Alternative formulation of the Wigner-Araki-Yanase theorem

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We study the Wigner-Araki-Yanase theorem by utilizing a slightly modified version of a formulation recently proposed by Ozawa [Phys. Rev. Lett. **67**, 1956 (1991)]. After characterizing the quantities that are conserved in the evolution of the total system (object plus apparatus), we suitably decompose the Hamiltonian and show that the Wigner-Araki-Yanase theorem can be formulated in terms of the conserved quantities of the object system and of some observables of the apparatus. A few examples clarify our general arguments.

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I. INTRODUCTION

The presence of an additive conserved quantity imposes very remarkable limitations [1, 2] on a quantum measurement process in the description given by von Neumann [3]. In particular, it is impossible to measure *exactly* those observables of an "object" system that do not commute with an additive conserved quantity of the total system (object plus measuring apparatus).

This argument is essentially due to Wigner, Araki, and Yanase and was initially proved for observables endowed with a discrete spectrum. The case of continuous spectrum was later considered by Ozawa [4], who introduced a description of the measurement process in the Heisenberg picture. The above theorem displays a limitation on the possibility of performing exact measurements of quantum mechanical observables even in principle [1, 2, 4, 5].

The quantum measurement problem [3, 6] is a central issue in the foundations and interpretation of quantum mechanics and is profoundly related to the problem of the internal consistency of quantum theory. The links between a quantum measurement process and a genuine irreversible behavior, leading eventually to dephasing ("decoherence") effects, have recently attracted the interest of many researchers [7, 6].

One of the most interesting aspects of the Wigner-Araki-Yanase (WAY) theorem is, so to speak, its independence from all particular interpretations of the measuring process: Even though the WAY argument deals with quantum measurements, it neither interferes with nor supports any interpretation of the measurement problem. In this sense, it can be stated that the Wigner-Araki-Yanase discovery is one of the most puzzling facets in the interpretation of quantum mechanics.

The aim of the present paper is to clarify the meaning of the WAY theorem and shed some light on Ozawa's formulation. In Sec. II we shall briefly summarize some previous results and emphasize that the process proposed by Ozawa is essentially based on von Neumann's formulation [3]. In Sec. III it will be shown how it is possible to characterize some conserved quantities of the object system. In Sec. IV we shall put forth a decomposition of the total Hamiltonian that depends and hinges essentially on Ozawa's process. In Sec. V we prove a theorem that has a very direct physical meaning and yields insights into the WAY argument. A few examples in Sec. VI will (hopefully) clarify these general arguments. Section VII gives a discussion and an outlook.

II. THE WIGNER-ARAKI-YANASE THEOREM IN OZAWA'S FORMULATION

It will be useful to review the main characteristics of Ozawa's measurement process by stressing at the same time the profound links with measurements in the manner of von Neumann. We shall also emphasize that some additional assumptions are required in order to prove a WAY-type theorem in Ozawa's formulation.

A. Formulation of the measurement process

Let A be an observable to be measured on system I, whose states belong to a separable Hilbert space $\mathcal{H}_1$. We shall consider a quantum measurement process in the following sense: An observable B of system II, whose states belong to a separable Hilbert space $\mathcal{H}_2$, yields the outcome of the measurement after the interaction between systems I and II has taken place. The interaction is described by a unitary operator U on $\mathcal{H}_1 \otimes \mathcal{H}_2$.

The measuring process in the Heisenberg picture, as proposed by Ozawa [4], is given by

$$\mathcal{A}(t) = U^\dagger(t)(A \otimes \mathbf{1})U(t) = A \otimes \mathbf{1}, \tag{2.1}$$

$$\mathcal{B}(t) = U^\dagger(t)(\mathbf{1} \otimes B)U(t) = KtA \otimes \mathbf{1} + \mathbf{1} \otimes N(t), \tag{2.2}$$

where K is a real constant, $\mathcal{A}(0) \equiv A \otimes \mathbf{1}$, and $\mathcal{B}(0) \equiv \mathbf{1} \otimes B$. [Henceforth we write A for the operator on $\mathcal{H}_1$, B for the operator on $\mathcal{H}_2$, and $\mathcal{A}, \mathcal{B}$ for the operators on

$\mathcal{H}_1 \otimes \mathcal{H}_2$. Although the operator $\mathcal{A}$ is time independent, we shall sometimes write $\mathcal{A}(t)$ in order to ease comparison with the above formulas.] The "noise" operator $N(t)$ is self-adjoint and satisfies

$$\lim_{\epsilon \to 0} \langle \xi_\epsilon | \exp[i\mu N(t)] | \xi_\epsilon \rangle = 1, \tag{2.3}$$

where $\{\xi_\epsilon\}_\epsilon$ is a set of normalizable states in $\mathcal{H}_2$ and $\mu \in \mathbb{R}$. The process (2.1) and (2.2) with (2.3) describes a measuring process [8], during which the properties of the observed system I are kept essentially unaltered. Although the process (2.1) and (2.2) was originally considered only at a certain time $t = \tau$ $(K\tau = 1)$, we shall assume in this paper that it holds true at any time during the finite interval $[0, \tau]$. We stress, in this context, that the three models introduced in Refs. [4] and [8], to be reanalyzed in this paper, obey the evolution (2.1) and (2.2). The description (2.1) and (2.2) will henceforth be referred to as the *extended Ozawa process* (EOP).

It was proved in Ref. [8] that the EOP yields a "measurement" in von Neumann's sense. This is readily disclosed by considering the simple case in which the operators A, B, and N have a discrete, nondegenerate spectrum: One initially prepares the detector in an eigenstate of N, say, $|\eta\rangle$ $(N|\eta\rangle = \eta|\eta\rangle)$, and defines the eigenvectors of A and B, respectively, $|n\rangle$ and $|\xi_i\rangle$ $(A|n\rangle = a_n|n\rangle$, $B|\xi_i\rangle = b_i|\xi_i\rangle)$. It is then straightforward, if lengthy, to see that

$$U\left(|n\rangle \otimes |\eta\rangle\right) \propto |n\rangle \otimes |\xi(n,\eta)\rangle \tag{2.4}$$

up to a phase factor (that depends on n and η), where $|\xi(n,\eta)\rangle$ is the eigenvector of B that corresponds to the eigenvalue $b(n,\eta) = a_n + \eta$ [8]. This is clearly a measurement of the first kind [3, 9] and satisfies the repeatability hypothesis [4, 8].

Notice that, in full agreement with von Neumann's description of the measurement process, the evolution is governed by a Hamiltonian and is therefore a strictly unitary process, in which the quantum mechanical coherence is fully kept because the phase correlation among different states of the object system is still present: In this sense, one can also regard the EOP as a spectral decomposition [1] performed on the object system. (This approach leads to the well-known von Neumann "chain" of measurements.) Finally, it is worth emphasizing that the process (2.4) can also be viewed as a quantum nondemolition measurement [10] on system I: The states of system I are set in a one-to-one correspondence with those of system II without modifying the properties of the former. It is very useful to observe that the EOP can be viewed as an *exact* measurement: The properties of system I are *faithfully* mirrored into those of system II after the interaction between the two systems is over. This characteristic makes the EOP very suitable for discussions on the WAY theorem.

B. The WAY argument

Let us now look at the WAY argument by making use of the scheme outlined above. Suppose that there exists an additive conserved quantity L for the total system

$$L = L_1 \otimes \mathbf{1} + \mathbf{1} \otimes L_2, \tag{2.5}$$

such that

$$[L, U(t)] = 0. \tag{2.6}$$

Then a WAY-type theorem might state that the existence of a unitary operator $U(t)$ satisfying Eqs. (2.1), (2.2), (2.5), and (2.6) implies that

$$[L, \mathcal{A}] = [L_1, A] \otimes \mathbf{1} = 0, \tag{2.7}$$

where we made use of the relation

$$[A_1 \otimes B_1, A_2 \otimes B_2] = [A_1, A_2] \otimes B_1 B_2 + A_2 A_1 \otimes [B_1, B_2]. \tag{2.8}$$

The above theorem does not hold in general: It has been proved in Refs. [4] and [8] that additional hypotheses must be called for. In particular, we have the following theorem.

Theorem. If, in addition to Eqs. (2.1), (2.2), (2.5), and (2.6), (i) L_1 is bounded and $\| N\xi_\epsilon \| \to 0$ $(\epsilon \to 0)$, or (ii) L_2 is bounded and $\| N\xi_\epsilon \| \to 0$ $(\epsilon \to 0)$, or (iii) N has at least one normalizable eigenstate, then $[L_1, A] = 0$.

Therefore, in "EOP language," the WAY theorem implies that, if $[L_1, A] \neq 0$, then either one of the hypotheses (i)–(iii) must be violated or it is impossible to devise an "exact" measurement process of the type (2.1) and (2.2). Observe that the conditions (i)–(iii) are independent of the explicit form of the Hamiltonian.

III. CONSERVED QUANTITIES

We will now show some consequences of the WAY theorem in the EOP formulation (2.1) and (2.2). The purpose of this section is to prove a theorem that links some conserved quantities of systems I and I plus II to some commutators of the "WAY type," such as $[L_1, A]$.

Equation (2.1) shows that the observable $\mathcal{A}(t) = A \otimes \mathbf{1}$ is a conserved quantity. Since L is also conserved, the commutator $L^{(1)}\mathcal{A} \equiv [L, \mathcal{A}]$ and, more generally, the quantities

$$L^{(n)}\mathcal{A} \equiv [L, [L, \ldots, [L, \mathcal{A}] \cdots]] \qquad (n = 1, 2, \ldots) \tag{3.1}$$

are also conserved. Using Eq. (2.8), we can rewrite $L^{(n)}\mathcal{A}$ as

$$L^{(n)}\mathcal{A} = [L_1, [L_1, \ldots, [L_1, A] \cdots]] \otimes \mathbf{1} \equiv L_1^{(n)} A \otimes \mathbf{1} \qquad (n = 1, 2, \ldots). \tag{3.2}$$

These operators $L_1^{(n)} A$ act on $\mathcal{H}_1$. Since the observable $N(t)$ belongs to $\mathcal{H}_2$, it follows from Eq. (2.2) that

$$\begin{aligned} Kt[L^{(n)}\mathcal{A}, \mathcal{A}] &= [L^{(n)}\mathcal{A}, \mathcal{B}(t) - \mathbf{1} \otimes N(t)] \\ &= [L^{(n)}\mathcal{A}, \mathcal{B}(t)] \\ &= [U^\dagger(t) L^{(n)}\mathcal{A} U(t), U^\dagger(t)\mathcal{B}(0)U(t)] \\ &= U^\dagger(t)[L_1^{(n)} A \otimes \mathbf{1}, \mathbf{1} \otimes B]U(t) = 0, \end{aligned} \tag{3.3}$$

so that, for $n = 1, 2, \ldots$,

$$[L^{(n)}\mathcal{A}, \mathcal{A}] = [L_1^{(n)}A, A] \otimes \mathbf{1} = 0. \tag{3.4}$$

If $[L, \mathcal{A}] = 0$, then there are no nonvanishing conserved quantities of the type $L^{(n)}\mathcal{A}$. On the other hand, if $[L, \mathcal{A}] \neq 0$, then some additional nontrivial conserved quantities of the type $L^{(n)}\mathcal{A}$, not proportional to the identity, can exist. Moreover, such quantities commute with $\mathcal{A}$. (Notice that these conserved quantities might be trivial even if $[L, \mathcal{A}] \neq 0$: For example, if $[L, \mathcal{A}] =$const, then $L^{(n)}\mathcal{A} = 0$.)

Finally, observe that if $L_1 \otimes \mathbf{1}$ is conserved, then we get $[L_1, A] = 0$. This can be proved in the same way as Eq. (3.3): By replacing $L^{(n)}\mathcal{A}$ in Eq. (3.3) by $L_1 \otimes \mathbf{1}$, we get $[L_1, A] = 0$. (Notice that this proof hinges upon the process (2.1) and (2.2) in an essential way. Moreover, no analogous theorem can be proved for the observable $L = L_1 \otimes \mathbf{1} + \mathbf{1} \otimes L_2$. It is easy to check, from the above proof, that this is due to the fact that, in general, $[L_2, B] \neq 0$. Needless to say, the operators A and B play a very different role in an EOP.) We have thus obtained the following theorem.

Theorem 1. The existence of a unitary operator $U(t)$ satisfying Eqs. (2.1) and (2.2) implies that (i) if $L_1 \otimes \mathbf{1}$ is conserved, then $[L_1, A] = 0$; (ii) if there exists an additive conserved quantity L, then there are some conserved quantities $L^{(n)}\mathcal{A} = L_1^{(n)}A \otimes \mathbf{1}$ such that $[L_1^{(n)}A, A] = 0$. These can be nontrivial only if $[L, \mathcal{A}] = [L_1, A] \otimes \mathbf{1} \neq 0$.

Notice that the proof of the above theorem makes use only of operator properties: no reference is made to the initial states of systems I and II.

The physical meaning of part (i) of the above theorem is best disclosed by considering the following two examples. We shall come back to part (ii) in Secs. V and VI.

Example 1. Let $H = gx \otimes P$, where x, p are the coordinate and momentum of particle I (object system), X, P the analogous quantities of particle II (apparatus), and g a constant. This is the celebrated von Neumann model [3]. Let $A \equiv x$ and $B \equiv X$. The EOP is readily calculated as

$$\mathcal{A}(t) = x \otimes \mathbf{1}, \tag{3.5}$$

$$\mathcal{B}(t) = gtx \otimes \mathbf{1} + \mathbf{1} \otimes X = KtA \otimes \mathbf{1} + \mathbf{1} \otimes N(t), \tag{3.6}$$

where $N(t) = B = X$ and K will henceforth be identified with g. Let $L_1 = p$; obviously $[L_1, A] \neq 0$. By Theorem 1, L_1 cannot be conserved: the reason is that H contains its conjugate variable x.

From the above example we can draw the following general conclusion: In an EOP the quantity A is conserved; since $\mathcal{B}$ must have a nontrivial dynamics (i.e., it must involve A itself in order to yield a measurement of A), the Hamiltonian H must contain A. As a consequence L_1, in order to be conserved (i.e., in order to commute with H), must commute with A. This is manifest in the following.

Example 2. Let $H = g\boldsymbol{\ell}^2 \otimes P$, where $\boldsymbol{\ell}$ is the angular momentum of particle I (object system) and X, P position and momentum of particle II (apparatus) along one direction [8]. The model is not necessarily one dimensional. Let $A \equiv \boldsymbol{\ell}^2$ and $B \equiv X$. The EOP reads

$$\mathcal{A}(t) = \boldsymbol{\ell}^2 \otimes \mathbf{1}, \tag{3.7}$$

$$\mathcal{B}(t) = gt\boldsymbol{\ell}^2 \otimes \mathbf{1} + \mathbf{1} \otimes X = gtA \otimes \mathbf{1} + \mathbf{1} \otimes N(t), \tag{3.8}$$

where $N(t) = B$. Let L_1 be a component of $\boldsymbol{\ell}$. Then L_1 is conserved and therefore $[L_1, A] = 0$.

These two examples shed light on the role played by the Hamiltonian in an EOP. First of all, it must contain the operator A in order to yield the right dynamics (2.1) and (2.2). Second, and consequently, the value of $[L_1, A]$ depends essentially (even though nontrivially, as we shall see later in this paper) on the value of $[L_1 \otimes \mathbf{1}, H]$: It can be important to know whether or not the quantity L_1 is conserved.

Finally, notice that if $[L_1, A] \neq 0$, then L_1 cannot be conserved in an EOP. Conversely, if L_1 is conserved and $[L_1, A] \neq 0$, it is impossible to design a suitable EOP yielding an *exact* measurement of A. (The word "exact" in the above must be understood in the same sense as in the WAY theorem.)

IV. EXPLICIT FORM OF THE HAMILTONIAN

We shall now put forth the explicit form of the Hamiltonian H of the total system so that an EOP can be obtained. This will clarify the role played by a measurement process (in von Neumann's sense) and bring to light some relations between the observables A, B, and H.

We shall start by decomposing the Hamiltonian. The problem of domain of operators may arise since we have to treat, in general, unbounded operators. To avoid this difficulty, we can introduce an appropriate rigged Hilbert space (Gel'fand triplets), where all algebraic operations with the operators are allowed and no questions arise with respect to the domain of definition [11]. We shall ignore for the moment the problem of the domain of definition and postpone a more rigorous analysis to a future publication.

Observe first that, by differentiating Eq. (2.2) with respect to t, we can obtain a relationship between A, B, and H:

$$\mathcal{A} = \frac{i}{K\hbar}[H, \mathcal{B}(t)] - \frac{1}{K}\mathbf{1} \otimes N'(t) \quad \forall t, \tag{4.1}$$

where $N'(t) = dN(t)/dt$ and the time evolution is assumed to be $U(t) = \exp(-iHt/\hbar)$. Notice that the implicit requirement that N be differentiable is not restrictive: N is a "noise" operator in the sense of the WAY theorem (that states the impossibility, even in principle, to perform exact quantum measurements of quantities that do not commute with an additive conserved quantity) and plays *no* role of "noise" in the stochastic sense.

Let the Hamiltonian be given by

$$H = \sum_{n=1}^{N} F_n \otimes G_n, \tag{4.2}$$

where F_n acts on $\mathcal{H}_1$ and G_n on $\mathcal{H}_2$. It is shown in Appendix A that all operators F_n and G_n can be assumed to be self-adjoint and that the above Hamiltonian can be decomposed into four parts:

$$H = A\otimes\Gamma + \mathbf{1}\otimes D + \sum_{n=1}^{N_2}\tilde{F}_n\otimes\tilde{G}_n + \sum_{n=1}^{N_4}\tilde{E}_n\otimes\tilde{D}_n, \tag{4.3}$$

where $N_2+N_4+2=N$. All operators on the right-hand side in Eq. (4.3) are linear combinations of F_n or G_n and as a result, self-adjoint. They satisfy $[B,\Gamma]=iK\hbar\mathbf{1}$, $[B,D]\neq 0$, $[L_1,\tilde{F}_n]\neq 0$, and $[B,\tilde{G}_n]=[L_1,\tilde{E}_n]=[B,\tilde{D}_n]=0$. All operators and nonvanishing commutators of the above type are linearly independent.

Several points are worth stressing: H contains A and an operator $\Gamma\neq 0$ satisfying the canonical commutation relations with B. In writing H as in (4.3) we implicitly assume that such an operator Γ exists. However, it may also happen that $\Gamma=0$ or, in other words, that H does not contain any operator obeying the canonical commutation relations with B. In such a case H does not contain A either, and it can be seen that in this case the EOP is trivial and A is proportional to the identity operator (see Appendix A). This could be expected on the basis of Theorem 1 of Sec. III (see, in particular, the discussion following the first example). We shall therefore henceforth assume that $\Gamma\neq 0$.

Let us next consider the role of the operator D in the Hamiltonian. Equation (4.1) is rewritten as

$$\begin{aligned}\mathbf{1}\otimes N'(t) &= \frac{i}{\hbar}U^\dagger(t)[H,\mathcal{B}(0)]U(t) - K\mathcal{A}\\ &= \frac{i}{\hbar}U^\dagger(t)\{A\otimes[\Gamma,B]+\mathbf{1}\otimes[D,B]\}U(t)-K\mathcal{A}\\ &= \frac{i}{\hbar}[\mathcal{D}(t),\mathcal{B}(t)]\\ &= \frac{i}{\hbar}[\mathcal{D}(t),\mathbf{1}\otimes N(t)],\end{aligned} \tag{4.4}$$

where we used Eqs. (4.3) and (2.2) and defined $\mathcal{D}(t)\equiv U^\dagger(t)(\mathbf{1}\otimes D)U(t)$ [12]. As a consequence of Eq. (4.4), if $D=0$ we have $N'(t)=0$, i.e., $N(t)=N(0)=B$. Conversely, if $N'(t)=0$ we have $[B,D]=0$ from Eq. (4.4). This implies that $D=0$, because D must satisfy $[B,D]\neq 0$ by assumption, if the Hamiltonian H contains D.

The results of this section can be summarized in the following theorem.

Theorem 2. Suppose that Eqs. (2.1) and (2.2) hold and that the Hamiltonian takes any form (4.2). Then H must have the form (4.3) and (i) if H contains A, then it must contain also $\Gamma\neq 0$ such that

$$[B,\Gamma]=iK\hbar\mathbf{1}, \tag{4.5}$$

K being the real constant introduced in Eq. (2.2). (ii) If $\Gamma=0$, the observable A in an EOP must be trivial, namely, $A=a\mathbf{1}$. In other words, in order that A be not a c number, the Hamiltonian must contain $A\otimes\Gamma$. (iii) If $\Gamma\neq 0$, the time evolution of $\mathbf{1}\otimes N(t)$ is given by Eq. (4.4) and

$$N(t)=B \Longleftrightarrow D=0. \tag{4.6}$$

V. ADDITIVE CONSERVED QUANTITY AND A WAY-TYPE THEOREM

Notice that no use has been made of Eqs. (2.5) and (2.6) in proving Theorem 2: L_1 is, at this stage, *any* operator acting on $\mathcal{H}_1$. The decomposition of H put forth in the preceding section is independent of the existence of an additive conserved quantity L of the total system.

We want to investigate the role played by a conserved quantity of the total system $L=L_1\otimes\mathbf{1}+\mathbf{1}\otimes L_2$, as in Eqs. (2.5) and (2.6). We shall show that the value of the WAY commutator $[L_1,A]$ can be written as a function of some operators contained in the Hamiltonian. More to this, it is possible to prove necessary and sufficient conditions for the above commutator to vanish: These conditions depend, as we shall see, on some observables of the "apparatus" system II.

In order to deal with the conserved quantities that appeared in Sec. III let us consider

$$\begin{aligned}&e^{isL_1\otimes\mathbf{1}}\mathcal{A}e^{-isL_1\otimes\mathbf{1}}\\ &\quad=\mathcal{A}+\sum_{n=1}^{\infty}\frac{(is)^n}{n!}(L_1\otimes\mathbf{1})^{(n)}\mathcal{A}\\ &\quad=\mathcal{A}+\sum_{n=1}^{\infty}\frac{(is)^n}{n!}\frac{1}{iK\hbar}[\mathcal{B}(0),(L_1\otimes\mathbf{1})^{(n)}H]\\ &\quad=\mathcal{A}+\frac{1}{iK\hbar}[\mathcal{B}(0),e^{isL_1\otimes\mathbf{1}}He^{-isL_1\otimes\mathbf{1}}-H].\end{aligned} \tag{5.1}$$

In Eq. (5.1), we have used the equality

$$(L_1\otimes\mathbf{1})^{(n)}\mathcal{A}=-\frac{i}{K\hbar}[\mathcal{B}(0),(L_1\otimes\mathbf{1})^{(n)}H], \tag{5.2}$$

which is proved in Appendix B.

Since, by Eq. (4.3), $[\mathcal{B}(0),H]=iK\hbar\mathcal{A}+\mathbf{1}\otimes[B,D]$, we can rewrite Eq. (5.1) as

$$\begin{aligned}&e^{isL_1\otimes\mathbf{1}}\mathcal{A}e^{-isL_1\otimes\mathbf{1}}\\ &\quad=\frac{1}{iK\hbar}[\mathcal{B}(0),e^{isL_1\otimes\mathbf{1}}He^{-isL_1\otimes\mathbf{1}}-\mathbf{1}\otimes D].\end{aligned} \tag{5.3}$$

Finally, since (see Appendix B)

$$e^{isL_1\otimes\mathbf{1}}He^{-isL_1\otimes\mathbf{1}}=e^{-is\mathbf{1}\otimes L_2}He^{is\mathbf{1}\otimes L_2}, \tag{5.4}$$

we arrive at

$$\begin{aligned}&e^{isL_1\otimes\mathbf{1}}\mathcal{A}e^{-isL_1\otimes\mathbf{1}}\\ &\quad=\frac{1}{iK\hbar}[\mathcal{B}(0),e^{-is\mathbf{1}\otimes L_2}He^{is\mathbf{1}\otimes L_2}-\mathbf{1}\otimes D].\end{aligned} \tag{5.5}$$

According to the analysis of Sec. III, the left-hand side of Eq. (5.5) is a conserved quantity, so that the right-hand side must be written in terms of some conserved quantities.

To this purpose, we again decompose the Hamiltonian (4.3) according to whether the operators of the object system contained in H are conserved quantities or not. Let us first rewrite the Hamiltonian (4.3) as

$$H = C_1 \otimes K_1 + C_2 \otimes K_2 + \sum_{n=1}^{N_2+N_4} \tilde{C}_n \otimes \tilde{K}_n, \quad (5.6)$$

where $C_1 = A$, $C_2 = \mathbf{1}$, $K_1 = \Gamma$, $K_2 = D$, and

$$\sum_{n=1}^{N_2+N_4} \tilde{C}_n \otimes \tilde{K}_n = \sum_{n=1}^{N_2} \tilde{F}_n \otimes \tilde{G}_n + \sum_{n=1}^{N_4} \tilde{E}_n \otimes \tilde{D}_n. \quad (5.7)$$

Notice that $[B, \tilde{K}_n] = 0$. Suppose that there exists a conserved quantity C_3 that can be expressed as $C_3 = \sum_{n=1}^{m} c_n \tilde{C}_n$ $(c_1 = 1, m \le N_2+N_4)$, where the summation includes $\tilde{C}_1$. Observe that

$$\sum_n \tilde{C}_n \otimes \tilde{K}_n = \sum_{n=1}^{m} c_n \tilde{C}_n \otimes \tilde{K}_1 + \sum_{n=2}^{m} \tilde{C}_n \otimes (\tilde{K}_n - c_n \tilde{K}_1) + \sum_{n=m+1}^{N_2+N_4} \tilde{C}_n \otimes \tilde{K}_n. \quad (5.8)$$

Changing the symbol for some operators $\tilde{K}_n$, we express the Hamiltonian as

$$H = \sum_{n=1}^{3} C_n \otimes K_n + \sum_{n=2}^{N_2+N_4} \tilde{C}_n \otimes \tilde{K}_n. \quad (5.9)$$

The quantities C_n are conserved and all operators C_n, $\tilde{C}_n$, K_n, and $\tilde{K}_n$ in Eq. (5.9) are independent. Proceeding in the same way, we can write the Hamiltonian as

$$H = \sum_n C_n \otimes K_n + \sum_n \tilde{C}_n \otimes \tilde{K}_n, \quad (5.10)$$

where C_n are conserved and all operators on the right-hand side are independent. Furthermore we cannot construct any conserved quantities from any linear combination of $\tilde{C}_n$.

Substituting Eq. (5.10) into Eq. (5.5) leads us to

$$e^{isL_1 \otimes \mathbf{1}} \mathcal{A} e^{-isL_1 \otimes \mathbf{1}} = \sum_n C_n \otimes \frac{1}{iK\hbar}[B, e^{-isL_2} K_n e^{isL_2} - D\delta_{n2}] + \sum_n \tilde{C}_n \otimes \frac{1}{iK\hbar}[B, e^{-isL_2} \tilde{K}_n e^{isL_2}], \quad (5.11)$$

where $C_2 = \mathbf{1}$ and $K_2 = D$. Define

$$c_n \mathbf{1} = \frac{1}{iK\hbar}[B, e^{-isL_2} K_n e^{isL_2} - D\delta_{n2}], \quad (5.12)$$

$$\tilde{c}_n \mathbf{1} = \frac{1}{iK\hbar}[B, e^{-isL_2} \tilde{K}_n e^{isL_2}]. \quad (5.13)$$

We can prove that the above two quantities are c numbers; the proof is the same as that of (A14)–(A17) in Appendix A.

Now Eq. (5.11) can be rewritten as

$$e^{isL_1} A e^{-isL_1} = \sum_n c_n C_n + \sum_n \tilde{c}_n \tilde{C}_n, \quad (5.14)$$

where we have used the equality

$$e^{isL_1 \otimes \mathbf{1}} \mathcal{A} e^{-isL_1 \otimes \mathbf{1}} = e^{isL_1} A e^{-isL_1} \otimes \mathbf{1}. \quad (5.15)$$

Since the left-hand side and the first summation on the right-hand side in Eq. (5.14) are conserved quantities, the second summation must also be conserved. However, we know from Eq. (5.10) that we cannot construct any conserved quantities from the linear combination of $\tilde{C}_n$. This shows that $\tilde{c}_n = 0\ \forall n$. Therefore we get the following theorem.

Theorem 3. Suppose that Eqs. (2.1), (2.2), (2.5), and (2.6) hold and that the Hamiltonian takes any form (4.2). Then we have

$$e^{isL_1} A e^{-isL_1} = \sum_n c_n C_n, \quad (5.16)$$

where C_n are the (independent) conserved quantities appearing in the Hamiltonian. By expanding the left-hand side of the above equation we infer that the conserved quantities $(L_1)^{(n)} A$ in Eq. (3.2) are all expressed in terms of the conserved quantities C_n appearing in the Hamiltonian.

Equation (5.16) contains therefore all interesting information about the quantities that are conserved in an EOP. It also yields a WAY-type theorem: From Eq. (5.6), we know that if $K_1 = \Gamma \neq 0$, then $C_1 = A$ and $C_2 = \mathbf{1}$. Moreover, the coefficients c_n depend on s as in Eq. (5.12). Differentiating Eq. (5.16) with respect to s and setting $s = 0$, we obtain

$$[L_1, A] = \frac{i}{K\hbar} \sum_n b_n C_n, \quad (5.17)$$

where

$$b_n \mathbf{1} = [B, [L_2, K_n]] \quad (5.18)$$

are easily shown to be c numbers (the technique is the same as that used in Appendix A).

The value of the commutator $[L_1, A]$ is therefore closely related to the structure of the Hamiltonian. Equation (5.17) yields the WAY-type theorem sought. In fact, since the operators C_n in Eq. (5.17) are independent, we can obtain an interesting necessary and sufficient condition in order that $[L_1, A] = 0$ or, in other words, we get the following theorem.

WAY-type theorem. Let the Hamiltonian take any form (4.2) and $L = L_1 \otimes \mathbf{1} + \mathbf{1} \otimes L_2$ be a conserved quantity. Then in an EOP we have

$$[L_1, A] = 0 \Longleftrightarrow b_n \mathbf{1} = [B, [L_2, K_n]] = 0 \quad \forall n, \quad (5.19)$$

where the operators K_n multiply the independent conserved quantities C_n of system I in the Hamiltonian H [see (5.10)].

Notice that L_1 and A are observables of the *object* system, while B, L_2, and K_n are observables of the *apparatus.* The above theorem shows that, *in general,* we cannot expect $[L_1, A] = 0$ in an EOP.

VI. EXAMPLES

In order to discuss the general and somewhat surprising consequences of the theorems proved in the previous sections, let us look at some examples.

A. An interesting case

Let us first consider a particular case of the WAY-type theorem of the preceding section. Assume the Hamiltonian (5.9) contains the nonvanishing operator $K_1 = \Gamma$. We infer by Eq. (5.19) that

$$b_1 \mathbf{1} = [B,[L_2,\Gamma]] \neq 0 \Longrightarrow [L_1, A] \neq 0. \quad (6.1)$$

On the other hand, because $[B,[L_2,\Gamma]]$ is a c number, Eq. (4.5) implies

$$[L_2,\Gamma] = i\gamma\Gamma + \widetilde{\Gamma}, \quad (6.2)$$

where γ is a c number (that may also vanish) and the operator $\widetilde{\Gamma}$ commutes with B. Moreover, using the Jacobi identity and Eq. (4.5), we easily obtain

$$[B,[L_2,\Gamma]] = [\Gamma,[L_2,B]], \quad (6.3)$$

which implies that the right-hand side is a c number and, by virtue of Eq. (4.5), that

$$[L_2,B] = -i\gamma B + \widetilde{B}, \quad (6.4)$$

where $\widetilde{B}$ commutes with Γ.

In conclusion, we get

$$\gamma \neq 0 \Longrightarrow [L_1,A] \neq 0. \quad (6.5)$$

In other words, if $[L_2,\Gamma]$ and $[L_2,B]$ contain Γ and B, respectively, then $[L_1,A] \neq 0$.

Consider now the following case. System I and system II are two particles of coordinates x, X and momenta p, P, respectively. They interact via the Hamiltonian

$$H = gp \otimes P. \quad (6.6)$$

Let $A = p$ (therefore $\Gamma = gP$) and $B = X$. The EOP reads

$$\mathcal{A}(t) = p \otimes \mathbf{1}, \quad (6.7)$$

$$\mathcal{B}(t) = gtp \otimes \mathbf{1} + \mathbf{1} \otimes X = KtA \otimes \mathbf{1} + \mathbf{1} \otimes N(t), \quad (6.8)$$

so that $N(t) = B = X$ and $g = K$ (K and g will always be identified in this section). Let $L_1 = -xp$ and $L_2 = XP$. Even though neither L_1 nor L_2 is conserved, $L = L_1 + L_2$ is. In agreement with Theorem 1, $[L_1,A] = -i\hbar p \neq 0$. It is easy to check that $[L_2,\Gamma] = [XP,gP] = i\hbar gP = i\hbar\Gamma$ and $[L_2,B] = [XP,X] = -i\hbar X = -i\hbar B$, so that this is a nontrivial example of Eqs. (6.2) and (6.4), with $\gamma = \hbar \neq 0$. Moreover, $b_1 \mathbf{1} = [B,[L_2,\Gamma]] = -\hbar^2 g \mathbf{1}$ and Eq. (5.17) is identically satisfied $[-i\hbar p = (i/g\hbar)(-\hbar^2 g)p]$. This is an interesting application of Eq. (6.5): The value of the commutator $[L_1,A]$ depends on γ.

B. Some conditions on B

Let us now show how some physical requirements on the observable B have an effect on the observable A and the commutator $[L_1,A]$. Observe that H is defined in terms of operators that depend (directly or indirectly) on A and B. Let us start by recalling von Neumann's theorem (see, for example, Chap. IV of Ref. [13]): All irreducible representations of the canonical commutation relations (more precisely, Weyl relations) on a separable Hilbert space are unitarily equivalent, while any reducible representation for them is the direct sum of a finite sequence of irreducible representations.

For the commutator (4.5) to be valid, the spectrum of both B and Γ must be the same as that of a position operator, i.e., $(-\infty,+\infty)$. Consequently, if the spectrum of $B \neq (-\infty,+\infty)$, then there exists no operator Γ such that $[B,\Gamma] \propto \mathbf{1}$, namely, $\Gamma = 0$. Hence Theorem 2 implies $A = a\mathbf{1}$. In other words, in an EOP, in order that A be not a c number (and the measurement be not trivial) the spectrum of B must be $(-\infty,+\infty)$. (This is the case of all the models considered in this paper.)

Let us consider two interesting examples of the situation outlined above. As a first example, consider the case where B has at least one *normalizable* eigenstate. Then $\Gamma = 0$ and, as a result, we get $A = a\mathbf{1}$ and $[L_1,A] = 0$.

As a second example, let B be a bounded operator from below or above. Then there exists no operator Γ such that $[B,\Gamma] \propto \mathbf{1}$, which again leads to, by virtue of Theorem 2, $A = a\mathbf{1}$ and $[L_1,A] = 0$. (This situation is similar to the case of a lower-bounded Hamiltonian, in which there exists no time operator T such that $[H,T] = i\hbar\mathbf{1}$.)

Therefore, in an EOP (2.1) and (2.2), if B has at least one *normalizable* eigenstate or if B is *semibounded*, then $A = a\mathbf{1}$ and $[L_1,A] = 0$. The above conclusion simply reflects some properties of operators endowed with a continuous spectrum.

C. Three models

Let us now look at the three models introduced in Refs. [4] and [8] by making use of the results obtained in the previous sections.

Model 1 [8]. System I is a two-dimensional particle of coordinates x, y and momenta p_x, p_y and system II is another two-dimensional particle of coordinates X, Y and momenta P_X, P_Y. The quantity $A = x$ is measured via the observable $B = X$. The Hamiltonian H must contain x, according to Eq. (4.3); we take

$$H = g(x \otimes P_X + y \otimes P_Y), \quad (6.9)$$

where g is the coupling constant and the evolution is given by $U(t) = \exp(-iHt/\hbar)$. This is a two-dimensional version of von Neumann's model [see Example 1 of Sec. III]. Let $L_1 = xp_y - yp_x$ and $L_2 = XP_Y - YP_X$, so that $[L,H] = 0$. The EOP is

$$\mathcal{A}(t) = x \otimes \mathbf{1}, \quad (6.10)$$

$$\mathcal{B}(t) = gtx \otimes \mathbf{1} + \mathbf{1} \otimes X = KtA \otimes \mathbf{1} + \mathbf{1} \otimes N(t), \quad (6.11)$$

where $N(t) = B = X$ and $g = K$. Since $[L_1,A] = i\hbar y$ it follows from Theorem 1 that L_1 cannot be conserved. Moreover, $[L_1,A] = [L_1,x] = i\hbar y$ and $[L_1,y] = -i\hbar x$, so that according to Theorem 1, there are some conserved quantities of the type $L^{(n)}\mathcal{A}$: $x \otimes \mathbf{1}$ and $y \otimes \mathbf{1}$. The Hamiltonian has the structure (4.3) with $\Gamma = gP_X$ [conjugate to $B = X$ according to (4.5)], $D = \tilde{E}_n = \tilde{D}_n = 0$, and $\tilde{F}_1 = y, \tilde{G}_1 = gP_Y$. Also, the Hamiltonian is written as $H = C_1 \otimes K_1 + C_3 \otimes K_3$ [see Eq. (5.10)], where $C_1 = x, K_1 = gP_X, C_3 = y$, and $K_3 = gP_Y$. Since $D = K_2 = 0$, we get, by Eq. (4.6), $N(t) = B = X$. Moreover, $b_1\mathbf{1} = [B,[L_2,\Gamma]] = 0$, $b_3\mathbf{1} = [B,[L_2,K_3]] = g\hbar^2\mathbf{1}$, and $[L_1, A] = (i/g\hbar)(g\hbar^2)y = i\hbar y$ has the structure (5.17).

This model is interesting because it provides a counterexample to a general WAY-type theorem in which no additional conditions on the operators are required [8]. [Notice that all hypotheses (i)–(iii) of the Theorem stated in Sec. II B are violated.] It shows that it is possible to measure A *exactly* (via an EOP) even though $[L_1, A] \neq 0$.

Model 2 [8]. System I is a three-dimensional particle of coordinates $\mathbf{r}$, momentum $\mathbf{p}$, and angular momentum $\boldsymbol{\ell}$, and system II is another three-dimensional particle of coordinates $\mathbf{R}$, momentum $\mathbf{P}$, and angular momentum $\mathbf{L}$. The measured quantity is $A = \boldsymbol{\ell}^2$ and we set $B = X$. The Hamiltonian is taken to be

$$H = g\boldsymbol{\ell}^2 \otimes P_X. \tag{6.12}$$

The EOP reads

$$\mathcal{A}(t) = \boldsymbol{\ell}^2 \otimes \mathbf{1}, \tag{6.13}$$

$$\mathcal{B}(t) = gt\boldsymbol{\ell}^2 \otimes \mathbf{1} + \mathbf{1} \otimes X = KtA \otimes \mathbf{1} + \mathbf{1} \otimes N(t), \tag{6.14}$$

where $N(t) = B$. Let $L_1 = \ell_x$ and $L_2 = L_X$, each of which is a conserved quantity, so that $[L, H] = 0$. Since $[L_1, A] = 0$, there are no other conserved quantities such as $L^{(n)}\mathcal{A}$. The Hamiltonian H is the simplest possible, with $\Gamma = gP_X$: $H = A \otimes \Gamma = C_1 \otimes K_1$, so that $b_1\mathbf{1} = [B,[L_2,\Gamma]] = 0$ (there are no other b_n's). The present model was originally introduced in order to show that one can have $[L_1, A] = 0$ even though (i)–(iii) of the Theorem in Sec. II B are violated. It also provides an example of the WAY-type theorem (5.19).

Model 3 [4]. System I is a one-dimensional particle of position x and momentum p_x and system II is made up of three one-dimensional particles of positions y, z, w and momenta p_y, p_z, p_w. The measured quantity is $A = x$. The measurement is performed via $B = p_z - p_w$.

The Hamiltonian is taken to be

$$H = \frac{g}{2}x \otimes (w - z) + \frac{g}{2}\mathbf{1} \otimes y(z - w) \tag{6.15}$$

and the EOP is

$$\mathcal{A}(t) = x \otimes \mathbf{1}, \tag{6.16}$$

$$\begin{aligned}\mathcal{B}(t) &= gt\,x \otimes \mathbf{1} + \mathbf{1} \otimes (p_z - p_w - gty)\\ &= KtA \otimes \mathbf{1} + \mathbf{1} \otimes N(t),\end{aligned} \tag{6.17}$$

where $N(t) = p_z - p_w - gty$. If we set $L_1 = p_x$ and $L_2 = p_y + p_z + p_w$, then $[L, H] = 0$. Moreover $[L_1, A] = -i\hbar\mathbf{1}$: According to Theorem 1, L_1 cannot be conserved and the conserved quantities $L^{(n)}\mathcal{A}$ reduce to the trivial quantity $\mathbf{1} \otimes \mathbf{1}$. In the Hamiltonian, we have $\Gamma = (g/2)(w - z)$ and $D = (g/2)y(z - w)$. All other quantities are zero. That is, $H = A \otimes \Gamma + \mathbf{1} \otimes D$. Since D is conserved, Eq. (4.4) gives $N(t) = p_z - p_w - gty$. Moreover, $H = C_1 \otimes K_1 + C_2 \otimes K_2$, where $C_1 = x, K_1 = (g/2)(w-z)$, $C_2 = \mathbf{1}$, and $K_2 = (g/2)y(z - w)$. We get $b_1\mathbf{1} = [B,[L_2,\Gamma]] = 0$ and $b_2\mathbf{1} = [B,[L_2,K_2]] = -g\hbar^2\mathbf{1}$, in agreement with Eq. (5.17). This model was introduced in Ref. [4] for the same purpose as Model 1 of the present section.

VII. DISCUSSION

We have discussed in some detail the Wigner-Araki-Yanase theorem in the "extended" Ozawa process and have shown that some additional hypotheses are required on the operators involved in the measurement process. In particular, the spectrum of some of these operators appears to play a significant role. The problems related to the boundedness of L_2 were already known to Araki and Yanase (see footnote 4 of the first paper in Ref. [2]). These problems become more manifest in Ozawa's formulation of the measurement process and in the EOP considered in this paper. The analysis of Ref. [8], here briefly reconsidered in Sec. II, displayed which additional assumptions must be called for: a WAY-type theorem holds true if one deals, roughly speaking, with operators endowed with a discrete spectrum. Observe, however, that the hypotheses (i)–(iii) of the Theorem in Sec. II B look, to say the least, rather contrived and one has the feeling that it ought to be possible to prove a WAY-type theorem by starting from different (possibly simpler) conditions.

The analysis of Sec. V holds true for any operator, not necessarily bounded and/or endowed with a discrete spectrum. As we have seen, this allows us to consider more general cases, such as, for instance, a von Neumann–like measurement process. The process invented by von Neumann in order to describe a measurement process is a milestone in the literature on the quantum measurement problem. This process has been briefly reviewed in the examples we considered (see, in particular, Example 1 of Sec. III). The formulation proposed in Refs. [4, 8] and the EOP considered in this paper clearly agree with it and recover it as a particular case. The EOP and the von Neumann measuring process are certainly not the most general quantum measurement processes one can conceive. On the other hand, it is interesting to realize under which conditions the argument by Wigner, Araki, and Yanase maintains its validity even when one performs "measurements" of continuous observables.

We saw that, at least in an EOP, the value of the commutator $[L_1, A]$ is directly related to the value of the commutators $[B,[L_2,K_n]]$, as in Eqs. (5.17)–(5.19). We stress again that L_1 and A are observables of the *object* system, while B, L_2, and K_n are observables of the *apparatus.*

Even though this curious relationship between apparatus and object system has been put forth in a particular case (the EOP), it is, in our opinion, more general that it appears at a first glance. A "quantum measurement," in von Neumann's sense, is a dynamical process that sets up a correlation between object system and apparatus. In the Heisenberg picture (such as, for instance, an EOP) such a correlation is mirrored in the properties of some operators that are involved in the measurement. In this context, it is not surprising, *a posteriori*, that the value of the commutator $[L_1, A]$ be related to some properties of the apparatus system. The WAY phenomenon, like several other subtleties involved in the interactions taking place in a quantum measurement, deserves further investigation. There is probably more to come.

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APPENDIX A

In this appendix we analyze the Hamiltonian (4.2) and show that it must be written as in (4.3). Then we show that the EOP is trivial if the Hamiltonian does not contain $A \otimes \Gamma$.

Since the Hamiltonian (4.2) must be self-adjoint, we can rewrite it as

$$H = \frac{1}{2}\sum_{n=1}^{N}[F_n \otimes G_n + F_n^\dagger \otimes G_n^\dagger]. \tag{A1}$$

Each term of Eq. (A1) can be expressed in terms of self-adjoint operators:

$$F_n \otimes G_n + F_n^\dagger \otimes G_n^\dagger = \frac{1}{2}[(F_n + F_n^\dagger) \otimes (G_n + G_n^\dagger) + i(F_n - F_n^\dagger) \otimes (-i)(G_n - G_n^\dagger)]. \tag{A2}$$

Thus, without loss of generality, all operators F_n and G_n in the Hamiltonian (4.2) can be assumed to be self-adjoint.

Let $c_1F_1 + \cdots + c_kF_k = 0$ $(c_j \in \mathbb{R})$. Then the operators $F_1, \ldots, F_k$ are said to be (linearly) independent if all coefficients c_j are zero. Now suppose that F_a $(a = m+1, \ldots, N)$ are independent and that $F_i = \sum_{a=m+1}^{N} c_{ia}F_a$ $(i = 1, \ldots, m)$. Then the Hamiltonian (4.2) is rewritten as

$$H = \sum_{a=m+1}^{N} F_a \otimes G'_a, \tag{A3}$$

where $G'_a = G_a + \sum_{i=1}^{m} c_{ia}G_i$. That is, the operators F_i, which are not independent of the others, can be eliminated from the Hamiltonian. We can thus assume that all operators F_n in Eq. (4.2) are independent. Next, if the operators G_a $(a = m+1, \ldots, N)$ are independent and $G_i = \sum_{a=m+1}^{N} c_{ia}G_a$ $(i = 1, \ldots, m)$, then we have

$$H = \sum_{a=m+1}^{N} F'_a \otimes G_a, \tag{A4}$$

where $F'_a = F_a + \sum_{i=1}^{m} c_{ia}F_i$. The operators F'_a are independent because F_i, F_a are assumed to be independent. Since G_a are also independent, we can assume that all operators in the Hamiltonian (4.2) are independent, without loss of generality.

Next we divide the Hamiltonian (4.2) into two parts, according to whether or not $[L_1, F_n]$ is zero. Changing the symbols for some operators, we may express the Hamiltonian as follows:

$$H = \sum_{n=1}^{N_a} F_n \otimes G_n + \sum_{n=1}^{N_b} E_n \otimes D_n, \tag{A5}$$

where $[L_1, F_n] \neq 0$, $[L_1, E_n] = 0$, and $N_a + N_b = N$. Suppose that $[L_1, F_a]$ $(a = m+1, \ldots, N_a)$ are nonvanishing and independent and that $[L_1, F_i] = \sum_{a=m+1}^{N_a} c_{ia}[L_1, F_a]$ $(i = 1, \ldots, m)$. Rewrite the Hamiltonian (A5) as

$$H = \sum_{a=m+1}^{N_a} F_a \otimes G'_a + \sum_{i=1}^{m} F'_i \otimes G_i + \sum_{n=1}^{N_b} E_n \otimes D_n, \tag{A6}$$

where $G'_a = G_a + \sum_i c_{ia}G_i$ and $F'_i = F_i - \sum_a c_{ia}F_a$. Note that $[L_1, F'_i] = 0$. Since F_a, F_i, E_n are independent, the operators F_a, F'_i, E_n are also independent; similarly, G'_a, G_i, D_n are independent. Thus the Hamiltonian (A5) can be divided into two parts in such a way that F_n, E_n are independent, G_n, D_n are also independent, $[L_1, E_n] = 0$, and the commutators $[L_1, F_n]$ are nonvanishing and independent.

The first term of Eq. (A5) can further be divided into two parts according to whether or not $[B, G_n]$ is zero. Changing the symbols for some operators of the first term in Eq. (A5), we rewrite the Hamiltonian as

$$H = \sum_{n=1}^{N_1} F_n \otimes G_n + \sum_{n=1}^{N_2} \tilde{F}_n \otimes \tilde{G}_n + \sum_{n=1}^{N_b} E_n \otimes D_n, \tag{A7}$$

where $[B, G_n] \neq 0$ and $[B, \tilde{G}_n] = 0$. Suppose that $[B, G_a]$ $(a = m+1, \ldots, N_1)$ are independent and that $[B, G_i] = \sum_{a=m+1}^{N_1} c_{ia}[B, G_a]$ $(i = 1, \ldots, m)$. Then the Hamiltonian (A7) can be rewritten as

$$H = \sum_{a=m+1}^{N_1} F'_a \otimes G_a + \sum_{i=1}^{m} F_i \otimes G'_i + \sum_{n=1}^{N_2} \tilde{F}_n \otimes \tilde{G}_n + \sum_{n=1}^{N_b} E_n \otimes D_n, \tag{A8}$$

where $F'_a = F_a + \sum_i c_{ia}F_i$ and $G'_i = G_i - \sum_a c_{ia}G_a$. Note that $F'_a, F_i, \tilde{F}_n, E_n$ are independent and $G_a, G'_i, \tilde{G}_n, D_n$ are also independent. Moreover, $[L_1, F'_a], [L_1, F_i], [L_1, \tilde{F}_n]$ are all nonvanishing and independent. Note also that $[B, G'_i] = 0$ and $[B, G_a]$ are nonvanishing and independent. Consequently, on the right-hand side of the Hamiltonian (A7), we can assume that all operators are independent, $[L_1, F_n], [L_1, \tilde{F}_n]$ are nonvanishing and independent, $[B, G_n]$ are nonvanishing and independent, and $[L_1, E_n] = [B, \tilde{G}_n] = 0$.

The third term of Eq. (A7) can also be divided into two parts according to whether or not $[B, D_n]$ is zero. After considerations similar to the above ones, the Hamiltonian (4.2) can finally be written

$$H=\sum_{n=1}^{N_1}F_n\otimes G_n+\sum_{n=1}^{N_2}\tilde F_n\otimes\tilde G_n+\sum_{n=1}^{N_3}E_n\otimes D_n+\sum_{n=1}^{N_4}\tilde E_n\otimes\tilde D_n. \tag{A9}$$

Here all the operators on the right-hand side of Eq. (A9) are independent; the commutators $[L_1,F_n],[L_1,\tilde F_n]$ are nonvanishing and independent, $[B,G_n]$, (and also $[B,D_n]$) are nonvanishing and independent, and $[L_1,E_n]=[L_1,\tilde E_n]=[B,\tilde G_n]=[B,\tilde D_n]=0$. Furthermore, as will be proved below, we can assume that the commutators $[B,G_n],[B,D_n]$ are all nonvanishing and independent.

To prove the above statement, suppose that $[B,G_n],[B,D_a]$ $(n=1,\dots,N_1;\ a=m{+}1,\dots,N_3)$ are independent and that the other commutators are expressed as

$$[B,D_i]=\sum_{n=1}^{N_1}c_{in}[B,G_n]+\sum_{a=m+1}^{N_3}d_{ia}[B,D_a], \tag{A10}$$

where $i=1,\dots,m$. Then the Hamiltonian (A9) can be rewritten as

$$H=\sum_{n=1}^{N_1}F'_n\otimes G_n+\sum_{n=1}^{N_2}\tilde F_n\otimes\tilde G_n+\sum_{a=m+1}^{N_3}E'_a\otimes D_a+\sum_{i=1}^{m}E_i\otimes D'_i+\sum_{n=1}^{N_4}\tilde E_n\otimes\tilde D_n, \tag{A11}$$

where $F'_n=F_n+\sum_i c_{in}E_i$, $E'_a=E_a+\sum_i d_{ia}E_i$, and $D'_i=D_i-\sum_n c_{in}G_n-\sum_a d_{ia}D_a$. It is easy to check that $F'_n,\tilde F_n,E'_a,E_i,\ \tilde E_n$ are independent and $G_n,\tilde G_n,D_a,D'_i,\tilde D_n$ are also independent. The commutators $[L_1,F'_n],[L_1,\tilde F_n]$ are nonvanishing and independent. Moreover, $[L_1,E'_a]=[L_1,E_i]=[L_1,\tilde E_n]=[B,\tilde G_n]=[B,D'_i]=[B,\tilde D_n]=0$. Finally, $[B,G_n],[B,D_a]$ are assumed to be nonvanishing and independent. Thus the operators D_i whose commutators $[B,D_i]$ are expressed in terms of other commutators have been eliminated from the Hamiltonian. This completes the proof. The Hamiltonian (4.2) has been decomposed into four parts as in Eq. (A9).

No use has been made so far of Ozawa's process (2.1) and (2.2). The decomposition of the Hamiltonian (4.2) in the form (A9) is quite general and does not utilize any specific properties of the operators L_1 and B (which could be, at this stage, any two operators acting on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively).

In order to obtain Eq. (4.3), consider the process (2.1) and (2.2) and observe that since, by Eq. (2.1), A is constant, we get by Eq. (4.1)

$$\begin{aligned}\mathcal{A}&=\frac{i}{K\hbar}[H,\mathcal{B}(0)]-\frac{1}{K}\mathbf{1}\otimes N'(0)\\&=\frac{i}{K\hbar}\sum_{n=1}^{N_1}F_n\otimes[G_n,B]+\frac{i}{K\hbar}\sum_{n=1}^{N_3}E_n\otimes[D_n,B]\\&\quad-\frac{1}{K}\mathbf{1}\otimes N'(0).\end{aligned} \tag{A12}$$

Thus the commutator $[L_1\otimes\mathbf{1},\mathcal{A}]$ is given by

$$[L_1\otimes\mathbf{1},\mathcal{A}]=\frac{i}{K\hbar}\sum_{n=1}^{N_1}[L_1,F_n]\otimes[G_n,B], \tag{A13}$$

where we have used the relations $[L_1,E_n]=0$ and $[L_1\otimes\mathbf{1},\mathbf{1}\otimes N'(0)]=0$.

Let $\{\xi_i\}$ be a complete orthonormal set in $\mathcal{H}_2$. Since $[L_1\otimes\mathbf{1},\mathcal{A}]=[L_1,A]\otimes\mathbf{1}$, we get $\langle\xi_i|[L_1\otimes\mathbf{1},\mathcal{A}]|\xi_i\rangle=\langle\xi_j|[L_1\otimes\mathbf{1},\mathcal{A}]|\xi_j\rangle\ \forall i,j$. It follows from Eq. (A13) that

$$\sum_{n=1}^{N_1}[L_1,F_n](\langle\xi_i|[B,G_n]|\xi_i\rangle-\langle\xi_j|[B,G_n]|\xi_j\rangle)=0. \tag{A14}$$

Since $[L_1,F_n]$ are nonvanishing and independent, we get

$$\langle\xi_i|[B,G_n]|\xi_i\rangle=\langle\xi_j|[B,G_n]|\xi_j\rangle\quad\forall i,j. \tag{A15}$$

On the other hand, using $\langle\xi_i|[L_1\otimes\mathbf{1},\mathcal{A}]|\xi_j\rangle=0$ $(i\neq j)$, we have

$$\langle\xi_i|[B,G_n]|\xi_j\rangle=0\qquad(i\neq j). \tag{A16}$$

Equations (A15) and (A16) lead to $[B,G_n]=ib_n\mathbf{1}$ $(b_n\in\mathbb{R})$. Suppose that there exists an index n_0 such that one of the previous commutators does not vanish

$$[B,G_0]=ib_0\mathbf{1}, \tag{A17}$$

where $G_0\equiv G_{n_0}$ and $b_0\equiv b_{n_0}\neq0$. Then the other commutators $[B,G_n]$ cannot be proportional to the identity because they are independent. On the other hand, since the above equations (A14)–(A17) hold for all n, they must also be proportional to the identity. This indicates that all operators G_n $(n\neq n_0)$ must vanish identically.

On the other hand, if there exists an index m_0 such that

$$[B,D_0]=id_0\mathbf{1}, \tag{A18}$$

where $D_0\equiv D_{m_0}$ and $d_0\equiv d_{m_0}\neq0$, then the other commutators $[B,D_n]$ $(n\neq m_0)$ cannot be proportional to the identity operator because they are independent. Moreover, the commutators $[B,G_0]=ib_0\mathbf{1}$ and $[B,D_0]=id_0\mathbf{1}$ are independent by assumption, so that if the former is nonvanishing, then the latter must vanish and vice versa. In conclusion, Eq. (A12) can be rewritten as

$$A=\frac{1}{K\hbar}(b_0F_0\vee d_0E_0)+\tilde A, \tag{A19}$$

where $F_0=F_{n_0}$, $E_0=E_{m_0}$, $(\alpha\vee\beta)$ means that we must take "either α or β," and $\tilde A$ satisfies

$$\frac{1}{K}\mathbf{1}\otimes N'(0)=-\tilde A\otimes\mathbf{1}+\frac{i}{K\hbar}\sum_{n\neq m_0}^{N_3}E_n\otimes[D_n,B]. \tag{A20}$$

Since the operators $\mathbf{1}$ and $[B,D_n]$ $(n\neq m_0)$ are independent, the above equation implies that $\tilde A$ and E_n $(n\neq m_0)$ are proportional to the identity: $\tilde A=a\mathbf{1}$ and $E_n=e_n\mathbf{1}$ $(a,\ e_n\in\mathbb{R},\ n\neq m_0)$. The proof is similar to that of (A14)–(A17): Let $\{\xi_i\}$ be a complete orthonormal set in $\mathcal{H}_1$ (not in $\mathcal{H}_2$). Then we have, for any i and j,

$$\langle\xi_i|\mathbf{1}\otimes N'(0)|\xi_i\rangle = \langle\xi_j|\mathbf{1}\otimes N'(0)|\xi_j\rangle. \tag{A21}$$

By substituting (A20) into (A21) we get

$$-(\langle\xi_i|\tilde{A}|\xi_i\rangle - \langle\xi_j|\tilde{A}|\xi_j\rangle)\otimes\mathbf{1} + \frac{i}{K\hbar}\sum_n(\langle\xi_i|E_n|\xi_i\rangle - \langle\xi_j|E_n|\xi_j\rangle)\otimes[D_n,B] = 0. \tag{A22}$$

Since the operators $\mathbf{1}$ and $[D_n,B]$ are independent, we obtain, for any i and j,

$$\langle\xi_i|\tilde{A}|\xi_i\rangle = \langle\xi_j|\tilde{A}|\xi_j\rangle,\ \langle\xi_i|E_n|\xi_i\rangle = \langle\xi_j|E_n|\xi_j\rangle. \tag{A23}$$

On the other hand, when $i \neq j$, we have

$$\langle\xi_i|1\otimes N'(0)|\xi_j\rangle = \langle\xi_i|\xi_j\rangle N'(0) = 0 \tag{A24}$$

because $\langle\xi_i|\xi_j\rangle = 0$. From Eq. (A20) it follows that

$$-\langle\xi_i|\tilde{A}|\xi_j\rangle\mathbf{1} + (i/K\hbar)\sum\langle\xi_i|E_n|\xi_j\rangle[D_n,B] = 0. \tag{A25}$$

Since $\mathbf{1}$ and $[D_n,B]$ are independent, we get (for $i \neq j$)

$$\langle\xi_i|\tilde{A}|\xi_j\rangle = 0,\ \langle\xi_i|E_n|\xi_j\rangle = 0. \tag{A26}$$

Equations (A23) and (A26) imply $\tilde{A} = a\mathbf{1}$ and $E_n = e_n\mathbf{1}$. This completes the proof.

Moreover, if there exists an index ℓ_0 such that $E_1 = e_1\mathbf{1}$, where $E_1 \equiv E_{\ell_0}$ and $e_1 \equiv e_{\ell_0} \neq 0$, then the other operators E_n must vanish because they are independent. Hence the observable A and the Hamiltonian (A9) are respectively expressed as

$$A = \frac{1}{K\hbar}(b_0F_0 \vee d_0E_0) + a\mathbf{1}, \tag{A27}$$

$$H = (F_0\otimes G_0 \vee E_0\otimes D_0) + e_1\mathbf{1}\otimes D_1 + \sum_{n=1}^{N_2}\tilde{F}_n\otimes\tilde{G}_n + \sum_{n=1}^{N_4}\tilde{E}_n\otimes\tilde{D}_n, \tag{A28}$$

where $D_1 \equiv D_{\ell_0}$ and either b_0 or d_0 (or both) must vanish. The observable $A - a\mathbf{1}$ must therefore be contained in the Hamiltonian (A28).

By writing $(b_0\Gamma/K\hbar)$ for G_0 or $(d_0\Gamma/K\hbar)$ for D_0 (in either case Γ does not commute with B), we are led to the Hamiltonian (4.3)

$$H = A\otimes\Gamma + \mathbf{1}\otimes D + \sum_{n=1}^{N_2}\tilde{F}_n\otimes\tilde{G}_n + \sum_{n=1}^{N_4}\tilde{E}_n\otimes\tilde{D}_n, \tag{A29}$$

where $D = e_1D_1 - a\Gamma$, $[B,\Gamma]\neq 0$, $[B,D]\neq 0$, $[L_1,\tilde{F}_n]\neq 0$, $[B,\tilde{G}_n] = [L_1,\tilde{E}_n] = [B,\tilde{D}_n] = 0$, and all operators and nonvanishing commutators are linearly independent. Observe that $[B,D]\neq 0$ is a consequence of the fact that $[B,\Gamma], [B,D_1]$ are nonzero and linearly independent. Notice that, by (A17), (A18) and the definition of Γ, one obtains $[B,\Gamma] = iK\hbar\mathbf{1}$. We have thus obtained Eq. (4.3).

Before proving that the EOP becomes trivial if H does not contain A, it is necessary to emphasize that we are implictly assuming that $\Gamma \neq 0$ (or, in other words, that the Hamiltonian H contains an operator Γ obeying the canonical commutation relation with B). In such a case, Eq. (A20) leads to

$$N'(0) = \frac{i}{\hbar}[e_1D_1 - a\Gamma, B] = \frac{i}{\hbar}[D,B]. \tag{A30}$$

On the other hand, Γ can vanish (or, in other words, the Hamiltonian H contains *no* operator Γ obeying the canonical commutation relation with B). In such a case, $D = e_1D_1$ and Eq. (A20) yields

$$N'(0) = -aK\mathbf{1} + \frac{i}{\hbar}[e_1D_1,B] = -aK\mathbf{1} + \frac{i}{\hbar}[D,B]. \tag{A31}$$

In both cases, from Eq. (A12) and (A29) we get

$$\mathcal{A} = \frac{i}{K\hbar}\{A\otimes[\Gamma,B] + \mathbf{1}\otimes[D,B]\} - (1/K)\mathbf{1}\otimes N'(0) \tag{A32}$$

(where Γ may or may not vanish) and by using Eq. (A30) we obtain

$$\mathcal{A} = \frac{i}{K\hbar}(A - a\mathbf{1})\otimes[\Gamma,B] + a\mathbf{1}\otimes\mathbf{1}. \tag{A33}$$

If $[B,\Gamma] = iK\hbar 1$, we get the identity $\mathcal{A} = A\otimes 1$. On the other hand, if $\Gamma = 0$, we find

$$\mathcal{A} = a\mathbf{1}\otimes\mathbf{1}. \tag{A34}$$

This shows that the EOP is trivial if H does not contain A, i.e., $\Gamma = 0$.

It is worth emphasizing that the condition $\Gamma = 0$ can reflect nontrivial physical requirements: For example, if the spectrum of $B \neq (-\infty,+\infty)$, there can exist no Γ such that $[B,\Gamma] = iK\hbar\mathbf{1}$. This is discussed in Sec. VI.

APPENDIX B

In order to prove Eq. (5.2), we start by noticing that the Hamiltonian (4.3) gives

$$[L_1\otimes\mathbf{1},H] = [L_1,A]\otimes\Gamma + \sum_{n=1}^{N_2}[L_1,\tilde{F}_n]\otimes\tilde{G}_n. \tag{B1}$$

Taking into account $[B,\Gamma] = iK\hbar\mathbf{1}$ and $[B,\tilde{G}_n] = 0$, we find

$$[\mathcal{B}(0),[L_1\otimes\mathbf{1},H]] = [L_1,A]\otimes[B,\Gamma] = iK\hbar[L_1\otimes\mathbf{1},\mathcal{A}], \tag{B2}$$

which leads us to

$$[L_1\otimes\mathbf{1},\mathcal{A}] = -\frac{i}{K\hbar}[\mathcal{B}(0),[L_1\otimes\mathbf{1},H]]. \tag{B3}$$

Using Eq. (B3) and the Jacobi identity for commutators, we find

$$\begin{aligned}[L_1 \otimes \mathbf{1},[L_1 \otimes \mathbf{1},\mathcal{A}]] &= -\frac{i}{K\hbar}[L_1 \otimes \mathbf{1},[\mathcal{B}(0),[L_1 \otimes \mathbf{1},H]]] \\ &= \frac{i}{K\hbar}[\mathcal{B}(0),[[L_1 \otimes \mathbf{1},H],L_1 \otimes \mathbf{1}]] + \frac{i}{K\hbar}[[L_1 \otimes \mathbf{1},H],[L_1 \otimes \mathbf{1},\mathcal{B}(0)]] \\ &= -\frac{i}{K\hbar}[\mathcal{B}(0),[L_1 \otimes \mathbf{1},[L_1 \otimes \mathbf{1},H]]]. \end{aligned} \tag{B4}$$

Similarly, we get, for $n = 1, 2, \ldots$,

$$(L_1 \otimes \mathbf{1})^{(n)}\mathcal{A} = -\frac{i}{K\hbar}[\mathcal{B}(0),(L_1 \otimes \mathbf{1})^{(n)}H], \tag{B5}$$

which is Eq. (5.2).

Let us now turn to Eq. (5.4). It follows from $[L, H] = 0$ and the Jacobi identity that

$$\begin{aligned}[L_1 \otimes \mathbf{1},[L_1 \otimes \mathbf{1},H]] &= -[L_1 \otimes \mathbf{1},[\mathbf{1} \otimes L_2,H]] \\ &= [1 \otimes L_2,[H,L_1 \otimes \mathbf{1}]] + [H,[L_1 \otimes \mathbf{1},\mathbf{1} \otimes L_2]] \\ &= [\mathbf{1} \otimes L_2,[\mathbf{1} \otimes L_2,H]]. \end{aligned} \tag{B6}$$

Similarly, we have, for $n = 1, 2, \ldots$,

$$(L_1 \otimes \mathbf{1})^{(n)}H = (-1)^n(\mathbf{1} \otimes L_2)^{(n)}H. \tag{B7}$$

Equation (B7) leads to Eq. (5.4).

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