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Simulations of Lévy flights

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Abstract
Lévy flights, also known as α-stable Lévy processes or heavy-tailed statistics, are becoming a commonly used tool in optics. Nonetheless, the different parametrizations and the absence of any analytic expression for the distribution functions (apart from some exceptions) makes it difficult to efficiently simulate such processes. We review and compare three algorithms for the generation of sequences of symmetric stable Lévy random variables.

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1. Introduction

In recent years, two seemingly unrelated fields have exchanged ideas and concepts: laser manipulation of atoms on the one hand, non-Gaussian statistics and anomalous diffusion processes on the other hand [1]. In addition, it has been demonstrated that a disordered optical material can be engineered, in which light waves perform a Lévy flight [2]. Non-Gaussian, heavy-tailed statistics is becoming a commonly used tool in several physical applications.

Lévy flights or anomalous diffusion processes are known in the mathematical literature under the name of α-stable Lévy processes. They have infinite variance (except for the Gaussian case $\alpha = 2$) and possess scale-invariance and self-similarity properties. In this paper, we shall first define (α-stable) Lévy processes (section 2), then describe some algorithms that can be used to generate symmetric α-stable Lévy random numbers and therefore Lévy flights (section 3), and finally compare the performances of these algorithms (section 4).

2. Definition of symmetric α-stable Lévy processes

A Lévy process $(l_t)_{t\geqslant 0}$ is a cadlag (right continuous with left finite limits, from the French 'continue à droite, limitée à gauche'), stochastically continuous (i.e. discontinuities occur at random times) stochastic process with independent and stationary increments, such that $l_0 = 0$ almost surely (a.s.—namely happening with probability one) [3]. We will study symmetric α-stable Lévy processes. These processes are parametrized by two parameters: α, called *index of stability*, and σ, or equivalently $\gamma = \sigma^\alpha$, called *scale factor*.

A symmetric α-stable random variable (r.v.) l is defined by its characteristic function [4]

$$\phi_{\alpha,\sigma}(z) = \mathbb{E}\left[\exp(\mathrm{i}zl)\right] = \exp(-\sigma^\alpha |z|^\alpha), \qquad (1)$$

where $\mathbb{E}[\cdot]$ denotes the expectation, $z \in \mathbb{R}$, $\sigma \geqslant 0$ and $\alpha \in (0, 2]$. Note that there are different options for the characteristic functions; see [5] for other parameterizations.

The probability density function (pdf) of a symmetric α-stable r.v. is then given by the inverse Fourier transform of (1)

$$\begin{aligned} L_{\alpha,\gamma}(l) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Phi_{\alpha,\sigma}(z) \mathrm{e}^{-\mathrm{i}zl}\, \mathrm{d}z \\ &= \frac{1}{\pi}\int_0^\infty \exp(-\gamma z^\alpha)\cos(zl)\, \mathrm{d}z. \end{aligned} \qquad (2)$$

Except for a few special cases [$\alpha = \frac{1}{2}$, $\alpha = \frac{2}{3}$, $\alpha = 1$ (Cauchy distribution) and $\alpha = 2$ (Gaussian distribution with mean 0 and variance 2γ)] there is no closed expression for the pdf (2) or its cumulative distribution function (cdf),

$$F_{\alpha,\gamma}(l) = \int_{-\infty}^{l} L_{\alpha,\gamma}(x)\, \mathrm{d}x. \qquad (3)$$

Symmetric Lévy stable r.v.'s have many interesting properties.

Firstly, symmetry

$$L_{\alpha,\gamma}(-l) = L_{\alpha,\gamma}(l) \qquad (4)$$

is a consequence of

$$\Phi_{\alpha,\sigma}(-z) = \Phi_{\alpha,\sigma}(z). \qquad (5)$$

Secondly, from (1) it follows that if $l \sim L_{\alpha,\gamma}$ (i.e. l is distributed according to the pdf $L_{\alpha,\gamma}$) and $c > 0$, then

$$cl \sim L_{\alpha,\gamma c^\alpha}. \tag{6}$$

Indeed,

$$\mathbb{E}\left[\exp(\mathrm{i}zcl)\right] = \Phi_{\alpha,\sigma}(cz) = \Phi_{\alpha,c\sigma}(z). \tag{7}$$

This explains why the parameter γ is called scale factor. Therefore, in order to generate a symmetric Lévy stable r.v. $l \sim L_{\alpha,\gamma}$ with a characteristic parameter α and a generic γ, it suffices to write an algorithm that generates a symmetric Lévy stable r.v. $\tilde{l} \sim L_{\alpha,1}$ with the same α and a scale factor $\gamma = 1$, since

$$l = \tilde{l}\gamma^{1/\alpha} \tag{8}$$

in the sense of distributions.

Thirdly, the sum $l^{(1)} + \cdots + l^{(n)}$ of n independent copies of l has the same distribution as l multiplied by the constant $n^{1/\alpha}$:

$$n^{1/\alpha} l = \sum_{i=1}^{n} l^{(i)}, \tag{9}$$

where the equality is in the sense of distributions. Indeed, by recalling that the characteristic function of n independent random variables is the product of their characteristic functions,

$$\mathbb{E}\left[\exp\left(\mathrm{i}z\sum l^{(i)}\right)\right] = \mathbb{E}\left[\exp(\mathrm{i}zl)\right]^n = \Phi_{\alpha,\sigma\, n^{1/\alpha}}(z). \tag{10}$$

Property (9), well known for Gaussian r.v.'s ($\alpha = 2$), is called stability under addition, and is very useful for implementing recursive algorithms for the generation of α-stable Lévy random numbers.

A symmetric α-stable Lévy process is a stochastic process $(l_t)_{t\geqslant 0}$ with independent increments distributed according to a symmetric α-stable distribution

$$l_t - l_s \sim L_{\alpha,|t-s|}, \tag{11}$$

such that $l_0 = 0$ a.s.

From this definition it follows that symmetric α-stable Lévy processes are self-similar processes, i.e. their distributions are invariant under a suitable scaling of time and amplitude, $l_{ct} = c^H l_t$, where $H = 1/\alpha$ is called the Hurst exponent. Indeed

$$l_{ct} - l_{cs} \sim L_{\alpha,c|t-s|}, \qquad c^{1/\alpha}(l_t - l_s) \sim L_{\alpha,c|t-s|}. \tag{12}$$

The Brownian motion or Wiener process is the symmetric two-stable Lévy process: it is self-similar with the Hurst exponent $\frac{1}{2}$.

3. Simulation of symmetric α-stable Lévy processes

The generation of random numbers is a widely investigated computational problem. Many efforts have been devoted to the generation of uniformly and Gaussian distributed r.v.'s and many computationally efficient and high-quality methods have been developed for this purpose. For a general r.v., the most commonly used methods to generate random numbers can be classified into four basic categories: cdf inversion, rejection, transformation and recursive methods. Let us briefly review them.

1. Let f be the desired pdf and F the corresponding cdf. The *cdf inversion method* generates an r.v. l with pdf f by using an r.v. u uniformly distributed in [0, 1] and making use of the fact that the transformation $l = F^{-1}(u)$ yields an r.v. l with cdf F. This method can be used if the inverse function F^{-1} exists, i.e. when l is a continuous random variable with a strictly increasing cdf.
2. The *rejection method* is an algorithm for generating r.v.'s with arbitrary distribution. A version of this method is the following. Let f be a bounded pdf with a bounded support, C the set of points under the curve of f (subgraph of f) and Z a finite-area domain containing the subgraph of f: $Z \supset C$. Random points are extracted uniformly from Z, they are accepted if they belong to C and their x-components are the desired random numbers with pdf f. This method uses an auxiliary r.v. and is often time consuming, because it extracts random numbers and rejects values that do not satisfy a specific relation.
3. The *transformation method* directly transforms uniform or Gaussian random numbers into numbers that are distributed according to the desired distribution. The transformation strongly depends on the desired distribution.
4. The *recursive method* makes use of linear combinations of previously generated random numbers that simulate the desired distribution, to yield new outputs. In our case, this is done according to (9).

We shall now briefly review four methods for the generation of symmetric Lévy stable distributed r.v.'s: a rejection (R) algorithm [6], two transformation methods, due to Chambers, Mallows and Stuck (CMS) [7] and Mantegna (M) [8], respectively, and a recursive method used in [8].

3.1. The R, CMS, M and recursive algorithms

The R algorithm generates two uniform r.v.'s l and u, the first belonging to the interval $[-M, M]$, with $M > 0$ the truncation level, and the second to the interval between 0 and the maximum of the symmetric Lévy stable pdf $L_{\alpha,\gamma}$. The symmetric stable Lévy pdf is approximated with a piecewise constant pdf $f_{\alpha,\gamma}$ normalized to 1 in the interval $[-M, M]$ [6]: on each interval of length δ one assigns to $f_{\alpha,\gamma}$ a constant value evaluated from (2) by numerical integration. Then, if $u < f_{\alpha,\gamma}(l)$, l is accepted; otherwise, it is rejected and another couple of numbers is generated. A more efficient algorithm can be designed if u is chosen to have a pdf with a profile similar to the symmetric Lévy stable pdf rather than to a uniform distribution, thus reducing the probability of rejection.

The CMS algorithm [7] is a transformation method that simulates the generation of a symmetric Lévy stable distributed r.v. l with characteristic exponent α and scale factor γ, by applying a nonlinear transformation to two uniformly distributed r.v.'s.[4]

The M algorithm [8] is a transformation method that generates an r.v. that simulates a Lévy stable-distributed r.v. l with characteristic exponent α and scale factor $\gamma = 1$,

[4] Another version of the CMS algorithm can generate asymmetric Lévy stable r.v.'s, see [7].

Table 1. Performance evaluation for $\alpha = 0.8$ and $\gamma = 1$. We set $\delta = 0.04$ and $M = 20$ for the R algorithm. The best performance for every recursion algorithm is indicated in boldface.

Method	Speed	$F^{(2)}$	$F^{(4)}$	P_{χ^2}	P_{KS}
R	1.17×10^{-6}	0.51	0.12	–	–
CMS1	$\mathbf{6.24} \times 10^{-2}$	0.28	0.06	**82%**	90%
CMS5	1.22×10^{-2}	0.29	**0.023**	78%	87%
CMS10	6.11×10^{-3}	0.45	0.15	80%	89%
CMS100	5.85×10^{-4}	**0.25**	0.096	74%	**92%**
M1	$\mathbf{2.51} \times 10^{-2}$	2.7	1.6	61%	**94%**
M5	4.94×10^{-3}	1.2	0.34	62%	86%
M10	2.53×10^{-3}	0.45	0.15	70%	93%
M100	2.60×10^{-4}	**0.3**	**0.08**	**76%**	**94%**

by applying a nonlinear transformation to two Gaussian distributed stochastic variables. This algorithm is efficient for $0.75 \leqslant \alpha \leqslant 1.95$ and can be extended outside this interval.

The recursive algorithm for symmetric stable Lévy r.v.'s relies on the stability property (9) of Lévy stable r.v.'s: linear combinations of Lévy stable processes are themselves Lévy stable distributed. One can check that the degree of accuracy of the M algorithm can be enhanced [8] if one uses n intermediate independent r.v.'s (the optimal n depends on the value of α, and can vary between 1 and 100). We applied this method also to CMS-generated r.v.'s, but not to R-generated r.v.'s because it can become inefficient due to its slowness (see table 1).

4. Simulations

By using the software Mathematica we implemented the rejection algorithm R, the recursive CMS algorithm with recursive number n (CMSn) and the recursive M algorithm with recursive number n (Mn), both for $n \in \{1, 5, 10, 100\}$. Table 1 displays the results of our analysis on the distribution of random numbers generated through the different algorithms for $\alpha = 0.8$ and $\gamma = 1$ (for the rejection algorithm, we set $\delta = 0.04$ and $M = 20$).

Column 2 shows the algorithmic speed (i.e. the number of generated samples per second) relative to Mathematica's uniform random numbers generator speed. The relative speed does not depend on the cardinality of the sample for all the algorithms investigated, as one could expect, except for the R algorithm (where we took a sample of cardinality 1000).

Columns 3 and 4 display the distance between the target and the simulated distribution computed according to the formula $F^{(k)} = |\mu^{(k)} - \mu^{(k)}_{\mathrm{alg}}|/\sigma^{(k)}_{\mathrm{alg}}$, where $\mu^{(k)}$ is the kth moment of the target distribution calculated by numerical integration[5], $\mu^{(k)}_{\mathrm{alg}}$ the average value (over 100 samples of cardinality 2^{15}) of the kth moment of the distribution of random numbers generated by a specific algorithm and $\sigma^{(k)}_{\mathrm{alg}}$ the corresponding standard deviation. Only the case of even k is of interest, since odd moments vanish (see property (4)) and the algorithms are symmetric. In columns 3 and 4 we report $F^{(2)}$ and $F^{(4)}$, respectively. Clearly, an exhaustive analysis would require the evaluation of all even-k moments, but this task will be left for future work. On the same sample, we evaluated the percentage of samples passing the χ^2 test at a 0.1 confidence level (P_{χ^2}—column 5).

Finally, in column 6 we report P_{KS}, the percentage of samples simulating symmetric α-stable Lévy random numbers of cardinality 100 that passed the variance stabilized Kolmogorov–Smirnov test [9, 10] at a 0.1 confidence level. Note that the tails are included in this case.

[5] For this analysis, we simulated a truncated distribution defined to be 0 for $|l| > M$ and (2) otherwise, normalized to 1. Note that for recursion algorithms, in order for (9) to be valid, the rejection of random numbers $|l| > M = 20$ was done after the recursion. The motivation beyond the truncation is that the R algorithm can only simulate a pdf defined on a finite interval.

5. Conclusions

We described how to simulate symmetric Lévy flights with given α and γ: one starts from $l_0 = 0$ and adds simulated symmetric Lévy random numbers one at time, according to (11). For the generation of symmetric Lévy stable random numbers with $\gamma = 1$, the most efficient algorithms are the CMS and the M algorithm, which are transformation methods capable of simulating Lévy random numbers by using uniform or Gaussian random numbers. For a generic γ, equation (8) must be used. Both CMS and M are significantly slower than a uniform random number generator, but they turn out to be nonetheless fast enough (see table 1, column 2). Rejection algorithms can be even slower because of the unavoidable probability of rejection. CMS is the fastest algorithm, even though the M algorithm has a very good performance. On the other hand, recursion algorithms can be useful when the performance of the CMS or the M algorithms is low (which happens for some values of α). In these cases, the recursion formula can increase the performance of the algorithms as measured in terms of P_{χ^2} and P_{KS} (by 97% for the M algorithm with $\alpha = 1.9$, by 11% for the CMS algorithm with $\alpha = 1.5$). In general, the CMS algorithm has better percentages P_{χ^2} and P_{KS}, and recursion versions of this algorithm in general do not enhance the performance. Nevertheless, in some cases, the M recursion algorithm yields the best performance for some n.

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