Phase Transitions of Bipartite Entanglement

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We analyze the statistical properties of the entanglement of a large bipartite quantum system. By framing the problem in terms of random matrices and a fictitious temperature, we unveil the existence of two phase transitions, characterized by different spectra of the reduced density matrices.

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The bipartite entanglement of small quantum systems (such as a pair of qubits) can be given a quantitative characterization in terms of several physically equivalent measures, such as entropy and concurrence [1]. The problem becomes more complicated for larger systems and/or higher dimensional qudits [2]. The interest of characterizing entanglement for these systems is twofold: on one hand, it has fascinating links with complexity [3] and a related definition of multipartite entanglement [4]; on the other hand, it has applications in quantum information and related fields of investigation [5].

In this Letter we intend to characterize the statistics of the entanglement of a large quantum system. We shall tackle this problem by studying a random matrix model that describes the statistical properties of the purity of a bipartite quantum system. In the context of quantum information this model was introduced in [6,7] in order to describe the statistics of the eigenvalues of the reduced density matrix of a subsystem and extract the first moments of some quantities of interest, like the entanglement entropy or the purity. We will obtain the exact generating function of the purity in the limit of large space dimension (large N in the matrix model) and will connect the entropy with the volume of the manifolds with constant purity (isopurity manifolds). We will also show that the matrix model undergoes two phase transitions: one at a negative and one at a positive (fictitious) temperature. The phase transition at negative temperature will be paralleled to another one, that is well known in the study of random matrix models and conformal field theory literature [8]. We notice that techniques related to those presented in this Letter have been recently employed [9] to analyze the statistics of the lowest eigenvalue of the reduced density matrix.

Consider a bipartite system in the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, with $\dim\mathcal{H}_A = N \le \dim\mathcal{H}_B = M$. Assume that the system is in a pure state $|\psi\rangle \in \mathcal{H}$. The reduced density matrix of subsystem A reads

$$\rho_A = \mathrm{Tr}_B|\psi\rangle\langle\psi| \tag{1}$$

and is a Hermitian, positive, unit-trace $N \times N$ matrix. Its purity

$$\pi_{AB} = \mathrm{Tr}_A\rho_A^2 \in [1/N, 1] \tag{2}$$

is a good measure of the entanglement between the two subsystems: its minimum is attained when all the eigenvalues are equal to $1/N$ (completely mixed state, maximal entanglement between the two bipartitions), while its maximum detects a factorized state (no entanglement). We consider a typical pure state $|\psi\rangle$ [6,7], sampled according to the unique, unitarily invariant Haar measure. The significance of this measure can be understood in the following way: let us fix a state vector $|\psi_0\rangle$ and consider a random unitary transformation $|\psi\rangle = U|\psi_0\rangle$. In the least set of assumptions on U one obtains the Haar measure on U which induces a uniform measure on $|\psi\rangle$, independent of $|\psi_0\rangle$. By tracing over subsystem B, this measure translates into the measure over the space of Hermitian, positive matrices of unit trace [6,7]

$$\begin{aligned} d\mu(\rho_A) &= \mathcal{D}\rho_A(\det\rho_A)^{M-N}\delta(1-\mathrm{Tr}\rho_A),\\ &= d^N\lambda\prod_{i<j}(\lambda_i-\lambda_j)^2\prod_{\ell}\lambda_\ell^{\mu N}\delta\Big(1-\sum_k\lambda_k\Big), \end{aligned} \tag{3}$$

where λ_k are the positive eigenvalues of ρ_A (Schmidt coefficients), we dropped the volume of the $SU(N)$ group (which is irrelevant for our purposes) and $\mu N \equiv M - N$.

We will consider the statistical properties of the rescaled quantity

$$R = R_{AB} = N^3\pi_{AB}. \tag{4}$$

The moments of this function can be obtained by lengthy, direct calculations. We will propose a different approach that makes use of a partition function:

$$Z_{AB} = \int d\mu(\rho_A) \exp(-\beta R_{AB}), \tag{5}$$

where β is a fictitious temperature. This approach is easily generalizable to any other measure of entanglement. In particular for $\beta = 0$ one obtains typical states, while for larger values of β one gets more entangled states (for $\beta \to \infty$ maximally entangled states).

Henceforth, we will assume $N \gg 1$. We will analyze in detail the case $\mu = 0$ and then give the results for $M - N = \mu N > 0$. Our problem has been translated into the study of random (reduced) density matrices ρ_A with

$$Z_{AB} = \int_{\lambda_i>0} d^N\lambda \prod_{i<j} (\lambda_i - \lambda_j)^2 \delta\Big(1 - \sum_{i=1}^{N} \lambda_i\Big) e^{-\beta N^3 \sum_i \lambda_i^2}. \tag{6}$$

As a first step, we introduce a Lagrange multiplier for the delta function

$$Z_{AB} = N^2 \int \frac{d\xi}{2\pi} \times \int_{\lambda_i>0} d^N\lambda e^{iN^2\xi(1-\sum_i \lambda_i) - \beta N^3 \sum_i \lambda_i^2 + 2\sum_{i<j} \ln|\lambda_i - \lambda_j|}. \tag{7}$$

By assuming N large we can look for the stationary point of the exponent with respect to both the λ_i's and ξ. The contour of integration for ξ lies on the real axis but we will soon see that the saddle point for ξ lies on the imaginary ξ axis. It is then understood that the contour needs to be deformed to pass by this point parallel to the line of steepest descent. The saddle point equations are

$$-2\beta N^3 \lambda_i + 2\sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j} - iN^2\xi = 0, \tag{8}$$

$$\sum_i \lambda_i = 1. \tag{9}$$

In the limit of large N, by adopting the natural scaling

$$\lambda_i = \frac{1}{N}\lambda(x_i), \qquad 0 < x_i = \frac{i}{N} \leq 1, \tag{10}$$

we can write Eq. (8) as

$$-\beta\lambda + -\!\!\!\!\!\!\int_0^\infty d\lambda' \frac{\rho(\lambda')}{\lambda - \lambda'} - i\frac{\xi}{2} = 0, \tag{11}$$

where

$$\rho(\lambda) = \int_0^1 dx \delta(\lambda - \lambda(x)) \tag{12}$$

is the density of eigenvalues. A similar equation, restricted at $\beta = 0$, was studied by Page [7].

We start at high temperatures $\beta \ll 1$ and assume a solution of the form [10] (see Fig. 1)

$$\rho(\lambda) = \frac{\beta}{\pi}\Big(\frac{b}{2} + \lambda\Big)\sqrt{\frac{a-\lambda}{\lambda}}, \tag{13}$$

for $0 \leq \lambda \leq a$ and 0 otherwise. This form satisfies the integral equation as can be promptly verified. The Lagrange multiplier ξ is related to the parameters a, b by $\xi = i\beta(a - b)$, and it is purely imaginary, as anticipated.

We can find a, b by imposing normalization and the constraint, which derive from (12) and (9),

$$\int_0^a d\lambda \rho(\lambda) = 1, \qquad \int_0^a d\lambda \rho(\lambda)\lambda = 1. \tag{14}$$

By imposing the form (13) we find

$$\frac{\beta}{8} a(a + 2b) = 1, \qquad \frac{\beta}{16} a^2(a + b) = 1. \tag{15}$$

For $\beta_- < \beta < \beta_+$ with

$$\beta_- = -2/27, \qquad \beta_+ = 2, \tag{16}$$

there is a unique solution of these equations that yields real, positive $\rho(\lambda)$:

$$a(\beta) = \sqrt{\frac{8}{3\beta}}\Big(\Delta - \frac{1}{\Delta}\Big), \qquad b(\beta) = \frac{4}{\beta a} - \frac{a}{2}, \tag{17}$$

where $\Delta = (\sqrt{-\beta/\beta_-} + \sqrt{1 - \beta/\beta_-})^{1/3}$. The average purity is given by

$$\langle \pi_{AB} \rangle = \frac{R}{N^3} = \sum_i \lambda_i^2 = \frac{1}{N}\frac{\beta}{128} a^3(5a + 4b). \tag{18}$$

By using (17) one shows that $R(\beta = 0) = 2N^2$, $R(\beta_+) = 5N^2/4$ and $R(\beta_-) = 9N^2/4$ (see later for the significance of this values).

One can also compute the free energy

$$F = R - \frac{2N^2}{\beta} \int_0^1 dx \int_0^x dy \log|\lambda(x) - \lambda(y)| \tag{19}$$

and using the saddle point equations (11) it is possible to show that

$$\int d\lambda\rho(\lambda) \int d\lambda' \rho(\lambda') \log|\lambda' - \lambda| = \int d\lambda \rho(\lambda)\Big(\log\lambda + \beta\frac{\lambda^2}{2} + i\frac{\xi}{2}\lambda\Big), \tag{20}$$

where we also used (14), and obtain

$$\frac{F}{N^2} = \frac{1}{8}(6 - a)a - \frac{2 + a\log(a/4)}{a\beta} + \frac{3a^4\beta}{256}, \tag{21}$$

in terms of the function $a(\beta)$ introduced above.

Notice that βF is the generating function for the connected correlations of R. The radius of convergence in the expansion around $\beta = 0$ defines the behavior of the late terms in the correlations.

One can find the values of all the cumulants of R, π_{AB} (or connected correlations, the derivatives of $\log Z_{AB}$) in the unbiased distribution at $\beta = 0$, when $\rho(\lambda) = (1/2\pi)\sqrt{(4-\lambda)/\lambda}$. One starts by observing that a series expansion of (17) yields

$$a(\beta) = \sum_{l\geq 0} 4^{l+1} 3^{1-3l} \frac{(3l-1)!}{(2l+1)!(l-1)!}\left(\frac{\beta}{\beta_-}\right)^l. \quad (22)$$

By making use of this expression one finds

$$\langle\langle \pi_{AB}^n \rangle\rangle = -\frac{(-1)^n}{N^{3n}} \frac{\partial^n}{\partial\beta^n}(\beta F)\bigg|_{\beta\to 0} = \frac{2^{n+1}}{N^{3n-2}}\frac{(3n-3)!}{(2n)!}. \quad (23)$$

The first three cumulants are of course the large-N limits of known results [6] (for small N exact expressions for the first 5 cumulants can be also found in [11]).

We are now ready to unveil the presence of two phase transitions. The most evident one is at the end of the radius of convergence of the small β expansion, which occurs at β_-. We can extend our equations smoothly down to β_- but not below. At β_- we have $\rho(\lambda) = 2/(27\pi)(6-\lambda)^{3/2}/\sqrt{\lambda}$ and $\pi_{AB} = 9/4N$ (see Figs. 1 and 2). The derivative at the right edge of eigenvalue density vanishes and some eigenvalues can evaporate to $+\infty$ [12]. The limits $\beta \to \beta_-$ and $N \to \infty$ can be combined (double-scaling limit) to interpret the free energy as the partition function of random 2D surfaces (a theory of pure gravity). Using (23) we see that around β_- the free energy $F \propto (\beta - \beta_-)^{5/2} +$ less singular [8]. In fact, if one relaxes the unit-trace condition, our partition function $\mathcal{Z}_{AB}$ has been studied in the context of random matrix theories [13]. The objects generated in this way correspond to chequered polygonations of surfaces. Our calculations show that the constraint $\mathrm{Tr}\rho_A = 1$ is irrelevant for the critical exponents.

The other phase transition occurs as β is increased (the temperature decreased). The value of b decreases continuously and eventually vanishes at β_+ (where $\pi_{AB} = 5/4N$), becoming $b < 0$ for $\beta > \beta_+$. The solution (13) is not valid anymore, since $\rho(\lambda)$ becomes negative for $\lambda < -b/2$. We have to look for another solution, and, by noting that at β_+, $\rho(\lambda) = (\beta_+/\pi)\sqrt{\lambda(2-\lambda)}$ (see Fig. 1), we do so in the usual semicircle form

$$\rho(\lambda) = \frac{\beta}{\pi}\sqrt{\lambda - b}\sqrt{a-\lambda}. \quad (24)$$

The normalization and the constraint yield

$$\frac{\beta}{8}(a-b)^2 = 1, \qquad \frac{\beta}{16}(a-b)^2(a+b) = 1. \quad (25)$$

This can be easily solved to find

$$a = 1 + \sqrt{\frac{\beta_+}{\beta}}, \qquad b = 1 - \sqrt{\frac{\beta_+}{\beta}} \quad (26)$$

and hence

$$R = N^2\left(1 + \frac{1}{2\beta}\right). \quad (27)$$

Moreover, from (19) and (20), one gets

$$\frac{F}{N^2} = 1 + \frac{3}{4\beta} + \frac{1}{2\beta}\log(2\beta). \quad (28)$$

We can now notice how the phase transition at β_+ is due to the restoration of a $\mathbb{Z}_2$ symmetry P ("parity") present in Eq. (11), namely, the reflection of the distribution $\rho(\lambda)$ around the center of its support ($\lambda = a/2$ for $\beta \leq \beta_+$ and 1 for $\beta > \beta_+$). For $\beta \leq \beta_+$ there are two solutions linked by this symmetry, and we picked the one with the lowest F; at β_+ these two solutions coincide with the semicircle (24), which is invariant under P and becomes the valid and stable solution for higher β.

The expression for the entropy $S = \beta(R - F)$, which counts the number of states with a given value of the purity, is obtained from Eqs. (18), (21), (27), and (28). In the critical region, $\beta \to \beta_+$, we find

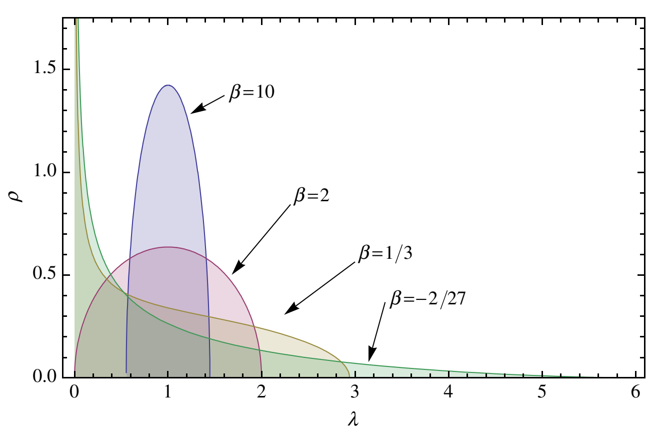

FIG. 1 (color online). Density of eigenvalues at different temperatures. The phase transitions occur at $\beta_+ = 2$ and at $\beta_- = -2/27$.

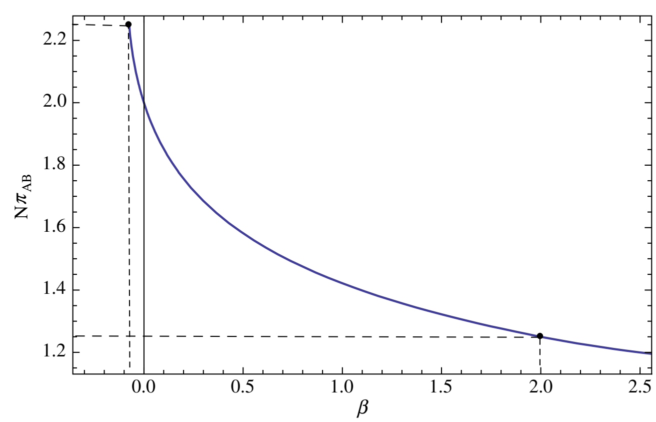

FIG. 2 (color online). $\langle\pi_{AB}\rangle$ as a function of the inverse temperature. Notice the value $\langle\pi_{AB}\rangle = 2/N$ at $\beta = 0$ (typical states). In the $\beta \to \infty$ limit we find the minimum $\langle\pi_{AB}\rangle = 1/N$. The phase transitions described in the text are at $\beta_- = -2/27$, $\langle\pi_{AB}\rangle = 9/4N$ (left point) and $\beta_+ = 2$, $\langle\pi_{AB}\rangle = 5/4N$ (right point).

$$\frac{S}{N^2} \sim -\frac{1}{4} - \log 2 - \frac{\beta - \beta_+}{4} + \theta(\beta - \beta_+)\frac{(\beta - \beta_+)^2}{16}, \tag{29}$$

where θ is the step function. We see that S is continuous at the phase transitions, together with its first derivative although the second derivative is discontinuous. So this is a second order phase transition.

Notice that the entropy is unbounded from below when $\beta \to +\infty$. The interpretation of this result is quite straightforward: the minimum value of π_{AB} is reached on a submanifold (isomorphic to $SU(N)/Z_N$ [14]) of dimension $N^2 - 1$, as opposed to the typical case vectors which form a manifold of dimension $2N^2 - N - 1$ in the Hilbert space $\mathcal{H}$. Since this manifold has zero volume in the original Hilbert space, the entropy, being the logarithm of this volume, diverges.

With the same techniques, starting from (3) we can find the cumulants of the purity for unbalanced bipartitions. Leaving the details for a forthcoming publication we report the results for the first five cumulants only:

$$\langle \pi_{AB} \rangle = \frac{1}{N}\frac{2+\mu}{1+\mu}, \quad \langle\langle \pi_{AB}^2 \rangle\rangle = \frac{1}{N^4}\frac{2}{(1+\mu)^2},$$
$$\langle\langle \pi_{AB}^3 \rangle\rangle = \frac{8}{N^7}\frac{2+\mu}{(1+\mu)^4}, \quad \langle\langle \pi_{AB}^4 \rangle\rangle = \frac{48}{N^{10}}\frac{6+6\mu+\mu^2}{(1+\mu)^6},$$
$$\langle\langle \pi_{AB}^5 \rangle\rangle = \frac{384}{N^{13}}\frac{22+33\mu+13\mu^2+\mu^3}{(1+\mu)^8}, \tag{30}$$

where $\mu = (M - N)/N$. For $\mu = 0$ these reduce to the results of the previous section.

Conclusions.—We have calculated the generating function of a typical entanglement measure, averaged over the Hilbert space. We have shown that, when interpreted as a partition function, it possesses multiple phase transitions. In the different phases the distribution of Schmidt coefficients have different profiles. Sudden changes of these profiles occur at the phase transitions.

We have studied these phase transition(s) as a function of a fictitious temperature β, introduced to define the generating function of the purity. This fictitious temperature can also be thought of as localizing the measure on set of states with entanglement larger or smaller than the typical one [14] (in the same way temperature is used in classical statistical mechanics to fix the energy to a given value in the thermodynamic limit).

Notice that the phase transitions investigated here, that appear in the study of the generating functions of any entanglement measure, are not quantum phase transitions (QPT). Since entanglement is known to be a good indicator of QPTs [15], it would be interesting to investigate the link, if any, between these different transitions.

In conclusion, by using techniques borrowed from the study of random matrix theory, we gave a complete characterization of the statistics of one entanglement measure. We also proposed one direction in which random matrix theory is likely to play a significant role in the study of entanglement, namely, the role of the phase transitions found in random matrix theory as describing the change in the profile of typical, less or more entangled states.

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