Dynamical algebra of observables in dissipative quantum systems

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Received 29 September 2016
Accepted for publication 16 December 2016
Published 10 January 2017

Abstract
Dynamics and features of quantum systems can be drastically different from classical systems. Dissipation is understood as a general mechanism through which quantum systems may lose part or all of their quantum aspects. Here we discuss a method to analyze behaviors of dissipative quantum systems in an algebraic sense. This method employs a time-dependent product between system’s observables which is induced by the underlying dissipative dynamics. We argue that the long-time limit of the algebra of observables defined with this product yields a contractive algebra which reflects the loss of some quantum features of the dissipative system, and it bears relevant information about irreversibility. We illustrate this result through several examples of dissipation in various Markovian and non-Markovian systems.

Keywords: dissipative quantum systems, irreversibility, algebra of observables

1. Introduction
Coherence is at the heart of the most genuinely non-classical aspects of a quantum-mechanical system. A dissipative process provokes deterioration of quantum features and
makes a quantum system more ‘classical’. For instance, coherence between the two branch waves in a double-slit experiment is a fundamental requisite to observe interference. A decohering environment affects the ability of a quantum system to interfere, making its behavior more ballistic and classical.

It is natural to expect that features of a dissipative dynamics heavily influence the quantum-to-classical transition. Intuitively, one would expect that during a dissipative process it becomes increasingly difficult to unearth some of non-classical aspects of a quantum system. These limitations should be reflected in the measurements that one can perform on the system and therefore should modify its algebra of observables.

In this article, we shall study this problem of dissipative quantum-to-classical transition from an algebraic perspective for Markovian and non-Markovian quantum systems. Markovian dynamics can be formulated according to Gorini, Kossakowski, Sudarshan [1], and Lindblad (GKSL) [2], within the framework of quantum dynamical semigroups [3]. Quantum dissipative systems [4, 5] have a wide range of applications and are essential in quantum information science and technologies [6]. We shall also extend some results to the non-Markovian case, by considering a rather general class of time-local master equations.

In contrast to studying (behavior of) fluctuations and coherence as a means to identify quantumness of a system, our algebraic approach relies on how commutators (as signatures of quantum behaviors) are affected in time by the underlying dissipative dynamics. In this sense, the algebraic approach implies what observables are relevant given a dynamics. Specifically, in this article we shall elaborate on a mathematical mechanism that yields a dynamical deformation and eventually a contraction of the algebra of the observables of a dissipative system. A precursor to this algebra contraction mechanism can be found in [8], where, however, only the Lie algebra was considered. Here we shall look at the whole associative product of observables, and find that the contraction must involve both the Lie and Jordan products. This will enable us to elucidate some subtle aspects of the algebra contraction. We shall also discuss more examples (including some non-Markovian ones).

The mechanism we shall investigate hinges upon an alternative definition of the product between system observables, induced by the dissipative evolution. We shall study the effects of the contraction on the full associative algebra of operators, and show that some (initially nonvanishing) commutators eventually vanish as a consequence of dissipation: the associated observables become simultaneously measurable and in this sense the system becomes more ‘classical’. Although the approach we propose here bears analogies with the macroscopic and semi-classical limit [9–11], it has clear distinctions with the macroscopic limit too. In both approaches, the system is considered ‘classical’ when some or all of its observables commute; whereas in our algebraic approach no explicit macroscopic limit appears (although the presence of a bath—comprised of infinite degrees of freedom—is implicitly assumed when one writes a GKSL equation).

We show that the asymptotic state (i.e. in the long-time limit) plays a key role in our algebraic description. Several features of the contracted algebra can be properly understood only if one computes the expectation value over the asymptotic state of the evolution. In addition, as already stressed, one needs to contract the full associative algebra of observables; following the Lie algebra alone does not suffice to draw proper conclusions on the underlying dissipative dynamics. This is a distinctive difference of our study with that of [8].

This article is organized as follows. The alternative product and the general framework are introduced in section 2. Section 3 is entirely devoted to some case studies, which help familiarize with the scheme and its physical consequences. A generic $d$-level system is analyzed in section 4. The non-Markovian case is considered in section 5. We conclude with some comments and an outlook in section 6.
2. An alternative product

Let the algebra of observables of a quantum system belong to the Banach space \( \mathcal{B}(\mathcal{H}) \) of bounded linear operators defined on its Hilbert space \( \mathcal{H} \). Assume the dynamics to be Markovian, so that the evolution equation for the density matrix \( \rho \) reads

\[
\dot{\rho}(\tau) = \mathcal{L}[\rho(\tau)],
\]

whose solution is

\[
\rho(\tau) = e^{\mathcal{L}\tau} [\rho(0)] = \Lambda(\tau) [\rho(0)] \quad (\tau \geq 0).
\]

Here and henceforth a dot denotes \( d/d\tau \). The adjoint dynamical equation for an observable \( A \) is defined through

\[
\dot{A}(\tau) = \mathcal{L}[A(\tau)] \iff A(\tau) = \Lambda(\tau)[A(0)] \quad (\tau \geq 0).
\]

The equivalence of the two descriptions hinges upon Dirac’s prescription \[12\]

\[
\text{Tr}[\rho(\tau)A(\tau)] = \text{Tr}[\rho(0)A(\tau)], \quad \forall \rho, A
\]

that connects the Schrödinger and Heisenberg pictures.

We will focus on the effects of the adjoint evolution \( \Lambda(\tau) \) on the product of observables. Let \( \mathcal{A} \subset \mathcal{B}(\mathcal{H}) \) be the algebra of observables and \( \{A_i\} \) a basis. The product \( \circ \circ \) between observables reads

\[
A_i \circ A_j = \alpha_{ij}^k A_k
\]

(summation over repeated indices implied), and is naturally defined as \( (A \circ B) \psi = A(B \psi) \), \( \forall \psi \in \mathcal{H} \). This enables one to define the commutators (Lie product) through the structure constants \( c \),

\[
[A_i, A_j] = c^k_{ij} A_k, \quad c^k_{ij} = \alpha^k_{ij} - \alpha^k_{ji},
\]

where \( [A, B] = A \circ B - B \circ A \), and the anticommutators (Jordan product), through

\[
[A_i, A_j] = s^k_{ij} A_k, \quad s^k_{ij} = \alpha^k_{ij} + \alpha^k_{ji},
\]

where \( \{A, B\} = A \circ B + B \circ A \).

In this article we focus on the time-dependent product \[8, 13\]

\[
A \circ_\tau B \equiv (\Lambda(\tau)^{-1} [\Lambda(\tau) \circ \Lambda(\tau)]), \quad \forall A, B \in \mathcal{A},
\]

that can be expressed as

\[
A \circ_\tau A_j = \alpha^k_{ij}(\tau) A_k.
\]

Clearly, \( \circ_{\tau=0} = \circ \) and \( \alpha^k_{ij}(0) = \alpha^k_{ij} \).

Note that, by definition, the new product \( \circ \circ \) is associative for any \( \tau \) (if the original product was associative):

\[
(A \circ_\tau B) \circ_\tau C = ([\Lambda(\tau)^{-1} [\Lambda(\tau) \circ \Lambda(\tau)]]) \circ_\tau C = (\Lambda(\tau)^{-1} [\Lambda(\tau) \circ \Lambda(\tau)]) \circ_\tau (\Lambda(\tau)^{-1} [\Lambda(\tau) \circ \Lambda(\tau)]) = A \circ_\tau (B \circ_\tau C).
\]

Moreover, we observe that the time-dependent product \( \circ \circ \) yields an algebra homomorphism between the associative algebras \( (\mathcal{A}, \circ, \circ) \) and \( (\Lambda(\tau)[\mathcal{A}], \circ) \), since \( \forall \tau \)}
$\Lambda_\tau^\tau : \mathcal{A} \rightarrow \Lambda_\tau^\tau[\mathcal{A}], \quad \Lambda_\tau^\tau(A \circ_\tau B) = \Lambda_\tau^\tau(A) \circ_\tau \Lambda_\tau^\tau(B). \tag{11}$

It is evident that definition (9) depends on the existence of the inverse $(\Lambda_\tau^\tau)^{-1}$, which is guaranteed for Markovian dynamics for all $\tau \geq 0$. Note that, in general, this is not valid in the non-Markovian case. Commutators and anticommutators are defined, respectively, as

$[A_i, A_j]_\tau \equiv (\Lambda_\tau^\tau)^{-1}([A_i^\tau(A_i), A_j^\tau(A_j)]) \equiv c_{ij}(\tau)A_k,$ \tag{12}

$\{A_i, A_j\}_\tau \equiv (\Lambda_\tau^\tau)^{-1}(\{A_i^\tau(A_i), A_j^\tau(A_j)\}) \equiv s_{ij}(\tau)A_k. \tag{13}$

The $\tau \rightarrow \infty$ limit (if exists) is naturally defined as

$A_i \circ_\infty A_j \equiv \lim_{\tau \rightarrow \infty} A_i \circ_\tau A_j = c_{ij}(\infty)A_k,$ \tag{14}

$[A_i, A_j]_\infty \equiv \lim_{\tau \rightarrow \infty} [A_i, A_j]_\tau = c_{ij}(\infty)A_k, \tag{15}$

$\{A_i, A_j\}_\infty \equiv \lim_{\tau \rightarrow \infty} \{A_i, A_j\}_\tau = s_{ij}(\infty)A_k. \tag{16}$

In general, the above equations yield a contraction $\mathcal{A}_\infty$ of the original algebra $\mathcal{A}$ [14–16].

Several remarks are in order. Note that some commutators will vanish in the $\tau \rightarrow \infty$ limit, so that the contracted algebra will always contain one or more Abelian subalgebras. In this sense, the system becomes more ‘classical’, in accord with physical intuition about dissipative dynamics. Observe also that the limiting product (14) captures (through the non-invertibility of the limiting map and the contraction of the algebra) a symptom of irreversibility that is not detected for any finite $\tau$. This can be viewed as a relic of the continuous interaction with an underlying infinite-dimensional dissipative bath. For finite-dimensional systems, one can observe different phenomena, such as collapses and revivals, that do not lead to any contraction.

In the following, the general idea that the contraction yields a method to discern the irreversibility of the dissipative dynamics will be tested on several examples (for some of these examples see, e.g. [4–7]). In particular, we shall illustrate the mechanism that affects the quantum features of the dissipative system.

### 3. Case studies: qubits and quantum oscillators

Let us illustrate the main ideas outlined in the previous section by looking at four case studies: (i) phase damping, (ii) energy damping, (iii) interaction with a thermal environment, and (iv) interaction with a squeezed environment. We shall see that in all cases the associative algebra of operators undergoes a contraction. These examples help elucidate a contraction through its various features.

#### 3.1. Phase damping

**3.1.1. Phase damping of a qubit.** Our first paradigmatic example is the phase damping of a qubit [17]. We shall analyze this example with particular care, by solving it in different ways in order to pinpoint the origin of the contraction and its consequences.
The evolution of the density matrix of the qubit is described by equation (1) (we drop
the explicit $\tau$-dependence hereinafter)

$$\dot{\rho} = \mathcal{L}[\rho] = -\frac{\gamma}{2}(\rho - \sigma_3 \rho \sigma_3),$$

(17)

where $\gamma > 0$, and $\sigma_i$ ($i = 0, 1, 2, 3$) are the Pauli matrices (with $\sigma_0 \equiv 1$). The
adjoint dynamics for an observable $A$ is simply obtained by replacing any quantum
jump operator with its adjoint (and eventually by changing $i$ in $-i$), so that

$$\dot{A} = \mathcal{L}[A] = -\frac{\gamma}{2}(A - \sigma_3 A \sigma_3).$$

(18)

This map is self-dual ($\mathcal{L} = \mathcal{L}^\dagger$ and $\Lambda_\tau = \Lambda_\tau^\dagger$) and yields

$$\Lambda_\tau^x[\sigma_{0,3}] = \Lambda_\tau^x[\sigma_{0,3}] = \sigma_{0,3},$$

(19)

$$\Lambda_\tau^z[\sigma_{1,2}] = e^{-\gamma \tau} \sigma_{1,2} \rightarrow \Lambda_\tau^z[\sigma_{1,2}] = 0.$$  

(20)

The associative product (9) reads

$$\sigma_{0,3} \circ_{\tau} \sigma_{0,3} = \sigma_{0,3} \circ \sigma_{0,3} \rightarrow \sigma_{0,3} \circ_{\infty} \sigma_{0,3} = \sigma_{0,3} \circ \sigma_{0,3}.$$  

(21)

$$\sigma_{1,2} \circ_{\tau} \sigma_{1,2} = e^{-2\gamma \tau} \sigma_{1,2} \circ \sigma_{1,2} \rightarrow \sigma_{1,2} \circ_{\infty} \sigma_{1,2} = 0,$$  

(22)

$$\sigma_{0,3} \circ_{\tau} \sigma_{1,2} = \sigma_{0,3} \circ \sigma_{1,2} \rightarrow \sigma_{0,3} \circ_{\infty} \sigma_{1,2} = \sigma_{0,3} \circ \sigma_{1,2}.$$  

(23)

and yields the following $\alpha(\infty)$ constants in equation (14)

$$\alpha_{ij}^k(\infty) = \begin{cases} 
\alpha_{ij}^k & \text{if } i \in \{0, 3\} \text{ or } j \in \{0, 3\}, \\
0 & \text{if } i, j \in \{1, 2\}.
\end{cases}$$

(24)

This leaves us with $\{\sigma_0, \sigma_3\}$, namely $\mathcal{A}_\infty = \mathbb{C}^2$, the Abelian algebra of
diagonal $2 \times 2$ matrices.

From the mathematical point of view, the explicit calculation of the contracted
associative product is all one needs. In particular, from the $\alpha$ constants one can derive
the structure constants characterizing both the Lie and Jordan algebras (15) and (16).
However, as we shall see in this and the following examples, the direct calculation of the
Lie algebra alone may motivate interesting observations. In this example, direct computation
would yield the Lie algebra

$$[\sigma_1, \sigma_2]_r = 2i e^{-2\gamma \tau} \sigma_3 \rightarrow [\sigma_1, \sigma_2]_\infty = 0,$$  

(25)

$$[\sigma_2, \sigma_3]_r = 2i \sigma_1 \rightarrow [\sigma_2, \sigma_3]_\infty = 2i \sigma_1,$$  

(26)

$$[\sigma_3, \sigma_1]_r = 2i \sigma_2 \rightarrow [\sigma_3, \sigma_1]_\infty = 2i \sigma_2.$$  

(27)

whose asymptotic structure constants $\epsilon_{ij}^k(\infty)$ (equation (16)) are of course in agreement
with the $\alpha(\infty)$ constants in equation (24). According to equations (25)–(27), the
original $\mathfrak{su}(2)$ algebra contracts to the Euclidean algebra $\mathfrak{e}(2)$ of the isometries of the plane. But this is not the end
of the story, as we have seen that the contraction must be pursued further, to yield the Abelian
algebra $\mathbb{C}^2$. This can only be done by considering the full associative product of observables.
One should also note that $\mathfrak{su}(2)$ is consistent with the Abelian algebra $\mathbb{C}^2$, if $\sigma_{1,2} \sim 0$, as
dicated by equation (22). See also the following remarks.

The above picture is in accord with the asymptotic solution of equation (17), that reads
\[ \varrho = \frac{1}{2}(\sigma_0 + x \cdot \sigma) \xrightarrow{\tau \to \infty} \Lambda_\infty[\varrho] = \varrho(\infty) = \frac{1}{2}(\sigma_0 + x_3 \sigma_3), \]

(28)

\( x \) being a vector in the unit three-dimensional ball, \( \|x\| \leq 1 \). All the preceding equations that involve operators in the \( \tau \to \infty \) limit must be understood in the weak sense, according to equation (4). For example, the expectation values of \( \sigma_1 \) and \( \sigma_2 \) on the asymptotic state (28) vanish: as time goes by, it becomes increasingly difficult to measure the coherence (off-diagonal operators) between the two states of the qubit. In the \( \tau \to \infty \) limit, coherence is fully lost. On the other hand, the expectation value of \( \sigma_3 \) does not vanish and the only nontrivial observables are the populations. We note also that, on the asymptotic state (28)

\[ (\Delta \sigma_{1,2})^2 = \langle (\sigma_{1,2})^2 \rangle - \langle \sigma_{1,2} \rangle^2 = 0 - 0 = 0, \]

(29)

where \( (\sigma_{1,2})^2 = \sigma_{1,2} \circ_{\infty} \sigma_{1,2} \), whereas

\[ (\Delta \sigma_3)^2 = 1 - \langle \sigma_3 \rangle^2. \]

(30)

The interpretation is consistent: \( \sigma_1 \) and \( \sigma_2 \) weakly vanish in the asymptotic state (28) and the only nontrivial observable besides \( 1 \) is \( \sigma_3 \). Thus \( A_{\infty} = \mathbb{C}^2 \). This also provides an alternative way of looking at fluctuations: they are codified in the algebraic structure, in the same way as Heisenberg’s uncertainty principle is codified in the nonvanishing commutator of position and momentum.

One can obtain a deeper insight into the contraction of the algebra described above by looking at the problem from a wider perspective. The GKSL equation (17) can be obtained from the following Hamiltonian:

\[ H = 1 \otimes \int_{\mathbb{R}} \omega \, a_\omega^\dagger a_\omega \, d\omega + \frac{\Gamma}{2} \sigma_3 \otimes \int_{\mathbb{R}} (a_\omega^\dagger + a_\omega^\dagger) \, d\omega, \]

(31)

where \( \Gamma = \sqrt{\gamma/2\pi} \), and \( a_\omega^\dagger \) (\( a_\omega^\dagger \)) are the bosonic annihilation (creation) operators of the bath.

The usual procedure [18, 19] is to expand up to the second order in \( \Gamma \) and trace out the bath degrees of freedom in order to obtain an evolution equation for the density matrix of the system. We shall, however, take in the following a different route.

The solutions to the Heisenberg operators read

\[ (\sigma_3 \otimes 1)(\tau) = e^{iH\tau}(\sigma_3 \otimes 1)e^{-iH\tau} = \sigma_3 \otimes 1, \]

(32)

\[ (\sigma_1 \otimes 1)(\tau) = e^{iH\tau}(\sigma_1 \otimes 1)e^{-iH\tau} = \sigma_1 \otimes \exp \left[ \Gamma \int \left( \frac{1 - e^{-i\omega \tau}}{\omega} a_\omega^\dagger - \frac{1 - e^{i\omega \tau}}{\omega} a_\omega^{\dagger} \right) \, d\omega \right], \]

(33)

\[ (1 \otimes a_\omega)(\tau) = e^{iH\tau}(1 \otimes a_\omega)e^{-iH\tau} = e^{-i\omega \tau} 1 \otimes a_\omega - \frac{\Gamma}{2} \frac{1 - e^{-i\omega \tau}}{\omega} \sigma_3 \otimes 1, \]

(34)

where \( \sigma_\pm \equiv (\sigma_1 \pm i\sigma_2)/2 \).

Clearly, the Schrödinger operators of the system in equations (32) and (33) evolve into the Heisenberg operators that contain contributions of the bath. The idea is to identify the evolved operators of the system under the adjoint evolution equations (3) and (18), with the trace of the full Heisenberg operators over the ground state \( |0\rangle \) of the bath:

\[ A(\tau) = \Lambda_\tau[A(0)] \equiv \langle (A \otimes 1)(\tau) \rangle_{\text{bath}} = \text{Tr}_{\text{bath}} [(A \otimes 1)(\tau)(1 \otimes |0\rangle \langle 0|)]. \]

(35)
We thus obtain
\[ \langle (\sigma_3 \otimes 1) (\tau) \rangle_{\text{bath}} = \sigma_3 \tag{36} \]
and
\[ \langle (\sigma_\tau \otimes 1) (\tau) \rangle_{\text{bath}} = \langle 0 | \exp \left[ \Gamma \int \left( \frac{1 - e^{-i \omega \tau}}{\omega} a_\omega - \frac{1 - e^{i \omega \tau}}{\omega} a_\omega^\dagger \right) d\omega \right] | 0 \rangle \sigma_\tau \]
\[ = (\Pi_\omega (0 | \alpha_\omega) \sigma_\tau \]
\[ = (\Pi_\omega e^{-|\alpha_\omega|^2/2}) \sigma_\tau \]
\[ = \exp \left[ -2 \Gamma^2 \int \frac{\sin^2(\omega \tau)}{\omega^2} d\omega \right] \sigma_\tau \]
\[ = e^{-2 \pi \Gamma^2 \tau} \sigma_\tau \]
\[ = e^{-\gamma \tau} \sigma_\tau \tag{37} \]
where \( | \alpha_\omega \rangle \) is a coherent state with \( \alpha_\omega = - (\Gamma / \omega)(1 - e^{i \omega \tau}) \). These results enable us to recover equations (19) and (20), as well as the contraction of the ensuing algebra, if one identifies \( \gamma = 2 \pi \Gamma^2 \), which is the Fermi golden rule\(^8\). This derivation offers an interesting perspective on the products (9), (12), (13), and their limits (14)–(16). As emphasized in section 2, the limit captures, through the contraction, a symptom of irreversibility that is not detected for any finite \( \tau \). Clearly, this can occur only with an infinite-dimensional dissipative bath.

The general features discussed for this particular simple model will be unaltered for other dissipative dynamical systems. We shall discuss other examples in the following.

3.1.2. Phase damping of a harmonic oscillator. Let
\[ \mathcal{L}[\varrho] = -\frac{\gamma}{2} ([N^2, \varrho] - 2 N \varrho N), \tag{38} \]
that describes a harmonic oscillator undergoing phase damping. Since \( \mathcal{L}^2 = \mathcal{L} \) and \( \Lambda^2_{\tau} = \Lambda_{\tau} \), one finds
\[ \Lambda^2_{\tau} [a] = e^{-\gamma \tau / 2} a, \quad \Lambda^2_{\tau} [a^\dagger] = e^{-\gamma \tau / 2} a^\dagger, \quad \Lambda^2_{\tau} [N] = N. \tag{39} \]
Note also that \( \Lambda^2_{\tau} [a^\dagger] \circ \Lambda^2_{\tau} [a] = e^{-\gamma \tau} a^\dagger a \), so that
\[ a^\dagger \circ_\tau a = (\Lambda^2_{\tau})^{-1} [\Lambda^2_{\tau} [a] \circ \Lambda^2_{\tau} [a]] = e^{-\gamma \tau} a^\dagger a \rightarrow 0. \tag{40} \]
Hence
\[ a^\dagger \circ_\infty a = 0 \quad \Rightarrow \quad a^\dagger = a = 0, \tag{41} \]
in agreement with equation (39). \( \Lambda^0_{\infty} [a^\dagger] = \Lambda^0_{\infty} [a] = 0 \), and the homomorphism (11). We note that (nonvanishing) \( N \) cannot be identified with (vanishing) \( a^\dagger \circ_\infty a \). This leaves us with the Abelian algebra generated by \( \{1, N\} \), similarly to the example of the phase damping of a qubit discussed in section 3.1.1, and is consistent with physical interpretation: a generic density matrix
\[ \varrho = \sum c_n |n\rangle \langle n| \xrightarrow{\tau \rightarrow \infty} \varrho (\infty) = \sum |c_n|^2 |n\rangle \langle n| \tag{42} \]
\(^8\)We assumed that the bath is initially in its ground state \( |0\rangle \). Different initial states are possible: for example, taking an initial thermal state of the bath would yield a different GKSL equation and a different decay rate \( \gamma \), proportional to the number of thermal photons.
becomes diagonal in the $N$-representation, so that populations do not change and the ladder operators $a^\dagger$ and $a$ must vanish (weakly) over the final state.

As in the example of section 3.1.1, we observe that the contraction of the original Heisenberg–Weyl oscillator algebra $h_4$ will yield the Lie algebra $iso(1,1)$ of the Poincaré group in $1 + 1$ dimensions,

\[
[a, a^\dagger]_\infty = 0, \quad [a, N]_\infty = a, \quad [a^\dagger, N]_\infty = -a^\dagger,
\]

as can be checked by direct calculation. This is consistent with the Abelian algebra generated by \{1, N\} if equation (41) is taken into account.

### 3.1.3. Comparison of the first two examples

A simple heuristic comparison between the first two examples shows that by the following substitution:

\[
a \rightarrow (\sigma_1 - i\sigma_2)/2 = \sigma_-, \quad N \rightarrow (\sigma_0 + \sigma_3)/2,
\]

the commutation relations will be preserved and the equation of motion of the example in section 3.1.2 will change into the equation of motion of the example in section 3.1.1—up to the rescaling factor $1/2$. The physical content of the two examples is, therefore, the same. However, mathematically this is only an analogy and should be taken with care, as the two algebras are different. The physical analogy discussed here will be valid for all the examples that follow and it is a hand-waving way to translate results obtained for qubits into analogous results for harmonic oscillators and vice versa.

### 3.2. Energy damping

#### 3.2.1. Energy damping of a qubit

Let

\[
\mathcal{L}[q] = -\frac{1}{2}(\{\sigma_+, \sigma_-\} - 2\sigma_- \sigma_+).
\]

Hence, we have

\[
\mathcal{L}[A] = -\frac{\gamma}{2}(\{\sigma_+, A\} - 2\sigma_+ A\sigma_-).
\]

We note that the evolution is not self-dual: $\mathcal{L}' \neq \mathcal{L}$. We obtain

\[
\Lambda^t_{1}[\sigma_{1,2}] = e^{-\gamma t^2} \sigma_{1,2}, \quad \Lambda^t_{1}[\sigma_3] = e^{-\gamma t}(\sigma_3 + \sigma_0) - \sigma_0, \quad \Lambda^t_{1}[\sigma_0] = \sigma_0
\]

and

\[
\Lambda^t_{\infty}[\sigma_{1,2}] = 0, \quad \Lambda^t_{\infty}[\sigma_3] = -\sigma_0, \quad \Lambda^t_{\infty}[\sigma_0] = \sigma_0.
\]

Observe also that $\sigma_1 \circ_\infty \sigma_1 = \sigma_0$. Thus the algebra is contracted to the one-dimensional Abelian algebra generated by the single element $\sigma_0$. This is in accord with the physical interpretation. The solution of equation (45) reads

\[
q = \frac{1}{2}(\sigma_0 + x \cdot \sigma) \overset{t \to \infty}{\longrightarrow} q(\infty) = P, \quad P = (\sigma_0 - \sigma_i)/2
\]

so that the final state is the projection $P = (\sigma_0 - \sigma_i)/2$ over the ground state.

#### 3.2.2. Energy damping of a harmonic oscillator

The following energy damping scenario can be attributed to the process of direct photodetection. For this dynamics, we have
\[ \mathcal{L}[\varrho] = -\frac{\gamma}{2}(\{a^\dagger a, \varrho\} - 2a^\dagger a\varrho), \]  
whence
\[ \mathcal{L}^t[A] = -\frac{\gamma}{2}(\{a^\dagger a, A\} - 2a^\dagger Aa), \]
and
\[ \Lambda_2^t[a] = e^{-\gamma t}a, \quad \Lambda_2^t[a^\dagger] = e^{-\gamma t}a^\dagger, \quad \Lambda_2^t[N] = e^{-\gamma t}N \quad (N = a^\dagger a). \]

Only unity survives the contraction. Thus, similarly to the qubit example in section 3.2.1, the oscillator algebra is contracted to the trivial Abelian algebra \( \mathcal{A}_\infty \) made up of a single element (unity) \( 1 \). This is consistent with physical interpretation, as the system decays to the ground state.

### 3.3. Interaction with a thermal field
#### 3.3.1. 2-level atom in a thermal field

In this example,
\[ \mathcal{L}[\varrho] = \frac{\gamma}{2}(n + 1)(2\sigma_+^0\sigma_+ - \{\sigma_+, \varrho\}) + \frac{\gamma}{2}n(2\sigma_+^0\sigma^- - \{\sigma_-, \varrho\}), \]  
where \( \sigma_\pm = (\sigma_1 \pm \sigma_2)/2 \) and \( n = (e^{\beta \Omega} - 1)^{-1} \), with \( \beta \) the inverse temperature and \( \Omega \) the energy difference of the two states of the qubit. This dynamics generalizes the example in section 3.2.1 for \( n \neq 0 \).

It can be easily checked that \( \mathcal{L}^t[\sigma_3] = -2\gamma(n + \frac{1}{2})\sigma_3 - \gamma\sigma_0 \) and \( \mathcal{L}^t[\sigma_0] = 0 \). Hence, \( \sigma_3 + \frac{1}{1 + 2n} \sigma_0 \) is an eigenoperator of \( \mathcal{L}^t \), \( \mathcal{L}^t[\sigma_3 + \frac{1}{1 + 2n} \sigma_0] = -2\gamma(n + \frac{1}{2})(\sigma_3 + \frac{1}{1 + 2n} \sigma_0) \), and the time evolutions of \( \sigma_3 \) and \( \sigma_0 \) are easily found to be
\[ \Lambda_2^t[\sigma_3] = e^{-\gamma t(1 + 2n)}\sigma_3 + \frac{1}{1 + 2n}(e^{-\gamma t(1 + 2n)} - 1)\sigma_0, \quad \Lambda_2^t[\sigma_0] = \sigma_0, \]
so that
\[ \Lambda_2^t[\sigma_3] = -\frac{1}{1 + 2n} \sigma_0, \quad \Lambda_2^t[\sigma_0] = \sigma_0. \]

The observables \( \sigma_1 \) and \( \sigma_2 \) are also eigenoperators of \( \mathcal{L}^t \), since \( \mathcal{L}^t[\sigma_{1,2}] = -\gamma(n + \frac{1}{2})\sigma_{1,2} \), and their time evolutions are
\[ \Lambda_2^t[\sigma_{1,2}] = e^{-\gamma(n + 1/2)\sigma_{1,2}} \rightarrow \Lambda_2^\infty[\sigma_{1,2}] = 0. \]

The contracted algebra \( \mathcal{A}_\infty \) is Abelian and is generated by the single element \( \sigma_0 \). This is in accord with the asymptotic solution of equation (53),
\[ \varrho = \frac{1}{2}(\sigma_0 + x \cdot \sigma) \stackrel{t \to \infty}{\longrightarrow} \Lambda_\infty[\varrho] = \varrho(\infty) = \frac{P_+ + e^{\beta \Omega}P_-}{1 + e^{\beta \Omega}}, \]
where the notation is as in equation (28), and \( P_\pm = (\sigma_0 \pm \sigma_3)/2 \) are the two projections.

Direct computation of the Lie algebra enables one to make a further remark. One obtains
\[ [\sigma_1, \sigma_2]_\infty = 2i(\sigma_3 + \frac{1}{1 + 2n} \sigma_0), \]
\[ [\sigma_2, \sigma_1]_\infty = 0, \]  
\[ [\sigma_3, \sigma_1]_\infty = 0. \]  
Curiously, we encounter a central extension of the Heisenberg–Weyl algebra \( h_3 \) (\( a, a^\dagger, 1 \), without \( N \)). However, interestingly, equation (56) forces the left-hand side of equation (58) to vanish. For consistency, the right-hand side must vanish too. This yields
\[ \sigma_3 + \frac{1}{1 + 2n}\sigma_0 \propto P_r - e^{-\beta m}P = 0, \]  
which is Boltzmann’s statistics.

### 3.3.2. Thermal damping of a harmonic oscillator.

Let
\[ \mathcal{L}[\rho] = \frac{\gamma}{2}(m + 1)(2a_0a - a_0^a - a_0^a a) + \frac{\gamma}{2}m(2a_0^a a - a_0^a a - a_0^a). \]  
Here again, \( m = (e^{\beta m} - 1)^{-1} \), with \( \beta \) inverse temperature and \( \Omega \) the oscillator frequency. The solution is
\[ \Lambda^2_{\gamma}[a] = e^{-\frac{\gamma}{2}a}, \quad \Lambda^2_{\gamma}[a^d] = e^{-\frac{\gamma}{2}a^d}. \]  
Since \( \mathcal{L}[N] = -\gamma(N - m1) \) and \( \mathcal{L}[1] = 0 \), one easily verifies that \( N - m1 \) is an eigenoperator of \( \mathcal{L} \) with eigenvalue \( -\gamma \). Thus, the time evolution of this operator can be easily obtained as
\[ \Lambda^2_{\gamma}[N - m1] = e^{-\gamma}(N - m1), \]  
whence the time evolution of \( N \) becomes
\[ \Lambda^2_{\gamma}[N] = e^{-\gamma}N + m(1 - e^{-\gamma})1. \]  
From this we conclude that
\[ \Lambda^2_{\gamma}[a] = 0, \quad \Lambda^2_{\gamma}[a^d] = 0, \quad \Lambda^2_{\gamma}[N] = m1, \]  
which implies that the contracted algebra \( A_{\infty} \) is Abelian and made up of a single element (unity): \( 1 \). As in the example of section 3.3.1, the final state is thermal and the result is consistent.

### 3.4. 2-level atom in a squeezed vacuum

Let
\[ \mathcal{L}[\rho] = \frac{\gamma}{2}(n + 1)(2\sigma_0^a\sigma_0 - \{\sigma_0, \sigma_0^a\}) + \frac{\gamma}{2}m(2\sigma_0^a\sigma_0 - \{\sigma_0, \sigma_0^a\}) \]
\[ - \gamma(m\sigma_0^a\sigma_0 + m^*\sigma_0^a\sigma_0), \]  
where \( |m|^2 \leq n(n + 1) \). Since \( \sigma_3 \) and \( \sigma_0 \) are the eigenoperators of the third term of \( \mathcal{L} \) with eigenvalue \( 0 \), the time evolution of these operators is not different from that found in the previous example (where \( m \) was zero). That is,
\[ \Lambda^2_{\gamma}[\sigma_3] = e^{-\gamma(1 + 2n^*\sigma_3} + \frac{1}{1 + 2n}(e^{-\gamma(1 + 2n^*\sigma_3} - 1)\sigma_0, \quad \Lambda^2_{\gamma}[\sigma_0] = \sigma_0. \]  
It can be easily calculated that
\[ \mathcal{L}^i[\sigma_1] = -\gamma((n + 1/2) + (m + m')/2)\sigma_1 - i\gamma(m - m')\sigma_2/2, \tag{69} \]

whence

\[ \mathcal{L}^i[\sigma_2] = -\gamma((n + 1/2) - (m + m')/2)\sigma_2 - i\gamma(m - m')\sigma_1/2. \tag{70} \]

After some algebra, one can find that \( \sigma_2 - i\frac{m - m'}{m + m' + 2|m|}\sigma_1 \) is an eigenoperator of \( \mathcal{L}^i \) with eigenvalue \(-\gamma(n + 1/2) \pm \gamma|m|\). Hence, we obtain the time evolutions of these two eigenoperators as

\[ \Lambda^i_{n}[\sigma_2] - i\frac{m - m'}{m + m' + 2|m|}\sigma_1] = e^{-\gamma((n + 1/2)^2}(\sigma_2 - i\frac{m - m'}{m + m' + 2|m|}\sigma_1), \tag{71} \]

\[ \Lambda^i_{n}[\sigma_2] - i\frac{m - m'}{m + m' - 2|m|}\sigma_1] = e^{-\gamma((n - 1/2)^2}(\sigma_2 - i\frac{m - m'}{m + m' - 2|m|}\sigma_1). \tag{72} \]

Thus the time evolutions of \( \sigma_{1,2} \) are obtained as

\[ \Lambda^i_{n}[\sigma_1] = e^{-\gamma((n + 1/2)^2}\left[ \sigma_1\left(\frac{m + m'}{2|m|}\sinh(\gamma|m|) + \cosh(\gamma|m|)\right) \right. \]

\[ + \sigma_2\left(\frac{m - m'}{2|m|}\sinh(\gamma|m|)\right) \]

\[ \Lambda^i_{n}[\sigma_2] = e^{-\gamma((n + 1/2)^2}\left[ \sigma_2\left(-\frac{m + m'}{2|m|}\sinh(\gamma|m|) + \cosh(\gamma|m|)\right) \right. \]

\[ + \sigma_1\left(\frac{m - m'}{2|m|}\sinh(\gamma|m|)\right) \] \tag{73}

\[ \Lambda^i_{n}[\sigma_1] = e^{-\gamma((n + 1/2)^2}\left[ \sigma_1\left(\frac{m + m'}{2|m|}\sinh(\gamma|m|) + \cosh(\gamma|m|)\right) \right. \]

\[ + \sigma_2\left(\frac{m - m'}{2|m|}\sinh(\gamma|m|)\right) \]

\[ \Lambda^i_{n}[\sigma_2] = e^{-\gamma((n + 1/2)^2}\left[ \sigma_2\left(-\frac{m + m'}{2|m|}\sinh(\gamma|m|) + \cosh(\gamma|m|)\right) \right. \]

\[ + \sigma_1\left(\frac{m - m'}{2|m|}\sinh(\gamma|m|)\right) \] \tag{74}

Having \(|m|^2 \leq n(n + 1)\) we find \( n \pm |m| + 1/2 > 0 \). As a result, when \( \tau \) goes to infinity we obtain

\[ \Lambda^i_{\infty}[\sigma_3] = -\frac{1}{1 + 2n}\sigma_0, \quad \Lambda^i_{\infty}[\sigma_0] = \sigma_0. \tag{75} \]

\[ \Lambda^i_{\infty}[\sigma_1] = 0, \quad \Lambda^i_{\infty}[\sigma_2] = 0. \tag{76} \]

This is the same algebra as in the previous examples, and hence the same considerations apply here too.

### 4. d-level quantum system

The examples in the preceding section enabled us to familiarize with the contracted product and its physical implications. We can now generalize some salient features of the technique by looking at a simple example involving a \( d \)-level quantum system.

Consider the following construction: let \( \mathcal{P} \) be a completely-positive trace preserving projection, i.e. \( \mathcal{P}^2 = \mathcal{P} \), and let \( \mathcal{P}^\perp = I - \mathcal{P} \), where \( I \) denotes the identity map (i.e. \( I[A] = A \) for all \( A \in \mathcal{A} \)). Note that \( \mathcal{P}^\perp \) contrary to \( \mathcal{P} \) is not completely positive. Now, define the generator

\[ \mathcal{L} = -\gamma\mathcal{P}^\perp, \tag{77} \]
with $\gamma > 0$. One easily finds the evolution

$$\Lambda_{\tau} = \mathcal{P} + e^{-\gamma \tau} \mathcal{P}^\perp.$$  \hfill (78)

Hence the asymptotic dynamical map reads as $\Lambda_{\infty} = \mathcal{P}$. The inverse map reads as

$$\Lambda_{\tau}^{-1} = \mathcal{P} + e^{\gamma \tau} \mathcal{P}^\perp.$$  \hfill (79)

and hence one finds the corresponding limiting product

$$A \circ_{\infty} B = \mathcal{P}^\perp[A \circ \mathcal{P}^\perp[B]] + \mathcal{P}^{\perp\perp}[A \circ \mathcal{P}^\perp[B] + \mathcal{P}^\perp[A] \circ \mathcal{P}^{\perp\perp}[B]],$$  \hfill (80)

for any $A, B \in \mathcal{A}$. This formula shows that $(\mathcal{A}_{\infty} \circ_{\infty})$ defines an associative algebra. Indeed, if $A, B \in \mathcal{P}^\perp[A]$, then $A \circ_{\infty} B \in \mathcal{P}^\perp[A]$. Moreover, if $1$ denotes the unit element in $\mathcal{A}$, then $\mathcal{P}^\perp[1] = 1$, and hence $1 \circ_{\infty} A = A$, for all $A \in \mathcal{A}$.

The asymptotic algebra is defined by the projection $\mathcal{P}$, that is, $\mathcal{A}_{\infty} = \mathcal{P}[\mathcal{A}]$. Hence, we can freely model the asymptotic algebra by choosing an appropriate projection $\mathcal{P}$.

Let us consider a completely-positive projector in $\mathcal{A} = B(\mathcal{H})$ represented as follows:

$$\mathcal{P}[A] = \sum_k P_k A P_k,$$  \hfill (81)

where $P_k$ is a family of mutually orthogonal projectors in $\mathcal{H}$ such that $\sum_k P_k = 1$. Note that $\mathcal{P}^\perp = \mathcal{P}$. One can easily check

$$\mathcal{P}^\perp[\mathcal{P}[A] \circ B] = \mathcal{P}^\perp[A \circ \mathcal{P}^\perp[B]] = \mathcal{P}^\perp[A] \circ \mathcal{P}^\perp[B],$$  \hfill (82)

which implies

$$A \circ_{\infty} B = \mathcal{P}^\perp[A] \circ \mathcal{P}^\perp[B] + \mathcal{P}^{\perp\perp}[A] \circ \mathcal{P}^\perp[B] + \mathcal{P}^\perp[A] \circ \mathcal{P}^{\perp\perp}[B],$$  \hfill (83)

and hence

$$A \circ_{\infty} B = A \circ B - \mathcal{P}^{\perp\perp}[A] \circ \mathcal{P}^{\perp\perp}[B].$$  \hfill (84)

In fact, we have the following proposition:

**Proposition 1.** $A \in \mathcal{A}_{\infty}$ if and only if

$$A \circ_{\infty} B = A \circ B,$$  \hfill (85)

for all $B \in \mathcal{A}$.

Indeed, due to equation (84), $A \circ_{\infty} B = A \circ B$ for all $B \in \mathcal{A}$ if and only if $\mathcal{P}^{\perp\perp}[A] = 0$ which means that $A \in \mathcal{A}_{\infty}$.

Let us look at two simple examples. Taking the Hilbert space $\mathcal{H} = \mathbb{C}^2$ and

$$\mathcal{P}[\sigma] = P_{\sigma} P_{\sigma} + P_{\sigma} P_{\sigma},$$  \hfill (86)

with $P_{\sigma} = (\sigma_0 \pm \sigma_3)/2$, one reconstructs equation (17). In fact, one obtains the following non-commutative deformed matrix multiplication [20]

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \circ \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{22}b_{22} \end{pmatrix},$$  \hfill (87)

namely,

$$(A \circ_{\infty} B)_{ij} = \begin{cases} (A \cdot B)_{ij} & \text{if } i \neq j \\ (A \cdot_{B} B)_{ii} & \text{if } i = j \end{cases}.$$  \hfill (88)
where $A \otimes B$ denotes the Hadamard product.

Taking $\mathcal{H} = \mathbb{C}^3$ and

$$\mathcal{P}[\varrho] = P \varrho P + P^\dagger \varrho P^\dagger,$$

with $P$ one-dimensional and $P^\dagger$ two-dimensional, one has the following example. Consider a 3-level system $\{|a\rangle, |b\rangle, |c\rangle\}$ and let

$$\mathcal{L}[\varrho] = \gamma (2P^\dagger \varrho P^\dagger - P^\dagger \varrho - \varrho P^\dagger),$$

where $P^\dagger = |b\rangle \langle b| + |c\rangle \langle c|$. It is straightforward to check that

$$|r\rangle \langle r| \xrightarrow{\tau \to \infty} |r\rangle \langle r| \quad (r = a, b, c)$$

$$|a\rangle \langle r| \pm |r\rangle \langle a| \xrightarrow{\tau \to \infty} 0 \quad (r = b, c)$$

$$|b\rangle \langle c| \pm |c\rangle \langle b| \xrightarrow{\tau \to \infty} |b\rangle \langle b| \pm |c\rangle \langle c|$$

Equation (91) guarantees population preservation in every level, equation (92) hinders transitions between level $a$ and the other two levels, whereas equation (93) shows that the dynamics within the subspace span{$|b\rangle, |c\rangle$} is preserved. Note that the original Lie algebra is $A = \mathfrak{su}(3)$, while the final algebra $A_\infty = \Lambda_\infty^3[A]$ contains $\mathfrak{su}(2)$ as a subalgebra (on span{$|b\rangle, |c\rangle$}). Summarizing, for a generic density matrix

$$\varrho = \begin{pmatrix} p_a & \varrho_{ab} & \varrho_{ac} \\ \varrho_{ba} & p_b & \varrho_{bc} \\ \varrho_{ca} & \varrho_{cb} & p_c \end{pmatrix} \xrightarrow{\tau \to \infty} \varrho(\infty) = \begin{pmatrix} p_a & 0 & 0 \\ 0 & p_b & \varrho_{bc} \\ 0 & \varrho_{bc} & p_c \end{pmatrix},$$

and

$$\begin{pmatrix} A_{aa} & A_{ab} & A_{ac} \\ A_{ba} & A_{bb} & A_{bc} \\ A_{ca} & A_{cb} & A_{cc} \end{pmatrix} \xrightarrow{\tau \to \infty} \begin{pmatrix} B_{aa} & B_{ab} & B_{ac} \\ B_{ba} & B_{bb} & B_{bc} \\ B_{ca} & B_{cb} & B_{cc} \end{pmatrix} = \begin{pmatrix} A_{aa} & A_{ab} & A_{ac} \\ A_{ba} & A_{bb} & A_{bc} \\ A_{ca} & A_{cb} & A_{cc} \end{pmatrix} \circ \begin{pmatrix} B_{aa} & B_{ab} & B_{ac} \\ B_{ba} & B_{bb} & B_{bc} \\ B_{ca} & B_{cb} & B_{cc} \end{pmatrix}$$

$$- \begin{pmatrix} 0 & A_{ab} & A_{ac} \\ A_{ba} & 0 & 0 \\ A_{ca} & 0 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & B_{ab} & B_{ac} \\ B_{ba} & 0 & 0 \\ B_{ca} & 0 & 0 \end{pmatrix}.$$ \hfill (95)

that generalizes the product (87). The dynamics between levels $|b\rangle$ and $|c\rangle$ is unitary and the asymptotic algebra reflects the underlying quantum coherence.

5. The non-Markovian case: time-local master equations

We focused so far on Markovian dynamics. In the non-Markovian case the analysis becomes more involved. We shall outline in this section some interesting differences between the two cases, by assuming that the dynamical map $\Lambda_\tau$ satisfies the following time-local master equation [5]

$$\dot{\Lambda}_\tau = \mathcal{L}_\tau \Lambda_\tau,$$ \hfill (96)

or, equivalently, the density matrix satisfies

$$\dot{\varrho}(\tau) = \mathcal{L}_\tau [\varrho(\tau)].$$ \hfill (97)
The corresponding solution reads \( \varrho(\tau) = \Lambda_\tau[\varrho(0)] \) with

\[
\Lambda_\tau = T \exp \left[ \int_0^\tau \mathcal{L}_\tau \, d\tau \right] \quad (\tau \geq 0),
\]

where \( T \) denotes the time-ordering operator. Note that the master equation (96) is general and covers both the Markovian and non-Markovian cases (see \([21, 22]\) for recent reviews). The time-local generator \( \mathcal{L}_\tau \) has the following well-known form

\[
\mathcal{L}_\tau[\varrho] = -i[H(\tau), \varrho] + \sum_k \gamma_k(\tau) \left( V_k(\tau) \varrho V_k^\dagger(\tau) - \frac{1}{2} [V_k(\tau) V_k(\tau), \varrho] \right),
\]

and gives rise to Markovian evolution if all decoherence/dissipation rates \( \gamma_k(\tau) \geq 0 \) \([21–24]\).

The adjoint map \( \Lambda_\tau^\# \) is always defined via Dirac’s prescription (4), but the existence of the inverse \( \Lambda_\tau^{-\#} \) is not guaranteed, in general, even for finite \( \tau \). Let us assume that

\[
\Lambda_\tau[F(\tau)] = \lambda_\tau F(\tau),
\]

and

\[
\Lambda_\tau^\#[G(\tau)] = \lambda_\tau^\dagger G(\tau),
\]

for \( i = 0, 1, \ldots, d^2 - 1 \). The time-dependent operators \( F_\tau(\tau) \) and \( G_\tau(\tau) \) provide bi-orthogonal basis

\[
\text{Tr}[G_i^\dagger(\tau) F_j(\tau)] = \delta_{ij}, \quad \tau \geq 0.
\]

Clearly \( \lambda_0(\tau) = 1 \) and \( G_0(\tau) = 1/\sqrt{n} \). One has the following spectral representation for the dynamical map

\[
\Lambda_\tau[\varrho] = \sum_{i=0}^{n^2-1} \lambda_i(\tau) F_i(\tau) \text{Tr}[G_i^\dagger(\tau) \varrho],
\]

and its dual

\[
\Lambda_\tau^\#[A] = \sum_{i=0}^{n^2-1} \lambda_i^\dagger(\tau) G_i(\tau) \text{Tr}[F_i^\dagger(\tau) A],
\]

and hence

\[
A \circ_\tau B = \sum_{i,j,k} \alpha_{ijk}(\tau) G_{ik}(\tau) \text{Tr}[F_j^\dagger(\tau) A] \text{Tr}[F_i^\dagger(\tau) B],
\]

with

\[
\alpha_{ijk}(\tau) = \frac{\lambda_{ij}(\tau) \lambda_{jk}(\tau)}{\lambda_k(\tau)} \text{Tr}[G_i(\tau) G_j(\tau) F_k^\dagger(\tau)].
\]

Note that if the dynamical map is commutative,

\[
[A_i, \Lambda_\tau] = 0,
\]

then \( F_i \) and \( G_i \) do not depend on time. Moreover, they also provide the corresponding eigenvectors for the generator \( \mathcal{L}_\tau \): 

\[
\mathcal{L}_\tau[F_i] = \mu_i(\tau) F_i,
\]

and
\[ L_t^s[G_t] = \mu_t^s(\tau)G_t, \]  
(109)

In this case one has
\[
\lambda_t(\tau) = \exp \left[ \int_0^\tau \mu_t(u) \, du \right].
\]  
(110)

Moreover, in the case of Markovian semigroup one finds \( \lambda_t(\tau) = e^{\mu \tau} \) and hence
\[
\alpha_t^{\beta}(\tau) = e^{\mu_t^{\beta}\tau - \mu_t^\beta\tau} \text{Tr}[G_t G_t^\dagger].
\]  
(111)

This implies
\[
G_t \circ \tau G_j = \sum_k \alpha_t^{\beta}(\tau)G_k.
\]  
(112)

Hence, by taking \( \{G_0, G_1, \ldots, G_{d-1}\} \) as the basis in \( \mathcal{B}(\mathcal{H}) \), one finds that commutators and anticommutators are defined, respectively, as
\[
\{G_t, G_j\}_t = \sum_k \alpha_t^{\beta}(\tau)G_k,
\]  
(113)
\[
\{G_t, G_j\}_t = \sum_k \alpha_t^{\beta}(\tau)G_k.
\]  
(114)

In conclusion, the \( \tau \rightarrow \infty \) limit (if exists) is naturally defined as
\[
G_t \circ \tau G_j = \lim_{\tau \rightarrow \infty} G_t \circ \tau G_j = \sum_k \alpha_t^{\beta}(\infty)G_k,
\]  
(115)
\[
\{G_t, G_j\}_\infty = \lim_{\tau \rightarrow \infty} \{G_t, G_j\}_t = \sum_k \alpha_t^{\beta}(\infty)G_k.
\]  
(116)
\[
\{G_t, G_j\}_\infty = \lim_{\tau \rightarrow \infty} \{G_t, G_j\}_t = \sum_k \alpha_t^{\beta}(\infty)G_k.
\]  
(117)

Let us look at what happens for the phase damping of a qubit, analyzed in section 3.1 in the Markovian case. The evolution of the density matrix of the qubit is given by equations (97) and (99),
\[
\dot{\rho}(\tau) = L_t^s[\rho(\tau)] = -\frac{1}{2} \gamma(\tau)(\rho - \sigma_3 \sigma_3),
\]  
(118)
where \( \gamma(\tau) \) is a real function. This generator gives rise to a legitimate dynamical map iff
\[
\Gamma(\tau) = \int_0^\tau \gamma(s) \, ds \geq 0,
\]  
with
\[
\rho(\tau) = \Lambda_t^\gamma[\rho(0)] = \frac{1}{2}(1 + e^{-\Gamma(\tau)})\rho(0) + \frac{1}{2}(1 - e^{-\Gamma(\tau)})\sigma_3 \sigma_3.
\]  
(119)

This map is self-dual \( L_t = L_t^\dagger \) and \( \Lambda_t = \Lambda_t^\dagger \) and yields
\[
\Lambda_t^\gamma[\sigma_{0,3}] = \Lambda_t^\gamma[\sigma_{0,3}] = \sigma_{0,3},
\]  
(120)
\[
\Lambda_t^\gamma[\sigma_{1,2}] = e^{-\Gamma(\tau)}\sigma_{1,2} \rightarrow \Lambda_t^\gamma[\sigma_{1,2}] = e^{-\Gamma(\tau)}\sigma_{1,2}.
\]  
(121)
where $\Gamma_\infty \equiv \Gamma(\infty)$. Note that $\Gamma_\infty$ controls the residual coherence in the asymptotic state. The associative product (115) yields

\[
\sigma_{0,3} \circ_T \sigma_{0,3} = \sigma_{0,3} \circ \sigma_{0,3} \quad \Rightarrow \quad \sigma_{0,3} \circ_\infty \sigma_{0,3} = \sigma_{0,3} \circ \sigma_{0,3},
\]

(122)

\[
\sigma_{1,2} \circ_T \sigma_{1,2} = e^{-2i\gamma} \sigma_{1,2} \circ \sigma_{1,2} \quad \Rightarrow \quad \sigma_{1,2} \circ_\infty \sigma_{1,2} = e^{-2i\gamma} \sigma_{1,2},
\]

(123)

\[
\sigma_{0,3} \circ_T \sigma_{1,2} = \sigma_{0,3} \circ \sigma_{1,2} \quad \Rightarrow \quad \sigma_{0,3} \circ_\infty \sigma_{1,2} = \sigma_{0,3} \circ \sigma_{1,2}.
\]

(124)

There are two different cases: if $\Gamma_\infty < \infty$ then $\mathcal{A}_\infty$ and $\mathcal{A}$ are isomorphic. On the other hand, if $\Gamma_\infty = \infty$ then

\[
\alpha^k_{ij}(\infty) = \begin{cases} 
\alpha^k_{ij} & \text{if } i \in \{0, 3\} \text{ or } j \in \{0, 3\} \\
0 & \text{if } i, j \in \{1, 2\}
\end{cases}
\]

(125)

and we recover the situation depicted in section 3.1. It is thus evident that for a Markovian semigroup the asymptotic algebra $\mathcal{A}_\infty$ contracts into the abelian $\mathbb{C}^2$. In the non-Markovian case the asymptotic coherence in general prevents such contraction.

The same situation occurs for the phase damping of a harmonic oscillator, analyzed in section 3.1, when the decoherence rate $\gamma(t)$ in equation (38) is time-dependent. Again, if $\Gamma_\infty < \infty$, then (41) is replaced by

\[
d^i_a \circ_a = e^{-\Gamma_\infty} d^i_a \circ a
\]

(126)

and $\{1, \hat{a} = e^{-\Gamma_\infty/2} a, \hat{a}^\dagger = e^{-\Gamma_\infty/2} a^\dagger, N\}$ reproduces the original Heisenberg–Weyl oscillator algebra $\mathfrak{h}_4$. If $\Gamma_\infty = \infty$, the Heisenberg–Weyl algebra is contracted to $\text{iso}(1, 1)$.

The examples investigated in section 3 and the $d$-level quantum system analyzed in section 4 can be generalized to the non-Markovian case by considering a time-dependent rate $\gamma(\tau)$. One recovers the Markovian case if $\Gamma_\infty = \infty$ and can partially preserve coherence if $\Gamma_\infty < \infty$. The analysis of yet more general (non-Markovian) evolutions will be performed in a future article. Additional difficulties will probably be encountered with general maps, that develop non-invertibility at finite times.

6. Conclusions and outlook

One of the main differences between classical and quantum systems lies in the commutation properties of its observables. Noncommutativity is a distinctive quantum feature: observables that can be simultaneously measured are in this sense ‘classical’ with respect to each other, and when all observables commute the system can be viewed as fully classical. These notions can be formulated in terms of the algebra of the operators of the system, and therefore can be traced back to the structure of the associative product among them.

Dissipation and decoherence generically tend to deteriorate quantum features of a system and make it increasingly classical. In this article, we have defined a product that detects the dissipative features of the evolution and the increasing difficulty in measuring those observables that are more affected by decoherence and dissipation. In the long-time limit, this procedure yields a contracted algebra of operators. The contracted algebra always contains commutative subalgebra(s) and is, therefore, more ‘classical’ than the original one. The whole procedure is based on Dirac’s prescription (4). We have worked out a number of examples with Markovian dynamics, and have seen that a key role is played by the asymptotic state of the evolution. The non-Markovian case is more involved, but a rather general class of time-local master
equations yields results that are consistent with those obtained in the Markovian case and recover the latter in a well-defined limit.

Observables that still do not commute after the contraction can be regarded as nonclassical, even in presence of the given dissipative process. They do not belong to the center of the contracted algebra and are associated with a nontrivial kernel of $\mathcal{L}$, which coincides with the commutant of the interaction algebra, that is the algebra generated by the Kraus operators and their adjoints [25–27]. These observables preserve (some or all of) their quantum features. The picture that emerges is interesting: the dissipative process will contribute to determine which observables become ‘classical’ and which ones remain ‘quantum’. This is to be contrasted with unitary evolutions, that do not change the nature of the observables of the system. We have analyzed in detail a paradigmatic case study for the dissipative evolution of a $d$-level quantum system. In general, the generator in equation (77) is determined by the physics of the problem. A part of the initial algebra will survive the contraction and will define those observables that preserve their quantum features after the dissipative evolution.

This clarifies that good testbeds for these ideas are physical systems in which decoherence and/or dephasing make it increasingly difficult to make measurements along some directions. In practice, one needs a system containing a subset of directions that are not heavily affected by dissipation (and are therefore stable on an appropriate time scale) and a dissipative process that acts along the other directions. The deformation of the algebra would reflect the increasing difficulty in extracting information (via measurements) along the latter directions. In the long-time limit, when the algebra contraction has taken place, measurements along these directions would yield no relevant information (e.g. because the expectation value of these observable in the asymptotic state would vanish). Relevant experimental platforms that provide such a noise process are superconducting circuits in waveguide QED [28], as well as photonic crystal waveguides [29] and optical fibers [30].

The contractions bear the consequences of the irreversibility of the dynamics. We will discuss in a future article the extension of the present framework to more general non-unitary evolutions such as (non-Markovian) quantum channels in the Kraus–Sudarshan representation [31, 32].

Acknowledgments

This work was partially supported by PRIN 2010LLKJBX on ‘Collective quantum phenomena: from strongly correlated systems to quantum simulators’, by INFN through the project ‘QUANTUM’, by the Italian National Group of Mathematical Physics (GNFM-INdAM), and by Sharif University of Technology’s Office of Vice President for Research. DC was partially supported by the National Science Centre project 2015/17/B/ST2/02026. G M would like to acknowledge partial support by the Excellence Chair Program, Santander-UCIIM.

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