e}}^{r} \omega_{m}(r') dr' \right) dr_{e},
\]

where

\[
p(r_{e}) = \frac{N(r_{e})}{N_{0}}
\]

\[
N(r_{e}) = \frac{1}{\lambda(r_{e})} \exp(-E/T) \exp \left( - \int_{r_{e}}^{\infty} \frac{1}{\lambda(r')} dr' \right), \quad N_{0} = \int dr_{e} N(r_{e}).
\]
Changing background - only matter

\[ p(r) = \delta(r - r_{\text{sphere}}) \]

\[ \langle \rho_{12} \rangle \]

\[ p \sin(2\theta) \times 10^{-10} \]

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Changing background - only matter

Include effect of damping, $D$:

$$\ddot{\mathbf{P}} = \mathbf{V} \times \dot{\mathbf{P}} - D \mathbf{P}_T.$$  

For $V_z$ and $D$ large and $P_z \approx 1$:

$$P_x \approx \frac{V_x V_z}{V_z^2 + D^2}, \quad P_y \approx \frac{-V_x D}{V_z^2 + D^2}.$$

Bell et al. 1998, Hannestad et al. 2012
Effect of non-adiabaticity

Solve $\dot{\vec{P}} = \vec{V} \times \vec{P} - D\vec{P}_T$ numerically:

Non-adiabatic effects for $D=0$:

$$\vec{P} = \left( \frac{V_x}{V_z} + \frac{V_x \partial_r V_z}{V_z^3} \sin \left( \int V_z(r')dr' \right) \right. \left. \frac{V_x \partial_r V_z}{V_z^3} \left( 1 - \cos \left( \int V_z(r')dr' \right) \right) \left( 1 + \frac{V_x^2 \partial_r V_z}{V_z^4} \sin \left( \int V_z(r')dr' \right) \right) \right) \right) \approx \frac{V_x \partial_r V_z}{2V_z^3} \approx \frac{V_x}{2V_z^2 r_0}$$

$|\rho_{12}| \approx \left| \frac{V_x \partial_r V_z}{2V_z^3} \right| \approx \left| \frac{V_x}{2V_z^2 r_0} \right|$
Linear stability analysis Banerjee, Dighe, Raffelt 2011

Linear analysis demonstrate stability or instability.

$$\rho = \frac{f_{\nu_e} + f_{\nu_x}}{2} + \frac{f_{\nu_e} - f_{\nu_x}}{2} \begin{pmatrix} s & S \\ S^* & -s \end{pmatrix}$$

$$|S| = |P_x + iP_y| \ll 1, \quad s^2 + S^2 = 1 \Rightarrow s = P_z \approx 1.$$  

Linearised equation:

$$i\dot{S} = (\omega + \lambda + \mu)S - \mu \int d\Gamma'(1 - v \cdot v')S',$$
Linear stability analysis

In Fourier space: \( S = e^{-i\Omega t} Q \)

\[ \Omega Q = (\omega + \lambda + \mu)Q - \mu \int d\Gamma' (1 - v \cdot v') Q' \]

Unstable if \( \text{Im}(\Omega) \neq 0 \). (See also Capozzi et al. 2017)

Discrete modes: solve matrix equation.

(Continuous modes: Decompose in independent functions.)

Can also be formulated as a dispersion relation. (Izaguirre, Raffelt and Tamborra, 2016)
Simple model

Linearised equation: ($\omega = -1, \lambda = 30, \mu = 3$)

$$i \dot{\vec{S}} = \begin{pmatrix} -\omega + \lambda - \mu & \mu \\ -\mu & \omega + \lambda + \mu \end{pmatrix} \vec{S}$$

Eigenvalues:

$$\Omega = \lambda \pm \sqrt{\omega (2\mu + \omega)}$$

Growth rate = $\text{Im}(\Omega)$.

Emission point = $\frac{2}{3} \Delta z$.

Start value = $\frac{1}{\Delta z \lambda} \sin 2\theta_m$.

Does not work for large $\Delta z$. 
Large $\Delta z$

- Take into account that $\mu$ changes inside the emission region.
- $\tilde{\mu}(z) = \mu \frac{z}{\Delta z}$ is so small for small $z$ that no instability exists.

The fastest possible growth that one can expect is:

$$S = \frac{1}{2} \frac{\mu \sin 2\theta_m}{\Delta z V_e}$$

$$\exp \left( \frac{\Delta z}{3\mu} \left( 2\mu \frac{z}{\Delta z} - 1 \right)^{3/2} \right).$$
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Very large $\Delta z$
Summary

- The non-negligible width of the neutrino sphere affects the neutrino state due to averaging over different emission points.

- The angle between the neutrino state and the Hamiltonian in polarization space is reduced by a factor $10^8$ at the neutrino sphere by the averaging.

- A small adiabaticity violation increases the angle significantly as the neutrino propagates out through the supernova.

- The onset of neutrino conversion can be analysed using linear stability analysis, and for a given model, it can be calculated if conversion has the potentially to occur.
Summary

- The non-negligible width of the neutrino sphere affects the neutrino state due to averaging over different emission points.

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Thank you for your attention!
Neutrino emission

Deleptonisation  Accretion  Cooling

Lang et al. 2016
Neutrino mixing

\[ \mathcal{L}_{\text{int}} = \frac{g}{\sqrt{2}} W^+_\mu \bar{\nu}_L \gamma^\mu l_L + \text{h.c.} \]

Interaction states and mass states are different:

\[ \nu^i = U \nu^i. \]

Mixing matrix:

\[
U = \begin{bmatrix}
1 & 0 & 0 \\
0 & c_{23} & s_{23} \\
0 & -s_{23} & c_{23}
\end{bmatrix}
\begin{bmatrix}
c_{13} & 0 & s_{13} e^{-i\delta_{CP}} \\
0 & 1 & 0 \\
-s_{13} e^{-i\delta_{CP}} & 0 & c_{13}
\end{bmatrix}
\begin{bmatrix}
c_{12} & s_{12} & 0 \\
-s_{12} & c_{12} & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]
Collective oscillations

Can collective oscillations still occur? YES!
Multiple angles, in-homogeneous

Linear stability analysis of a more realistic model:

$$(\Omega + \mathbf{v} \cdot \mathbf{k})Q = (\omega + \lambda + \mu(\epsilon - \mathbf{v} \cdot \phi))Q - \mu \int d\Gamma' (1 - \mathbf{v} \cdot \mathbf{v}') g' Q'$$

- $\mu$ and $\lambda$ functions of $r$.
- Multi angle matter effect.
- Homogeneous mode: $k = 0$

Chakraborty, RSLH, Izaguirre and Raffelt, 2015
Very fast flavour conversion  

R. F. Sawyer

Chakraborty, RSLH, Izaguirre and Raffelt, 2016

Conversion on meter-scale.
Can also occur in a supernova.  
Dasgupta, Mirizzi and Sen, 2016
Very fast flavour conversion

Dasgupta, Mirizzi and Sen, 2016