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ARTICLES

Wave-function collapse by measurement and its simulation

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The many-Hilbert-space approach to the measurement problem in quantum mechanics is applied to a typical "yes-no" experiment relative to two branch routes corresponding to mutually exclusive propositions. First, we reformulate the notion of wave-function collapse by measurement as a dephasing process between the two branch waves of an interfering particle (from our own point of view as opposed to the conventional Copenhagen interpretation). In this way, the concept of "wave-function collapse" is replaced by that of a *statistically defined* dephasing process. One of the most important points of this paper is the introduction of an order parameter ϵ that quantitatively describes the degree of decoherence. Its value ranges from $\epsilon=0$ (which describes the case in which the two waves are perfectly coherent) to $\epsilon=1$ (which describes the case in which coherence is totally lost); for this reason ϵ is named the "decoherence parameter." In terms of this parameter we formulate a definite *criterion* to judge whether an instrument works well or not as a measuring apparatus. Then, we study the interaction between a microscopic particle and a macroscopic system (a detector), by modeling the macrosystem with a linear array of complex δ potentials, which undergo several kinds of statistical fluctuations. This leads us, under particular conditions, to the so-called wave-function collapse, which is attained in the limit $\epsilon=1$. We also examine in some detail which kind of elastic and/or inelastic collisions can give the wave-function collapse. Some connections with recent experimental results in neutron interferometry and quantum optics are also stressed.

I. INTRODUCTION

It is widely known that the Copenhagen approach to the measurement problem in quantum mechanics is based on the conventional notion of wave-function collapse by measurement. For several decades many serious questions have been raised against the Copenhagen interpretation and von Neuman's postulate, which require the presence of an external observer in order to explain the evolution from a pure to a mixed state. Indeed, it seems that the presence of a classical observer is an essential requirement of the Copenhagen approach. However, this is unsatisfactory from several points of view, because, for instance, the necessity of introducing classical concepts in order to explain the quantum postulates prevents quantum mechanics itself from being a self-contained theory.

On the other hand, Machida and one of the present authors pointed out some years ago[1,2] that it is possible to explain the evolution from a pure state to a mixed state without resorting to classical concepts. In their many-Hilbert-space (MHS) approach, the wave-function collapse is indeed described within quantum mechanics itself via a continuous superselection rule whose precise mathematical meaning was later discussed by Araki.[3]

Besides some rather old but still open questions against the above-mentioned Copenhagen interpretation, we are now facing new problems coming from recent experiments, such as neutron interferometry,[4] mesoscopic phenomena,[5] and others, which seem to be hardly understandable in terms of the naive Copenhagen interpretation. In order to explain these kinds of experiments, we have to reconsider the meaning of wave-function collapse by measurement, apart from the conventional Copenhagen interpretation, and furthermore, we should reformulate our measurement theory so as to give a definite *criterion* to judge whether an instrument can work well or not as a measuring apparatus. Nowadays, therefore, any measurement theory has to be considered unsatisfactory unless it is capable not only of describing the measurement process in an abstract way but also of analyzing concretely the most recent experiments. In this context, the concept of measurement in quantum mechanics has changed over the past ten years from a few simple postulates to a very challenging problem.

The MHS theory has already formulated a criterion for the wave-function collapse in terms of an *inequality*,[1,2] by means of which Namiki and his collaborators have analyzed some crucial points of neutron-interference experiments.[2,6]

In this paper we first discuss what the wave function collapse is, apart from the naive Copenhagen interpretation, along the basic line of thought of the MHS theory. Based on this discussion, we formulate a *criterion* for the wave-function collapse, not as an inequality but as a

quantitative condition, in terms of an *order* parameter ϵ, which gives an estimate of the *degree of decoherence* of a quantum system. We have already applied this new criterion to the neutron-interference experiments with an absorber.[7] The general theoretical formulation of the problem will be followed by a numerical simulation in which a detector will be modeled with a Dirac comb of complex potentials. A similar simulation has been performed by Murayama, by making use of a linear array of real potentials.[8] Throughout our numerical analyses we will investigate how elastic and inelastic collisions inside the detector yield the wave-function collapse, specified by the condition $\epsilon=1$. A definition of *imperfect measurement* will also be given, which will be shown to correspond to a *partial collapse* of the wave function. We will start by sketching the main differences between the MHS theory and the conventional Copenhagen interpretation.

II. FORMULATION OF "WAVE-FUNCTION COLLAPSE" IN THE MHS APPROACH

Let us start by discussing which kind of physical processes take place in a quantum-mechanical measurement, from our own point of view as opposed to the conventional Copenhagen interpretation. For this purpose it is convenient to consider a yes-no experiment, because most measurements can be decomposed into a set of many yes-no experiments. A yes-no experiment is usually divided into two subsequent steps, the first being responsible for the spectral decomposition and the second for the detection. We shall show that the so-called "wave-function collapse" by measurement does not occur in the spectral decomposition step but in the detection step.

A typical yes-no experiment of the Stern-Gerlach type is schematized in Fig. 1. The particles are brought into the measuring apparatus *one* by *one,* by a beam of very weak intensity, or, in other words, the next particle comes into the experimental setup only after a measurement on the foregoing particle has taken place, and has been followed by a certain recovery process of the apparatus. The wave function of a particle sent in the measuring apparatus by an emitter E is decomposed by a divider V into $\psi=\psi_1+\psi_2$, where ψ_1 and ψ_2 are the two branch waves running through the two spatially separated routes I and II. We place a detector D on route II but no detector on route I. A coincidence detection of E (with "yes") and D (with "yes") means that the particle has taken route II, while an anticoincidence detection of E (with "yes") and D (with "no") means that it has taken route I. The former possibility is named case "yes" and the latter case "no." The latter is sometimes called negative-result measurement. Let routes I and II be so designed as to correspond to two mutually exclusive measurement propositions $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. In the Stern-Gerlach experiment to measure the particle spin σ_z, $\mathcal{P}_1$ ($\mathcal{P}_2$) corresponds to spin up (down), and ψ_1 (ψ_2) is the corresponding spin eigenfunction. The measurement to observe which proposition is true, $\mathcal{P}_1$ or $\mathcal{P}_2$, is completed by the determination of the particle path, I or II, and, consequently, the corresponding wave-function collapse takes place.

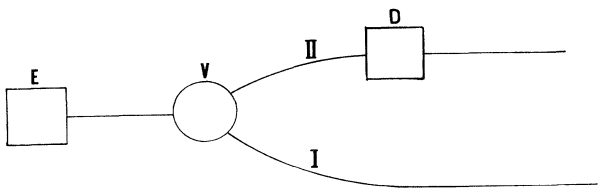

FIG. 1. "Yes-no" experiment.

The negative-result measurement has often been used as a paradoxical argument against the ergodic-amplification theory of quantum measurements,[9,10] because the corresponding wave-function collapse is realized without resort to any actual thermal-irreversible processes in D (such as counter triggering). We should remark that the wave-function collapse (i.e., the determination of the particle path to be I) is provoked, even in this case, by the interaction of ψ_2 with the constituents of D. In other words, ψ_2 interacts with D and does *not* disappear even in the negative-result measurement. This is the only possible solution to this paradox. We must strictly distinguish the wave-function collapse itself from the thermal-irreversible processes in D. The latter is only a secondary process, following the wave-function collapse, which is set in the apparatus in order to display the result of the measurement. "Triggering" and "no triggering" should be regarded as *displays* on an equal footing. For details, see Refs. 1 and 2.

Suppose that D is an ideal *nondestructive* detector for what we will call the *first* kind of measurement, by which ψ_2 is changed to

$$\psi_2'=T\psi_2 , \tag{1}$$

T being a complex number with modulus very close to (but a little less than) 1. If we follow the conventional Copenhagen interpretation, we have to accept the following transition of the wave function (wave-function collapse):

$$\psi\rightarrow\psi_2' \quad (\psi_1 \text{ disappears}) , \tag{2}$$

in case "yes" and

$$\psi\rightarrow\psi_1 \quad (\psi_2 \text{ and therefore } \psi_2' \text{ disappear}) , \tag{3}$$

in case "no," under the basic postulate that the probabilities of finding cases "yes" and "no" be proportional to $P_2=|\psi_2'|^2$ (very close to $|\psi_2|^2$) and $P_1=|\psi_1|^2$, respectively. Equations (2) and (3) are sometimes accepted as the "measurement postulate." In this paper we shall call this kind of behavior, characterized by the disappearance of one branch wave, the "naive Copenhagen interpretation."

We know that the process (2) or (3) is an acausal and probabilistic event and cannot be regarded as any kind of wave motion in which ψ is continuously shrinking into ψ_1 or ψ_2. It should also be remarked that quantum mechanics can never predict the definite result of a single measurement on one dynamical system in a superposed state, but it gives only a probabilistic prediction for the accumulated distribution of measured values obtained by many independent single measurements on many independent systems, each of which is described by the

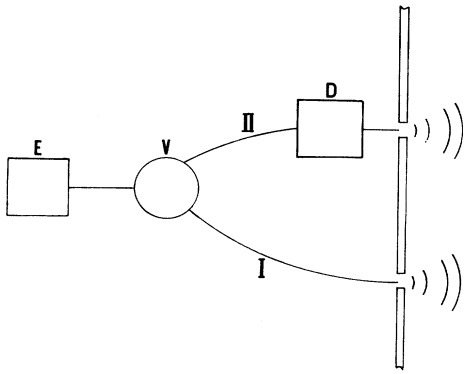

FIG. 2. Double-slit interference experiment.

same wave function. Hence, there is no way of talking about the measurement problem except in terms of an accumulation of many experimental results. We shall briefly discuss this point again in Sec. XI.

In order to examine the physical effect of the change $\psi_2 \rightarrow \psi_2'$ given by D, let us introduce the Young-type "two-step" experiment, in which the two branch waves are recombined, as in Fig. 2.

In this case, both branch waves are forwarded to small slits in order to make two spherical waves traveling towards the screen. Each particle makes a single spot on the screen, and can be observed there with a probability distribution proportional to

$$|\psi_1+\psi_2'|^2=P_1+P_2+2\,\mathrm{Re}(\psi_1^*\psi_2') \ , \tag{4}$$

in which for simplicity we have used the same notations ψ_1 and ψ_2' for the spherical waves as for the original branch waves. We can obtain a particle distribution proportional to (4) on the screen, if we accumulate many spots, one after another, corresponding to many particles brought into the experimental setup by a stationary beam. The last addendum in the right-hand side is the interference term and reflects the coherence between the two branch waves.

The naive Copenhagen interpretation forces us to accept that one of the branch waves actually disappears in each measurement. The disappearance results in the erasing of the interference term in each measurement, by which the probability distribution (4) becomes

$$P_1+P_2 \ . \tag{5}$$

As was mentioned above, however, quantum mechanics never gives us a definite answer for the result of a single measurement on one system, so that no one can know whether a branch wave has disappeared or not in a single measurement. This implies that we can observe (4) and (5) *only* on the *accumulated* distribution over many particles. Consequently, the wave-function collapse is not to be considered as the above-mentioned disappearance in a single measurement, but is rather to be formulated for the accumulated distribution as follows:

$$\sum_{\text{accumul}} \mathrm{Re}(\psi_1^*\psi_2')=0 \ . \tag{6}$$

For this reason, in this paper, we describe the notion of wave-function collapse by (6) [or equivalently, (5)] instead of (2) and (3); this can be realized if the detector gives, for the ensemble of accumulated particles, a random sequence of phase shifts between the two branch waves in an experimental run. Under this notion of wave-function collapse, therefore, we do not require a branch wave to disappear in a single measurement, but assume instead that both branch waves are still alive after the interaction with D. The essential ingredient to obtain (6) is not the disappearance of one branch wave but the decoherence between the two branch waves. In other words, a perfect detector is an apparatus that yields (6) by provoking a perfect decoherence between the two branch waves. By observing that (5), obtained by erasing the interference term, is a sum of probabilities of finding one of two mutually exclusive events, we can easily understand the natural result, usually referred to as wave-function collapse, that once we have found one event, another event should never occur. This is the same result obtained by the naive Copenhagen interpretation. In this context, one may say that, in our case, the notion of wave-function collapse still remains, but within a wider framework than the Copenhagen one.

We have often met some misleading discussion in which the spectral decomposition itself is identified with the very measurement process. Note that our wave-function collapse to yield (5) or (6) takes place in the detection step but not in the spectral decomposition. The latter is only a preparatory step of the whole measurement, in which the phase correlation between the two branch waves is fully kept.

It is, of course, true that the naive Copenhagen interpretation can work well as a simple calculational rule for quantum-mechanical expectation values, but we know that it leads sometimes to strange behavior of the wave function, such as the retrogressive disappearance of a branch wave in a space-time region. Furthermore, if we rely upon the idea of disappearance we can hardly understand the negative-result measurement, as was briefly mentioned above and in Refs. 1 and 2, as well as some of the neutron-interferometry experiments, as was discussed in Refs. 2, 6, and 7, and the mesoscopic phenomena[5] including *partial collapse* of the wave function, as will be discussed later.

As an additional reason against the disappearance of the branch waves, let us introduce a different kind of gedanken experiment, in the following way. Suppose that instead of D, we place, in route II, equipment controlled by a parameter ϵ; the equipment is empty if $\epsilon=0$ but becomes a perfect detector in the case $\epsilon=1$, through a sequence of intermediate steps, in which ϵ changes continuously from 0 to 1. As an example, one can consider ϵ to be proportional to the density of material in the detector. In the case $\epsilon=0$, of course, we have full coherence between two branch waves. But what happens for any finite value of ϵ, smaller than unity? It is natural to assume that the two branch waves do not disappear and keep the phase correlation, at least up to a certain extent, even though the modulus and the phase of ψ_2 are modified by the interaction with the constituents of the equipment. On the other hand, we can design the equipment so as to generate a signal for certain arbitrary values of ϵ smaller

than unity, through an appropriate physical process. In this case we are performing an *imperfect* measurement, which yields a *partial collapse* of the wave function, because the interference term, though reduced, is still nonvanishing. In an imperfect measurement, the two branch waves do not disappear, but only lose partially their relative phase correlation (their coherence). Even when ϵ is close to 1, we cannot accept that one branch wave disappears in an imperfect measurement. Why must we accept the disappearance only in the limit $\epsilon=1$? We must not and need not accept it. What actually happens in this limit is only the total dephasing or the complete decoherence between the two branch waves, and this is enough to yield the wave-function collapse, given by (5) or (6). It would be interesting to invent such equipment controlled by the parameter ϵ. This is a proposal for experimentalists.

This implies that the same equipment can work well as a measuring apparatus in some cases but simply acts as a phase shifter or an absorber in other cases, depending on the condition specified by parameters like the abovementioned ϵ. We shall reconsider this situation in detail when performing numerical simulations for a detector, modelled with a Dirac comb of complex potentials.

The physical process in an imperfect measurement is essentially similar to those observed in *mesoscopic* phenomena, which recently came to our attention.[5] We have also analyzed the same kind of phenomenon in the neutron-interference experiments with an absorber.[7]

The reason we have a random sequence of phase differences between the branch waves in an experimental run is that different incoming particles will interact with the detector system or with some of its local systems (with a huge number of degrees of freedom) in different microscopic states and will then undergo different phase shifts, because the internal motion of the detector system will change its microscopic state during each intermediate time interval between subsequent measurements on different particles. The whole ensemble of the microscopic states of the detector or of its local systems relevant to an experimental run cannot be represented within the framework of a *single* Hilbert space, so that we are inevitably led to a direct sum of *many* Hilbert spaces for their representation.[1,2] This is the reason we called our theory the "many-Hilbert space" (MHS) theory. In the limit of infinite degrees of freedom, our wave-function collapse is described by a continuous superselection rule.[1-3] In Sec. III we will derive a definite criterion for the wave-function collapse in terms of a parameter ϵ, named the *decoherence parameter,* based on the MHS approach. The wave-function collapse, finally yielding (5) or (6), is, of course, considered to be an evolution from a pure state to a mixed state which is generally described in terms of density matrices.[1,2] We will give an outline of the density-matrix description of the MHS theory in Sec. IV, in which the functioning of a destructive detector is also discussed.

III. CRITERION FOR THE WAVE-FUNCTION COLLAPSE

Following the line of thought outlined in Sec. II, we shall try to reformulate a "yes-no" experiment in terms of many-Hilbert-space quantities, and shall see that when the MHS structure of the detector is taken into account, new effects come to light. We shall also find that by recasting the quantum-mechanical quantities into "MHS language," it will be possible to define a new *order* parameter, that will be named *decoherence parameter,* in terms of which a precise *quantitative* definition of wave-function collapse can be given.

In a "yes-no" experiment an incoming "particle" is split in two states ψ_1 and ψ_2, corresponding to two possible routes. (By "particle" we mean, in the following discussion, a Schrödinger wave function. We will sometimes make use of quotes in order to remind the reader of the limits inherent in our terminology.) We will place our "detector" along the second path, so that the relative wave function will be modified according to $\psi_2 \rightarrow T\psi_2$, where T is the detector's "transmission coefficient." The total wave function will be

$$\psi=\psi_1+T\psi_2 \ , \tag{7}$$

and the intensity after recombination

$$\begin{aligned}|\psi|^2&=|\psi_1+T\psi_2|^2\\&=|\psi_1|^2+|T|^2|\psi_2|^2+2\,\mathrm{Re}(\psi_1^* T\psi_2) \ .\end{aligned} \tag{8}$$

So far, the MHS structure of the detector has been completely neglected. Let us take it into account: Equations (7) and (8) hold for *every single incoming* "particle." Let us label the incoming "particle" with j ($j=1,\ldots,N_p$, where N_p is the total number of particles in an experimental run) and rewrite the transmission coefficient as

$$T \rightarrow T_j, \quad j=1,\ldots,N_p \ . \tag{9}$$

This reflects a fundamental property of our approach: every incoming "particle" is described by the sum of the *same* branch waves $\psi=\psi_1+\psi_2$, immediately before interacting with the detector. But after the interaction, the detector transmission coefficient T will depend on the particular detector state at the *very instant* of the passage of the "particle." Being the detector subject to random fluctuations (which reflect the internal motion of its elementary quantum constituents and its MHS structure), the same macroscopic state of the detector will correspond to many different microscopic ones. Consequently, different incoming "particles" will be affected differently by the interaction with the detector, and will be described by slightly different values of T. Accordingly, Eq. (8) becomes

$$\begin{aligned}|\psi^{(j)}|^2&=|\psi_1+T_j\psi_2|^2\\&=|\psi_1|^2+|T_j|^2|\psi_2|^2+2\,\mathrm{Re}(\psi_1^* T_j\psi_2) \ .\end{aligned} \tag{10}$$

If we define $P^{(j)}=|\psi^{(j)}|^2$ as the probability of detecting the jth particle after recombination, then, after many particles have been detected, the average probability will be given by

$$P_{\mathrm{ave}}=\frac{1}{N_p}\sum_{j=1}^{N_p} P^{(j)}=|\psi_1|^2+\bar{t}|\psi_2|^2+2\,\mathrm{Re}(\psi_1^*\bar{T}\psi_2) \ , \tag{11}$$

where we have defined the *average* transmission probability

$$\bar{t}=\frac{1}{N_p}\sum_{j=1}^{N_p}|T_j|^2 , \tag{12}$$

and the *average* transmission coefficient

$$\bar{T}=\frac{1}{N_p}\sum_{j=1}^{N_p}T_j . \tag{13}$$

Note that, in general,

$$|\bar{T}|^2\neq\bar{t} ; \tag{14}$$

moreover, from Eq. (11), a sufficient and necessary condition for observing no interference (wave-packet collapse) is

$$\bar{T}=0 \text{ or, equivalently, } |\bar{T}|^2=0 . \tag{15}$$

In order to better clarify the above considerations, let us write the jth "particle" transmission coefficient as

$$T_j=T_0(1+\Delta_j), \quad j=1,\ldots,N_p \tag{16}$$

where T_0 is an ideal parameter and Δ_j depends on the particle. The average transmission coefficient [Eq. (13)] will be

$$\bar{T}=T_0(1+\bar{\Delta}) , \tag{17}$$

where, as usual, a bar denotes an average over j. On the other hand, the average transmission probability [Eq. (12)] will be given by

$$\bar{t}=t_0(1+2\,\mathrm{Re}\bar{\Delta}+\overline{|\Delta|^2}) , \tag{18}$$

where $t_0=|T_0|^2$. Note that $\bar{\Delta}$ does not necessarily vanish; this can be understood by realizing that $\bar{\Delta}$ represents an effect due to the statistical fluctuations in the macroscopic detector, which, in general, do not vanish even at zero temperature.

By combining Eqs. (17) and (18), we obtain

$$|\bar{T}|^2=\bar{t}(1-\epsilon) , \tag{19}$$

where

$$\epsilon=\frac{(\delta\Delta)^2}{\overline{|1+\Delta|^2}}>0 , \quad (\delta\Delta)^2=\overline{|\Delta|^2}-|\bar{\Delta}|^2=\overline{|\Delta-\bar{\Delta}|^2} . \tag{20}$$

Note that the condition $\epsilon<1$ is easily obtained from Eq. (19). The limit $\epsilon=1$ corresponds to the *full fluctuation* case, in which, by Eq. (20), $\bar{\Delta}=-1$, and, therefore, by Eq. (17), $\bar{T}=0$; this is, by Eq. (15), the condition for a *total* loss of coherence between the two branch waves. Equation (11) can now be rewritten as

$$P_{\mathrm{ave}}=|\psi_1|^2+\bar{t}|\psi_2|^2+2\sqrt{\bar{t}}\sqrt{1-\epsilon}\,\mathrm{Re}(\psi_1^* e^{i\beta}\psi_2) , \tag{21}$$

where we have written $\bar{T}=|\bar{T}|e^{i\beta}$. The meaning of our approach should now be evident: In Eq. (21), the interference term contains the new factor $\sqrt{1-\epsilon}$, which is absent if the MHS structure of the detector is not taken into account, namely, if its statistical fluctuations are neglected. We stress that it is possible to have $\epsilon=1$ irrespectively of $\bar{t}$ and, *vice versa*, $\bar{t}=0$ irrespectively of ϵ.

The standard quantum-mechanical formula for the intensity at the screen [Eq. (8), with $t=|T|^2$],

$$P=|\psi_1|^2+t|\psi_2|^2+2\sqrt{t}\,\mathrm{Re}(\psi_1^* e^{i\beta}\psi_2) , \tag{22}$$

is recovered in the limit $\epsilon=0$. On the other hand, from Eqs. (15), (19), and (21), we find that interference is lost, and hence the wave-function collapse takes place, in the limit $\epsilon=1$. For this reason, ϵ will be named *decoherence parameter*. The condition

$$\epsilon=1 \tag{23}$$

is a *definite numerical criterion* for the wave-function collapse. It will be of great interest, in this paper, to study all the intermediate cases in which $0<\epsilon<1$, and coherence is *partially* lost or, stated differently, the wave function is *partially* collapsed. The parameter ϵ is, in fact, an order parameter for the wave-function collapse.

We shall conclude this section with an ergodic hypothesis: The average over many particles going through the detector (denoted hitherto with a bar) will be assumed equal to the *statistical ensemble average* over all the possible detector's microstates. If we denote the latter with $\langle\cdots\rangle$, our assumption reads

$$\overline{\cdots}=\langle\cdots\rangle . \tag{24}$$

We shall come back to this point in Sec. X. This ergodic hypothesis leads to a natural interpretation of the formulas of the present section. It is worth stressing that our ergodic assumption makes sense only if N_p, the total number of particles in an experimental run, is *very large*. In other words, and this is the main point of our approach to the quantum measurement problem, it makes no sense to speak of wave-function collapse for a single particle. Moreover, we shall see that a random sequence T_j, for which $\bar{T}=0$ or $\epsilon=1$, can be obtained for a detector with a huge number of degrees of freedom.

IV. DENSITY-MATRIX FORMULATION OF THE MHS THEORY

Let us denote density matrices of the total system, the object particle, and the detector system by ρ^{tot}, ρ^Q and ρ^D, respectively. The whole detection process can be written as follows;

$$\rho_t^{\mathrm{tot}}=e^{-iHt/\hbar}\rho_I^Q\otimes\rho_I^D e^{iHt/\hbar} \underset{t\to\infty}{\sim} e^{-iH_0t/\hbar}S\{\rho_I^Q\otimes\rho_I^D\}S^\dagger e^{iH_0t/\hbar} , \tag{25}$$

where we have introduced the S matrix defined by

$$e^{-iHt/\hbar}\underset{t\to\infty}{\sim}e^{-iH_0t/\hbar}S , \tag{26}$$

H and H_0 being the total and free Hamiltonians of the total system, respectively. Note that $H_0=H_0^Q+H_0^D$ is given by the sum of the free Hamiltonians of Q and D without an interaction between them. The density matrices ρ_I^Q and ρ_I^D represent, respectively, the initial states of Q and D before the detection and after the spectral decomposition. In particular, we have

$$\rho_I^Q=\sum_{k,l=1,2}|\psi_k\rangle\langle\psi_l| , \tag{27}$$

corresponding to the decomposition $\psi\to\psi_1+\psi_2$, given in Sec. II.

Following the discussion in Secs. II and III, we add the subscript j to all the relevant quantities in (25) and (26),

because the jth particle in an experimental run will meet the detector in a (micro)state described by $\rho^D_{I,j}$ and will undergo a transition described by S_j. Hence, the accumulated distribution on the screen, in an experimental run, is described by the following average of $\rho^{\rm tot}_{t,j}$ over j:

$$\Xi^{\rm tot}_t \equiv \frac{1}{N_p}\sum_{j=1}^{N_p}\lim_{t\to\infty}\rho^{\rm tot}_{t,j} = \sum_{k,l=1,2}\Xi^{k,l}_t \,, \tag{28}$$

where

$$\Xi^{k,l}_t \equiv \frac{1}{N_p}\sum_{j=1}^{N_p} e^{-iH_0 t/\hbar} S_j \{ |\psi_k\rangle\langle\psi_l| \otimes \rho^D_{I,j}\} S_j^\dagger e^{iH_0 t/\hbar} \,. \tag{29}$$

As mentioned in Sec. III, we can replace the average over j with the statistical ensemble average over all the possible detector's microstates on the basis of the ergodic hypothesis (24).

Taking into account the fact that the interaction takes place only between ψ_2 and D, we can write (by dropping the index j for simplicity)

$$\begin{aligned}
&S\{|\psi_1\rangle\langle\psi_1|\otimes\rho^D_I\}S^\dagger = |\psi_1\rangle\langle\psi_1|\otimes\rho^D_I \,,\\
&S\{|\psi_2\rangle\langle\psi_2|\otimes\rho^D_I\}S^\dagger = |\psi_{2F}\rangle\langle\psi_{2F}|\otimes\rho^D_F \,,\\
&S\{|\psi_1\rangle\langle\psi_2|\otimes\rho^D_I\}S^\dagger = |\psi_1\rangle\langle\psi_{2F}|\otimes(\rho^D_I S^\dagger) \,,\\
&S\{|\psi_2\rangle\langle\psi_1|\otimes\rho^D_I\}S^\dagger = |\psi_{2F}\rangle\langle\psi_1|\otimes(S\rho^D_I) \,,
\end{aligned} \tag{30}$$

where $|\psi_{2F}\rangle$, $\langle\psi_{2F}|$ and ρ^D_F stand for the corresponding final states. On the other hand, the general structure of the S matrix is known to be

$$S = e^{i\Theta} S' e^{i\Theta} \tag{31}$$

in the channel representation,[1,2,11] in which Θ is a diagonal matrix representing the main part of the phase shift and is proportional to some parameter characterizing the size of the target (such as, for instance, N, the number of elementary constituents of the target), and S' is responsible for possible channel couplings including the reduction of the transmission probability. Therefore, if we restrict ourselves to a simple case consisting of two channels ("with" and "without" the object particle, in order to include the case of destructive detector), we can write down $|\psi_{2F}\rangle$ as follows:

$$\begin{aligned}
&|\psi_{2F}\rangle = T|\psi_2\rangle + T'|0\rangle, \quad \text{with } T = e^{2i\Theta_{\rm el}}\langle\psi_2|S'|\psi_2\rangle\\
&\qquad\qquad \text{and } T' = e^{i(\Theta_{\rm el}+\Theta_{\rm inel})}\langle 0|S'|\psi_2\rangle \,,
\end{aligned} \tag{32}$$

where $|\psi_2\rangle$ and $|0\rangle$ (both normalized to 1) stand for one-particle and zero-particle states, respectively. Note that T is just the same coefficient given in Sec. III Eq. (7) (apart eventually from a trivial factor) and that T' is written in terms of $\langle 0|S'|\psi_1\rangle$ ($|\langle 0|S'|\psi_1\rangle|^2$ being the absorption probability) and includes the same phase shift as T ($\Theta_{\rm el}$), plus another one ($\Theta_{\rm inel}$). Needless to say, we have to add the subscript j to all the relevant quantities in Eqs. (30), (31), and (32), and then take the averages of $\Xi^{k,l}_t$ in (29) over j. Since, for large values of N, the previously mentioned parameter characterizing the size of the target, the phase shift $\Theta_{\rm el}$ can become completely random, the condition $\epsilon = 1$ can be obtained. We can easily understand that under essentially the same conditions yielding $\epsilon = 1$, and, therefore, $\overline{T} = 0$, $\overline{T'}$ also vanishes. Thus, all the off-diagonal and cross-correlated components of (28) and (29) (with respect to the routes and the channels) vanish under the condition $\epsilon = 1$ because they are proportional to $\overline{T}$ and/or $\overline{T'}$, while all the diagonal and autocorrelated ones are kept nonvanishing because they depend only on $\overline{|T|^2}$ and/or $\overline{|T'|^2}$. The same discussion can be applied to the "emulsion" case, if we identify $|\psi_2\rangle$ and $|0\rangle$ with the bound and the dissociated states of an AgBr molecule, respectively.

Thus we obtain from Eq. (28), under the condition $\epsilon = 1$,

$$\Xi^{\rm tot}_t = \Xi^{1,1}_t + \Xi^{2,2}_t \,, \tag{33}$$

which is an explicit expression for the wave-function collapse. It is interesting to rewrite (33) for the two types of detector, nondestructive and destructive. Equation (33) becomes

$$\Xi^{\rm tot}_t = \xi^Q_{1,t}\otimes\sigma^D_{I,t} + \overline{|T|^2}\,\xi^Q_{2,t}\otimes\sigma^D_{F,t} + O(1-\overline{|T|^2}) \tag{34}$$

for an ideal nondestructive detector, with $\overline{|T|^2} \simeq 1$, while it becomes

$$\Xi^{\rm tot}_t = \xi^Q_{1,t}\otimes\sigma^D_{I,t} + (1-\overline{|T|^2})|0\rangle\langle 0|\otimes\sigma^D_{F,t} + O(\overline{|T|^2}) \tag{35}$$

for an ideal destructive detector, with $\overline{|T|^2} \simeq 0$, where we have used

$$\xi^Q_{k,t} \equiv e^{-iH^Q_0 t/\hbar}|\psi_k\rangle\langle\psi_k|e^{iH^Q_0 t/\hbar}, \quad (k=1,2) \tag{36}$$

and

$$\begin{aligned}
&\sigma^D_{F,t} \equiv \overline{e^{-iH^D_0 t/\hbar}\rho^D_F e^{iH^D_0 t/\hbar}} \,,\\
&\sigma^D_{I,t} \equiv \overline{e^{-iH^D_0 t/\hbar}\rho^D_I e^{iH^D_0 t/\hbar}} \,.
\end{aligned} \tag{37}$$

In (34) and (35) we have neglected the possibility of reflection at D, and have assumed that $|T'|^2 = 1-|T|^2$.

We stress that (33), (34), and (35) give an exact representation of the wave-function collapse, which is characterized by the lack of off-diagonal and cross-correlated components. In order to generalize the previous result to the measurement of the observable F of a system Q in a superposed state $\psi^Q = \sum_k c_k u_k$ (u_k being the eigenstate of F relative to the eigenvalue λ_k), let us consider an apparatus A, eventually made up of several different detectors. The final state of the total system after the wave-function collapse has taken place can be written as

$$\Xi^{\rm tot}_t = \sum_k |c_k|^2 \xi^Q_{k,t}\otimes\sigma^A_{F(k),t} \,, \tag{38}$$

where $\sigma^A_{F(k),t}$ stands for the final density matrix of the apparatus displaying the kth eigenvalue of F. It is exactly on this point that we disagree with other attempts at formulating the wave-function collapse, in which the disappearance of off-diagonal and cross-correlated components is not explicitly shown.[12] For example, some authors

have often identified the notion of wave-function collapse with the asymptotic orthogonality

$$(\Phi_k,\Phi_l)=\delta_{kl}+O(\epsilon),\quad \epsilon \xrightarrow[N\to\infty]{} 0 \tag{39}$$

(N being the degrees of freedom of the system A), for the apparatus wave function in the von Neumann-Wigner measurement processes of the first kind, i.e.,

$$\Psi=\psi^{Q}\otimes\Phi_0\to\tilde{\Psi}=\sum_k c_k T_k u_k\otimes\Phi_k\ , \tag{40}$$

where Φ_0 and Φ_k are, respectively, the initial and final states of A, and we may write $T_k=e^{i\gamma_k}$ (with real γ_k) for ideal measurements of the first kind. If we decompose the final-state density matrix into the sum of its diagonal and off-diagonal parts with respect to k, as

$$\tilde{\rho}=|\tilde{\Psi}\rangle\langle\tilde{\Psi}|=\tilde{\rho}_{\rm diag}+\tilde{\rho}_{\rm off}\ ,$$

we know that by calculating the traces with respect to the A states before taking the limit for $N\to\infty$, we obtain

$$\begin{aligned}&\mathrm{Tr}_A\tilde{\rho}_{\rm off}=O(\epsilon)\ ,\\ &\mathrm{Tr}_A\tilde{\rho}^2_{\rm off}=\sum_k |c_k|^2(1-|c_k|^2)|u_k\rangle\langle u_k|+O(\epsilon)\ .\end{aligned} \tag{41}$$

This means that even though its trace vanishes, $\tilde{\rho}_{\rm off}$ itself does not vanish, even in the infinite N limit. Thus we conclude that this kind of approach can never give the exact wave-function collapse as formalized by us in this section [Eq. (33) or (38)]. We should also remark that the secondary processes, such as counter triggering, following the wave-function collapse, are described in (34) and (35) through the time evolution of $\sigma^D_{F,t}$.

V. IDEAL CASES

As discussed in Sec. II, the same kind of equipment can work well or not as a detector, depending on the physical processes involved. It will be useful to consider some ideal cases which will help us to understand the role of the equipment.

Let T_j and R_j ($j=1,\ldots,N_p$) be the transmission and reflection coefficients for the jth particle and $t_j=|T_j|^2$, $r_j=|R_j|^2$ the transmission and reflection probabilities, respectively. We define, following Eqs. (12) and (13),

$$\begin{aligned}&\bar{T}=\frac{1}{N_p}\sum_{j=1}^{N_p} T_j\ ,\\ &\bar{R}=\frac{1}{N_p}\sum_{j=1}^{N_p} R_j\ ,\\ &\bar{t}=\frac{1}{N_p}\sum_{j=1}^{N_p} t_j=\frac{1}{N_p}\sum_{j=1}^{N_p}|T_j|^2\ ,\\ &\bar{r}=\frac{1}{N_p}\sum_{j=1}^{N_p} r_j=\frac{1}{N_p}\sum_{j=1}^{N_p}|R_j|^2\end{aligned} \tag{42}$$

as the *average* transmission-reflection coefficients and probabilities, respectively. In general, absorption shall be taken into account. For this purpose, we define the absorption probability for the jth particle,

$$a_j=1-|T_j|^2-|R_j|^2=1-r_j-t_j\ , \tag{43}$$

and its average value

$$\bar{a}=\frac{1}{N_p}\sum_{j=1}^{N_p} a_j=1-\bar{r}-\bar{t}\ . \tag{44}$$

By definition, $a_j+r_j+t_j=1$ and $\bar{a}+\bar{r}+\bar{t}=1$. We are now ready to consider some limiting cases.

(a) A phase shifter will be a device for which

$$\bar{r}=\bar{a}=0,\quad \bar{t}=1$$
$$\text{and } T_j=T^{(0)}=\text{const (independent of } j)\ . \tag{45}$$

Moreover, since $\overline{|T|^2}=|\bar{T}|^2=|T^{(0)}|^2=1$, we obtain, by Eq. (19), the value $\epsilon=0$ for the decoherence parameter.

(b) An ideal detector will be a device for which

$$|\bar{T}|^2=\bar{T}=0\ . \tag{46}$$

This means that every particle, by interacting with the detector, acquires a *random* phase, so that the average in Eq. (13) vanishes. In this case, by Eq. (19), we obtain the value $\epsilon=1$ for the decoherence parameter. Note that the parameter $\overline{T'}$, as defined in Eq. (32), *must* vanish as well, due to the presence of the random phase. Observe that nothing is said about the values of $\bar{r}$, $\bar{a}$, and $\bar{t}$. We will define therefore the following:

(c) An ideal nondestructive detector is an "ideal detector" for which

$$\bar{r}=\bar{a}=0\ \text{ and }\ \bar{t}=1\ . \tag{47}$$

This means that besides the property in Eq. (46), we must have, from Eq. (12), $t_j=|T_j|^2=1$, $\forall_j=1,\ldots,N_p$. In other words, *every* particle is transmitted with a random phase.

(d) A perfect absorber will be a device for which

$$\bar{t}=\bar{r}=0\ \text{ and }\ \bar{a}=1\ . \tag{48}$$

Note that nothing can be said about the decoherence parameter ϵ defined in Eqs. (19) and (20). We will define therefore the following:

(e) An ideal destructive detector is a "perfect absorber" for which the condition

$$\epsilon=1 \tag{49}$$

holds. The reason for this definition should be clear in the light of our previous definition (b) and of the discussion of Sec. IV: Indeed, an ideal detector, as defined in (b), is an object for which the quantities $\bar{T}$ and $\overline{T'}$, defined in Eq. (32), are such that

$$\bar{T}=\overline{T'}=0\ , \tag{50}$$

due to the presence of statistical fluctuations; observe that this effect is *completely independent* of the value of the transmission probability $\bar{t}=|T|^2$. In the limit $\bar{t}=0$ (perfect absorber), we obtain an ideal (perfectly absorbing) destructive detector.

Given these definitions and having clarified the mechanism engendering the loss of coherence in quantum mechanics (wave-function collapse), we are now ready to

investigate the problem from a numerical point of view. This will be done by means of a simple model in the following sections.

VI. MODELING THE DETECTOR WITH A DIRAC COMB

We will study the interaction between the particle and the "detector," by making use of a simplified one-dimensional Dirac-comb model. A particle interacts with a macroscopic object (a detector), made up of elementary constituents, according to the laws of quantum mechanics (Schrödinger equation). The particle will interact with many different elementary constituents, or with many different bunches of elementary constituents. We will describe every interacting constituent (or every bunch of constituents) with a δ potential and the whole detector with an array of δ potentials (a Dirac comb). The total barrier will be given by

$$V(x)=\sum_{i=1}^{N}\Lambda_i\delta(x-b_i)\ , \tag{51}$$

where N is the total number of δ potentials ("elementary interactions"), Λ_i their (complex) strengths, and b_i their positions. In order to avoid confusion, we shall stick henceforth to the following convention: the index i ($i=1,\ldots,N$) will label the δ potentials, while the index j ($j=1,\ldots,N_p$) will label the incombing particles in an experimental run.

The wave function will be

$$\begin{aligned}
&\psi_0=A_1e^{ik(x-b_1)}+B_1e^{-ik(x-b_1)}\ ,\\
&\vdots\\
&\psi_{i-1}=A_ie^{ik(x-b_i)}+B_ie^{-ik(x-b_i)}\ ,\\
&\psi_i=A_{i+1}e^{ik(x-b_{i+1})}+B_{i+1}e^{-ik(x-b_{i+1})}\\
&\quad=C_ie^{ik(x-b_i)}+D_ie^{-ik(x-b_i)}\ ,\\
&\vdots\\
&\psi_N=C_Ne^{ik(x-b_N)}+D_Ne^{-ik(x-b_N)}\ ,
\end{aligned} \tag{52}$$

where we have denoted the wave function between the ith and the $(i+1)$th potential as ψ_i and have adopted the convention of using the coefficients A_i,B_i (C_i,D_i) to write the wave function on the very left (right) of the ith potential. Obviously,

$$\begin{aligned}
B_i&=A_i\mathcal{R}_i+D_i\mathcal{T}_i'\\
C_i&=A_i\mathcal{T}_i+D_i\mathcal{R}_i'\ ,
\end{aligned} \tag{53}$$

where $\mathcal{R}$ and $\mathcal{T}$ are the reflection and transmission coefficients of a single δ potential (a survey of elementary properties of the scattering by a δ potential is given in the Appendix) and a prime denotes the wave impinging on the δ from the right. By solving for C_i,D_i, we obtain

$$\begin{bmatrix}C_i\\D_i\end{bmatrix}=\frac{1}{\mathcal{T}_i'}\begin{bmatrix}\mathcal{T}_i\mathcal{T}_i'-\mathcal{R}_i\mathcal{R}_i' & \mathcal{R}_i'\\ -\mathcal{R}_i & 1\end{bmatrix}\begin{bmatrix}A_i\\B_i\end{bmatrix}=Z_i\begin{bmatrix}A_i\\B_i\end{bmatrix}\ . \tag{54}$$

Since our problem is invariant under space reflection, $\mathcal{R}_i=\mathcal{R}_i'$ and $\mathcal{T}_i=\mathcal{T}_i'$, so that

$$Z_i=\frac{1}{\mathcal{T}_i}\begin{bmatrix}\mathcal{T}_i^2-\mathcal{R}_i^2 & \mathcal{R}_i\\ -\mathcal{R}_i & 1\end{bmatrix} \tag{55}$$

and, by making use of the explicit expressions for $\mathcal{R}$ and $\mathcal{T}$ given in the Appendix [Eq. (A3)],

$$Z_i=\frac{1}{\omega_i}\begin{bmatrix}\omega_i+1 & 1\\ -1 & \omega_i-1\end{bmatrix}\ , \tag{56}$$

with $\omega_i=i\hbar v/\Lambda_i\in\mathbb{C}$ (the set of complex numbers), where v is the particle speed and Λ_i the potential strength. The general formula for the interaction with N δ potentials can be cast into a compact form: By setting

$$A_1=1,\quad B_1=R,\quad D_N=0,\quad C_N=T\ , \tag{57}$$

we obtain

$$\begin{bmatrix}T\\0\end{bmatrix}=Z_N\prod_{i=1}^{N-1}e^{ikd_i\tau_3}Z_i\begin{bmatrix}1\\R\end{bmatrix}\ , \tag{58}$$

where τ_3 is the third Pauli matrix and $d_i=b_{i+1}-b_i$. The coefficients T and R are the *whole* barrier's transmission and reflection coefficients, and we can define the barrier absorption coefficient

$$a=1-|R|^2-|T|^2=1-r-t\ . \tag{59}$$

So far, the internal motions of the elementary constituents of the detector have not been taken into account. These internal motions will give rise to an intrinsic stochasticity of the parameters describing the constituents themselves. In terms of our Dirac-comb model, this stochasticity will be modeled as follows: The strength of every single δ potential will vary according to a statistical (say, Gaussian) law. This means that the Λ_i's in Eq. (51) and therefore the ω_i's in Eq. (56) will undergo random fluctuations. Moreover, the interactions between the detector's constituents and the particles will take place in different parts of the detector. This means also that b_i, and $d_i=b_{i+1}-b_i$ will be subject to statistical fluctuations. Therefore, in Eq. (58) (here rewritten in different form)

$$\begin{aligned}
\begin{bmatrix}T\\0\end{bmatrix}&=\frac{1}{\omega_N}\begin{bmatrix}\omega_N+1 & 1\\ -1 & \omega_N-1\end{bmatrix}\\
&\quad\times\prod_{i=1}^{N-1}e^{ikd_i\tau_3}\frac{1}{\omega_i}\begin{bmatrix}\omega_i+1 & 1\\ -1 & \omega_i-1\end{bmatrix}\begin{bmatrix}1\\R\end{bmatrix}\ ,
\end{aligned} \tag{60}$$

there will be several sources of stochasticity:

$$\begin{aligned}
&\omega_i=\frac{i\hbar v}{\Lambda_i}=\frac{i\hbar v}{\Omega_i+i\Gamma_i}\in\mathbb{C}\ ,\\
&d_i=b_{i+1}-b_i\ ,
\end{aligned} \tag{61}$$

as well as N, of course. We will assume a Gaussian distribution for Ω_i, Γ_i, and d_i and will study the global effect on the outgoing wave as a function of N. Also, we will

suppress the randomness of N, the total number of δ's (i.e., the total number of "interactions"). This is a conservative approximation: Assuming, say, a Gaussian distribution also for N would lead to an even stronger dependence of T on the MHS structure of the detector's density matrix.

Our problem is thus reduced to solving Eq. (60) for T for many different incoming particles [so that $T \rightarrow T_j$ $(j=1,\ldots,N_p)$ as in Eq. (9)] and then applying the averaging procedure described in Eqs. (12) and (13) in order to obtain information on the average probability P_{ave}, in Eq. (21), and on our decoherence parameter ϵ, defined in Eqs. (19) and (20). Notice that, due to our ergodic assumption [Eq. (24)], the above-mentioned averages over many particles ($\overline{\cdots}$) will be equivalent to the "ensemble averages" over the values of the parameters undergoing the statistical fluctuations ($\langle \cdots \rangle$).

VII. LIMITING CASES

Before proceeding to the numerical simulation, we should investigate whether and how closely the ideal cases considered in Sec. V can be reproduced by means of our Dirac-comb model. Let us consider the five cases separately and make use of the properties derived in the Appendix for the scattering by a single δ potential:

(a) Phase shifter. A good phase shifter should transmit every particle with probability 1, and should contribute a *constant* (i.e., particle-independent) phase to the wave functions of the different particles. Consider the coefficients ρ, τ, and α of Eqs. (A6) and (A7). These refer to a *single* δ potential. In order to minimize the losses, we require $\Gamma=0$ in Eq. (A7), so that $\alpha=0$. Moreover, for small Ω, Eq. (A6) reads

$$\rho=\left[\frac{\Omega}{\hbar v}\right]^2 , \qquad \tau=1-\left[\frac{\Omega}{\hbar v}\right]^2 , \tag{62}$$

so that, with every effect on the *second order* in Ω, the conditions (45) can be fulfilled by an array of N δ scatterers of strength $\Lambda=\Omega$ $(\Omega \simeq 0^+)$, if N is small compared to $(\Omega/\hbar v)^{-2}$, (so that the global reflection is negligible), and, *most important,* if the fluctuations of the parameters in Eq. (61) are "frozen."

(b) Ideal detector. In this case the statistical fluctuations of the parameters in Eq. (61) should be *wide enough* in order to obtain $|\overline{T}|^2=0$ and satisfy condition (46). The quantitative meaning of the words "wide enough" will be given in Sec. VIII.

(c) Ideal nondestructive detector. Everything should be identical to case (a), but the statistical fluctuations of the parameters should make $|\overline{T}|^2$ vanish, as in case (b). This will be shown numerically to work nicely already for $N \simeq 100$.

(d) Perfect absorber. Consider again the coefficients ρ, τ, and α of Eqs. (A6) and (A7). The condition $\alpha=1$ cannot be fulfilled, but α is bigger if $\Omega=0$. Moreover, for Γ small (and negative), we have

$$\tau=1+2\frac{\Gamma}{\hbar v} , \qquad \rho=\left[\frac{\Gamma}{\hbar v}\right]^2 , \qquad \alpha=-2\frac{\Gamma}{\hbar v} . \tag{63}$$

Observe that ρ is of second order, while both τ and α are of first order in Γ; moreover, the absorption effect is cumulative, when the number of δ potentials is increased. Therefore, provided N is big enough, an array of N δ potentials of strength $\Lambda=i\Gamma$ $(\Gamma \simeq 0^-)$ will reflect very little, transmit very little, and almost completely absorb the wave. The conditions in Eq. (48) can thus be fulfilled.

(e) Ideal destructive detector. Everything should be identical to the previous case, but the stochasticity, introduced via the fluctuations of the parameters in Eq. (61), should yield a value of ϵ close to unity. The above arguments will be shown numerically to work with great accuracy for relatively small values of N ($\alpha > 98\%$ for $N=200$ and $\epsilon > 99.5\%$ for $N=70$).

The above discussion should make it clear that the role played by the statistical fluctuations is of drastic importance as far as the behavior of the detector is concerned. It will also be useful to stress that in our picture an ideal nondestructive detector and a phase shifter turn out to be different aspects of the same object: If we consider the statistical fluctuations as a measure of the "noise" or the "temperature" of the system, then the phase shifter becomes an ideal nondestructive detector if the noise level is high enough. It is at any rate quite peculiar that the simple Dirac-comb model proposed here be able to reproduce correctly so many different ideal physical situations, ranging from a phase shifter to a "nonabsorbing detector," and from an absorber to an "absorbing detector."

VIII. NUMERICAL RESULTS

We are now ready to proceed with the numerical simulation. We start from Eqs. (7) and (8), and assume infinitesimal slits. The wave function at the screen in Fig. 2 will be given by

$$\psi=\psi_1+T\psi_2=\frac{e^{ik|z_1|}}{|z_1|}+T\frac{e^{ik|z_2|}}{|z_2|} , \tag{64}$$

where $|z_n|$ $(n=1,2)$ is the distance between the nth slit and the point at the screen and $k=2\pi/\lambda$, where λ is the wavelength. The intensity at the screen is

$$|\psi|^2=|\psi_1+T\psi_2|^2 = \frac{1}{z_1^2}+\frac{|T|^2}{z_2^2}+\frac{2}{|z_1||z_2|}\mathrm{Re}(Te^{ik(|z_1|-|z_2|)}) . \tag{65}$$

For the sake of clarity, in the following discussion, Eq. (65) will be assumed to hold true even when the condition $L \gg \lambda$ (where L is the slit-screen distance) is not strictly satisfied. This simplification does not alter our discussion and final conclusions, and will display the physical mean-

ing of the process under investigation in a more intelligible way.

The coefficient T is given by Eq. (60), where the parameters in Eq. (61) undergo Gaussian statistical fluctuations. We will focus our attention on thermal neutrons interacting with a crystal, so that

$$\lambda=\frac{2\pi}{k}=2\ \text{Å}\ ,$$
$$\langle d\rangle=4.5\ \text{Å}\ , \tag{66}$$

where the brackets denote the statistical ensemble average defined in Eq. (24), whose meaning becomes now evident: The average value of every parameter, over the N_p incoming neutrons, will be given just by the mean value of the parameter's Gaussian distribution.

The parameters Ω and Γ are somewhat more arbitrary. According to the discussion of Sec. VII we require

$$\frac{\langle\Omega\rangle}{\hbar v}=-\frac{\langle\Gamma\rangle}{\hbar v}=10^{-2}\ . \tag{67}$$

The noise level (the width of the Gaussian distributions) will always be kept as low as 2% of the average value, for every parameter. The average transmission probability and average transmission coefficient of the detector [Eqs. (12) and (13)] are always computed for $N_p=5000$.

The results are displayed in Figs. 3, 4, and 5. In every figure, the function

$$P_{\text{ave}}=\overline{|\psi_1+T\psi_2|^2}$$
$$=\frac{1}{z_1^2}+\frac{\overline{|T|^2}}{z_2^2}+\frac{2}{|z_1||z_2|}\text{Re}(\bar{T}e^{ik(|z_1|-|z_2|)}) \tag{68}$$

is plotted versus the screen coordinate, in arbitrary units. The interference pattern gradually disappears (wave-packet collapse) as N, the number of δ potentials ("elementary interactions"), increases from 0 to ∞. In all the cases considered (real, purely imaginary, and complex potentials), the interference disappears for $N\simeq 100$. In the two latter cases, the particle is (almost) completely absorbed for $N\simeq 200$.

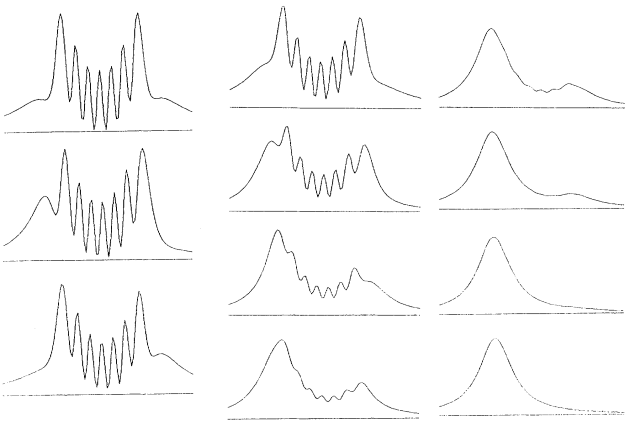

FIG. 4. Pure imaginary potentials: $\langle\Lambda\rangle=i\langle\Gamma\rangle=-i10^{-2}\hbar v$. From top to bottom and left to right, $N=0,2,5,10,20,35,50,70,100,200,\infty$.

In Fig. 3, for $N=100$, we have, with very good approximation, what we referred to as an ideal nondestructive detector in Secs. V and VII; Figs. 4 and 5, for $N=200$, are good examples of ideal destructive detectors. The intermediate cases are relative to the case of "partial collapse" of the wave function: Coherence is only *partially* lost, and one can say that an "imperfect measurement" has taken place.

In order to clarify better the mechanism underlying the wave-function collapse, we have drawn, in Figs. 6, 7, and 8 the phase diagrams for T. In Fig. 6, for instance, the transmission coefficients T_j of $N_p=150$ particles are displayed for N (number of δ's) increasing from 2 to 100. In the undisturbed case, when no δ is present, the T_j's would all be equal to $\bar{T}$ (which has conventionally been set equal to $e^{i\pi/4}$). The more potentials the incoming particles go through (i.e., the more interactions they un-

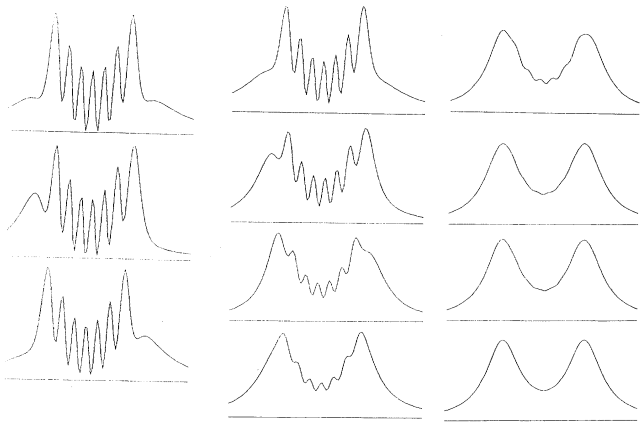

FIG. 3. Real potentials: $\langle\Lambda\rangle=\langle\Omega\rangle=10^{-2}\hbar v$. From top to bottom and left to right, $N=0,2,5,10,20,35,50,70,100,200,\infty$.

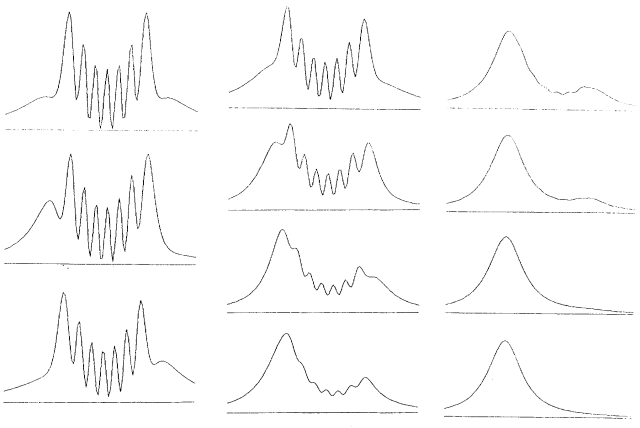

FIG 5. Complex potentials: $\langle\Lambda\rangle=\langle\Omega\rangle+i\langle\Gamma\rangle=(10^{-2}-i10^{-2})\hbar v$. From top to bottom and left to right, $N=0,2,5,10,20,35,50,70,100,200,\infty$.

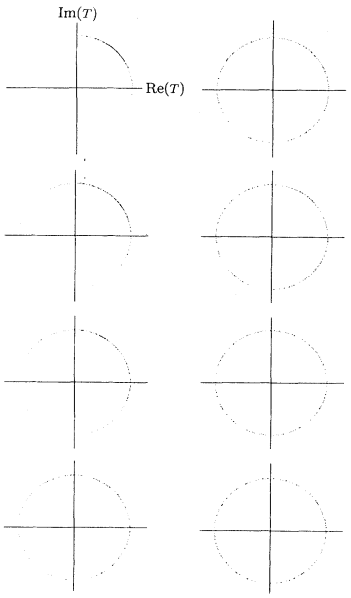

FIG. 6. Phase diagrams for T. The transmission coefficients T_j of $N_p=150$ particles are displayed for $N=2,5,10,20,35,50,70,100$ (from top to bottom and left to right). The parameters are equal to those in Fig. 3. (real case).

dergo), the more their T_j's spread. In the very-large-N limit, $\overline{T}=0$: Coherence is completely lost. An analogous situation occurs for Figs. 7 and 8, in which also absorption is present.

Our decoherence parameter is given in Table I for the various cases. According to Eq. (19), it is defined as

$$\epsilon=1-\frac{|\overline{T}|^2}{\overline{t}}=1-\frac{|\overline{T}|^2}{\overline{|T|^2}} . \tag{69}$$

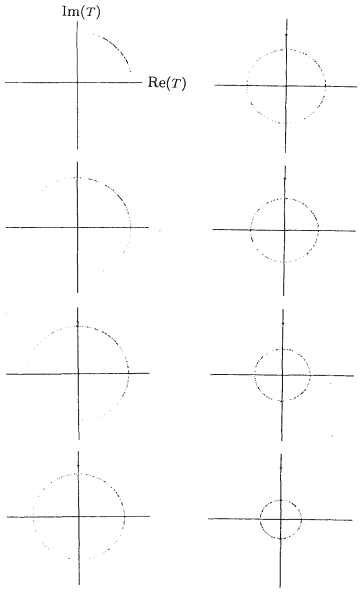

FIG. 7. Same as in Fig. 6, but with the parameters equal to those of Fig. 4 (pure imaginary case).

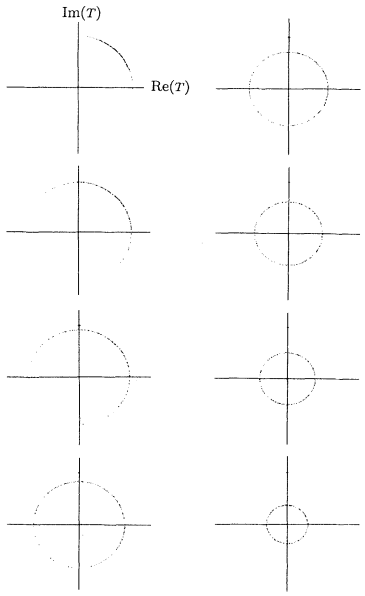

FIG. 8. Same as in Fig. 6, but with the parameters equal to those of Fig. 5 (complex case).

Observe that, at fixed N, the value of ϵ does *not* depend significantly on the type of interaction inside the detector: The three cases (real, purely imaginary, and complex potentials) yield the same values of ϵ up to the second significant digit. This means that, contrary to what is

TABLE I. Decoherence parameter ϵ.

Potential Type	$\lvert\overline{T}\rvert$	$\overline{t}$	ϵ	Number of Potentials
Real	0.96	1.0	0.07	
Imaginary	0.94	0.96	0.08	$N=2$
Complex	0.94	0.96	0.08	
Real	0.86	1.0	0.26	
Imaginary	0.81	0.90	0.27	$N=5$
Complex	0.81	0.90	0.27	
Real	0.72	1.0	0.49	
Imaginary	0.64	0.82	0.50	$N=10$
Complex	0.63	0.82	0.51	
Real	0.47	1.0	0.76	
Imaginary	0.39	0.67	0.77	$N=20$
Complex	0.38	0.67	0.78	
Real	0.25	1.0	0.94	
Imaginary	0.19	0.50	0.92	$N=35$
Complex	0.18	0.50	0.94	
Real	0.15	1.0	0.98	
Imaginary	0.09	0.37	0.98	$N=50$
Complex	0.08	0.37	0.98	
Real	0.07	1.0	1.0	
Imaginary	0.03	0.25	1.0	$N=70$
Complex	0.03	0.25	1.0	
Real	0.02	1.0	1.0	
Imaginary	0.01	0.13	1.0	$N=100$
Complex	0.01	0.13	1.0	
Real	0.01	1.0	1.0	
Imaginary	0.00	0.02	1.0	$N=200$
Complex	0.00	0.02	1.0	

sometimes believed, absorption has *no significant effect* on the collapse of the wave function, because the loss of coherence (of which ϵ is an estimate) stems only from the noise and the number of elementary "interactions" in the detector (our parameter N).

The case referred to as perfect absorber in Secs. V and VII is rather trivial, and has not been simulated. Indeed, if we "switch off" every fluctuation of the parameters in Eq. (61), we obtain the value $\epsilon=0$ for the decoherence parameter and Eq. (21) becomes identical to the standard quantum-mechanical formula (22): In other words, if we neglect the MHS structure of the macroscopic detector, we recover the standard quantum-mechanical formulas, and no wave-function collapse takes place.

This point, seemingly trivial, is far from being obvious: There may well be cases in which the MHS structure of the detector *cannot* be neglected, even in principle. We shall come back to this point in Sec. IX, in relation to neutron interferometry and quantum optics.

Let us conclude this section by showing our model "phase shifter" [see Eq. (45)]. As anticipated in Sec. VII, a phase shifter can be mimicked by an array of real δ potentials of low strength ($\Lambda=\Omega=10^{-4}\ \hbar v$) if no fluctuations are present. The phase χ can be varied by simply adding to the barrier other δ potentials, one by one.

In Fig. 9(a) the real part of the transmission coefficient $T=e^{i\chi}$ of one particle is plotted versus N, for N going from 100 to 200: The increasing thickness of the phase shifter is simulated by an increasing number of potentials. In this case, the noise is "frozen;" what happens if we "switch on" the noise is shown in Fig. 9(b), in which everything is analogous to Fig. 9(a), but the parameters undergo statistical (Gaussian) fluctuations of width 2% of the average value.

IX. APPLICATIONS: NEUTRON INTERFEROMETRY AND QUANTUM OPTICS

The above results are easily generalized to some recent experiments performed in Vienna,[4] in which an absorber is placed in one of the two arms of a perfect-crystal neutron interferometer. Let us observe first that if ψ_1 and ψ_2, in Eq. (21), are the wave functions corresponding to the two different paths in the interferometer, normalized to $\frac{1}{2}$ and in phase, and if we place a χ-phase shifter along, say, the second route, we obtain, for the intensity of the ordinary ray,[7]

$$I^{(\mathrm{ord})}\propto|\psi^{(\mathrm{ord})}|^2=\tfrac{1}{4}[1+\bar{t}+2\sqrt{\bar{t}}\sqrt{1-\epsilon}\cos(\chi+\beta)]\ , \tag{70}$$

so that the visibility, defined as

$$V=\frac{I_{\max}-I_{\min}}{I_{\max}+I_{\min}}\ , \tag{71}$$

is given, in our approach, by

$$V_{\mathrm{MHS}}=V_{\mathrm{QM}}\sqrt{1-\epsilon}\ , \tag{72}$$

where $V_{\mathrm{QM}}=2\sqrt{\bar{t}}/(1-\bar{t})$ is the standard quantum-mechanical value. The maximum degree of coherence of the two neutrons' branch waves is obtained, obviously, for $\epsilon=0$; the wave function is collapsed for $\epsilon=1$, in which case the visibility is zero.

A similar effect may be shown to be present in some fourth-order interference experiments performed in quantum optics.[13] If two photons are produced in a process of parametric down-conversion in a nonlinear crystal, then superposed on a beam splitter, and finally detected by two detectors, the number of photon coincidences is given by

$$N_C=C(t^2+r^2)[1-Vf(\alpha)]\ , \tag{73}$$

where C is a constant; t and r the transmission and reflection coefficients, respectively; $V=2tr/(t^2+r^2)$ is a "visibility," and $f(\alpha)$ is a function of some experimental parameters α and is always less than unity. By following an analogous reasoning, it can be shown that[14]

$$V_{\mathrm{MHS}}=V_{\mathrm{QM}}(1-\epsilon)^2\ , \tag{74}$$

where, again, $V_{\mathrm{QM}}=2\bar{t}\,\bar{r}/(\bar{t}^2+\bar{r}^2)$ is the standard quantum-mechanical value.

These look like standard situations, from the point of view of the MHS approach, but a closer look shows that there is no reason why the limit $\epsilon=0$ should be attainable in every conceivable case. Indeed, since the decoherence parameter ϵ is an estimate of the level of noise in a macroscopic system, it may well happen that the value $\epsilon=0$ (the total absence of noise) may not be attainable, even in principle. It has been recently shown that this is indeed the case, in neutron interferometry, when a strong absorber is present in one of the two paths of the interferometer, and in quantum optics, when the macroscopicity of the beam splitter is taken into account. In the light of our MHS analysis, such a situation corresponds to an imperfect measurement, in which coherence is only *partially* lost, ϵ being an estimate of such a loss. More details on this approach are given in Sec. X.

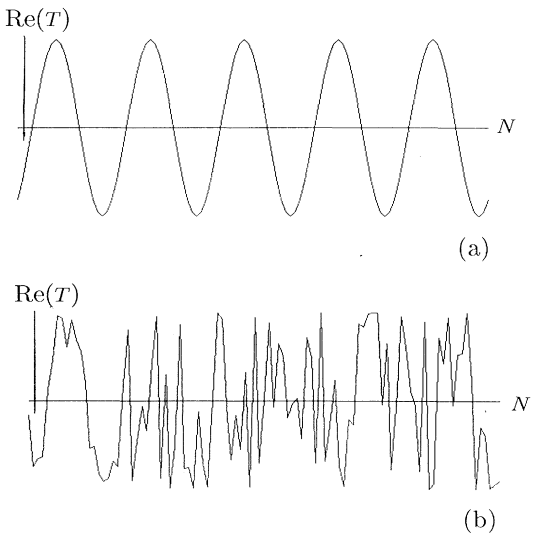

FIG. 9. (a) Phase shifter; (b) phase shifter with "noise."

X. AN ALTERNATIVE APPROACH

Following the line of thought given in Ref. 7, we give now an outline of derivation of $\bar{T}$ and $\bar{T}'$ based on the in-

teraction Hamiltonian:

$$H'=\sum_{n=1}^{N} G_n V(\mathbf{r}-\mathbf{r}_n)=\int d^3r'\rho(\mathbf{r}-\mathbf{r}')V(\mathbf{r}') , \quad (75)$$

where V is the potential between the particle (the neutron) and an elementary constituent of the detector, $\mathbf{r}_n$ the position of the nth constituent, and

$$\rho(\mathbf{r})=\sum_{n=1}^{N} G_n\delta(\mathbf{r}-\mathbf{r}_n) \quad (76)$$

the constituents' density. We can write $G_n=g\mathbf{1}_n+(1-g)\sigma_{1,n}$, g being a positive number smaller than 1, in which $\mathbf{1}_n$ and $\sigma_{1,n}$ stand for the unity and the first Pauli matrix associated with the nth constituent. In this way we can take into account, respectively, simple potential scatterings by the first term and transitions to another channel ($|0\rangle$) by the second one. In the infinite-N limit, we can discuss the whole problem by replacing $\sigma_{1,n}$ with a c number representing the transition probability of the elementary inelastic process and fluctuations around it. For details, see Ref. 2. Here, for the sake of simplicity, we shall assume $G_n=1$ by which the essential procedure is not altered.

Let us decompose the density function $\rho(\mathbf{r})$ as

$$\rho(\mathbf{r},t)=\langle\rho\rangle+\delta\rho(\mathbf{r},t) , \quad (77)$$

where $\langle\rho\rangle$ is a constant background density and $\delta\rho$ is a "noise" subject to the following statistical properties:

$$\langle\delta\rho(\mathbf{r},t)\rangle=0 ,$$
$$\langle\delta\rho(\mathbf{r},t)\delta\rho(\mathbf{r}',t')\rangle=F_\theta(\mathbf{r}-\mathbf{r}',t-t') . \quad (78)$$

Here the brackets denote the statistical ensemble average over all the possible constituents' states, and F is, in general, an increasing function of temperature θ of the detector as well as a function of other variables and parameters. Note that the t dependence of the density function comes from the internal motion of the constituents' positions, $\mathbf{r}_n(t)$, generated by H_0^D, the free Hamiltonian of the detector.

By writing $\mathcal{H}_0^Q=H_0^Q+\mathcal{V}$ and $\mathcal{H}'=H'-\mathcal{V}$ (with $\mathcal{V}=\langle\rho\rangle\int Vd^3\mathbf{r}$) and rewriting the total Hamiltonian as $H=\mathcal{H}_0^Q+H_0^D+\mathcal{H}'$, we obtain the S matrix,

$$S=S_0U_I(T) , \quad (79)$$

in the interaction representation, where T is the transit time of the particle in the detector D, S_0 is a scalar, and $U_I(t)$ obeys

$$U_I(t)=1+\frac{1}{i\hbar}\int_0^t dt'\mathcal{H}_I'(t')U_I(t') ,$$
$$\text{with } \mathcal{H}_I'(t)=e^{(i/\hbar)\mathcal{H}_0t}\mathcal{H}'e^{-(i/\hbar)\mathcal{H}_0t} . \quad (80)$$

Thus, via Eq. (78), we have given a definite meaning to the statistical average $\langle\cdots\rangle$, introduced in Eq. (24), and we can identify S_0 and $\langle\psi_2|S|\psi_2\rangle$ [Eq. (79)], with T_0 and $\bar{T}$ [Eq. (17)], respectively. Moreover, Eq. (80) allows us to follow a perturbative approach, by writing

$$\langle\psi_2|U_I(T)|\psi_2\rangle=1+\bar{\Delta}=1+\bar{\Delta}^{(1)}+\bar{\Delta}^{(2)}+\cdots . \quad (81)$$

For more details about this perturbative solution for $U_I(T)$, see Ref. 7.

In a similar way we can describe the interaction between a photon and a beam splitter by starting from the total Hamiltonian

$$H=\sum_{n=1}^{N_e}\left[\frac{1}{2m}\left[\mathbf{p}_n-\frac{e}{c}\mathbf{A}(\mathbf{r}_n)\right]^2+V(\mathbf{r}_n)\right]+\sum_k\hbar\omega_k a_k^\dagger a_k , \quad (82)$$

where N_e is the total number of electrons in the beam splitter, V is a binding potential for the electrons, and the notation is standard. The density function of the electrons in the beam splitter is written as

$$\rho(\mathbf{r},t)=\rho_S(\mathbf{r})+\delta\rho(\mathbf{r},t) , \quad (83)$$

where ρ_S is a "static" electron density and is a function of $\mathbf{r}$, in this case. An analogous calculation based on the ansatz (78) yields the perturbative expansion

$$\langle\phi,\psi|U_I(T)|\phi,\psi\rangle=(1+\bar{\Delta})=(1+\bar{\Delta}_1+\bar{\Delta}_2+\cdots) , \quad (84)$$

where ϕ is the photon wave packet and ψ the wave function of the electrons in the beam splitter. More details can be found in Ref. 14.

XI. CONCLUSIONS

We have shown that the MHS approach to the quantum measurement problem explains in a natural way the loss of coherence in quantum mechanics. Coherence is lost at a *statistical level,* when many interfering particles interact with a macroscopic object: This is ascribable to the effect described by Eq. (14), which is *de facto* responsible for the loss of coherence and which allows us to define the parameter ϵ of Eqs. (19) and (20) and Eq. (69). In terms of this *decoherence* parameter it has been possible to formulate a definite quantitative criterion for the wave-function collapse, which is obtained in the limit $\epsilon=1$. We wish to stress that our approach differs radically from the original theory of von Neumann, in which the wave function is collapsed due to the intervention of an external observer who is supposed to provoke an *acausal* change of the wave function by simply observing the quantum system. No external observer is needed in our case, in order to explain the collapse of the wave function, for the evolution from a pure state to a mixture is simply a statistical effect due to the macroscopicity of the detector. It is just this macroscopicity which is responsible for the difference between the coefficients $|\bar{T}|^2$ and $\bar{t}=|T|^2$. A careful analysis shows indeed that $|\bar{T}|^2$ and $\bar{t}$ do not depend on the noise in the same way: $\bar{T}$ is more influenced than $\bar{t}$ by the statistical fluctuations, because, by definition, it depends on the phases acquired by the different particles, and this is, in conclusion, the reason why the reduction of the wave packet takes place, in our approach.

Against the above argument, however, one may wonder whether a *mean value* obtained statistically, by performing an experiment many times, reflects a property of reality. According to a classical picture of reality, the

answer to this question should be negative. However, the dynamical laws of quantum mechanics are known to give quantitative predictions when the data relative to many repetitions of the same experiment are accumulated, and no prediction for single events (except in the special case in which the quantum system is in an eigenstate of the observable to be measured). This is the essential origin of the uncertainty principle, by which the very concept of reality is blurred. It is the authors' opinion that a new description of reality has to be sought, which should be based on quantum mechanics and will probably turn out to be very different from the classical one. The problem of the meaning of "reality" is much too philosophical to be addressed and resolved by the present paper. In this sense, we preferred to limit our purpose to the much more humble task of understanding whether the evolution from a pure to a mixed state (a quantum-mechanical "measurement") can be *derived* within the existing theory, instead of being simply postulated.

Note added in proof. An exact evaluation of the decoherence parameter ϵ was recently given for a solvable model of one-dimensional emulsion by M. Namiki and S. Pascazio, Found. Phys. Lett. (to be published).

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APPENDIX: SCATTERING BY A COMPLEX δ POTENTIAL

Let us recall some properties and formulas relative to the one-dimensional scattering of a plane wave by a δ-shaped potential. Assume the potential to be centered around $x=0$, and the wave to be normalized at 1 for $x=-\infty$. Then,

$$\psi^{-}=e^{ikx}+\mathcal{R}e^{-ikx}\ ,$$
$$\psi^{+}=\mathcal{T}e^{ikx}\ , \tag{A1}$$

where $\mathcal{R}$ and $\mathcal{T}$ are the reflection and transmission coefficients, respectively. The potential barrier will be given by

$$V(x)=\Lambda\delta(x) \tag{A2}$$

($\Lambda=\lim_{l\to 0,V_0\to\infty} lV_0$, where l is the barrier "thickness" and V_0 the barrier "height"). In this approximation, we have

$$\mathcal{R}=\left[-1+i\frac{\hbar v}{\Lambda}\right]^{-1}\ ,$$
$$\mathcal{T}=\left[1+i\frac{\Lambda}{\hbar v}\right]^{-1}\ , \tag{A3}$$

where v is the particle's speed.

We will be interested in the general case in which Λ is a complex number. It will be useful to remember that the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi+V\psi\ , \tag{A4}$$

when $V=V_R+iV_I\in\mathbb{C}$, yields the following continuity equation:

$$\frac{\partial P}{\partial t}+\nabla\cdot\mathbf{S}=\frac{2}{\hbar}V_I P\ , \tag{A5}$$

where $P=|\psi|^2$ and

$$\mathbf{S}=(\hbar/2im)(\psi^*\nabla\psi-\psi\nabla\psi^*)$$

are the probability density and current, respectively. If $V_I>0$ the r.h.s. of Eq. (A5) acts as a "source;" if $V_I<0$ it acts as a "sink." We will mainly be interested in the latter case.

If we set $\Lambda=\Omega+i\Gamma$, in Eq. (A3), we obtain for the reflection (ρ) and transmission (τ) probability, respectively,

$$\rho=|\mathcal{R}|^2=\frac{\Gamma^2+\Omega^2}{(\hbar v-\Gamma)^2+\Omega^2}\ ,$$
$$\tau=|\mathcal{T}|^2=\frac{(\hbar v)^2}{(\hbar v-\Gamma)^2+\Omega^2}\ , \tag{A6}$$

so that it is possible to define the absorption coefficient

$$\alpha=1-\rho-\tau=-\frac{2\hbar v\Gamma}{(\hbar v-\Gamma)^2+\Omega^2}\ . \tag{A7}$$

By definition, $\rho+\tau+\alpha=1$ and $\alpha>0$ if $\Gamma<0$ ("sink").

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