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Van Hove's "$\lambda^2 t$" limit in nonrelativistic and relativistic field-theoretical models ☆

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Abstract

Van Hove's "$\lambda^2 t$" limiting procedure is analyzed in some interesting quantum field-theoretical cases, both in nonrelativistic and relativistic models. We look at the deviations from a purely exponential behavior in a decay process and discuss the subtle issues of state preparation and initial time.

1. Introduction

In 1955 Van Hove proposed a remarkable time rescaling procedure [1] that enabled him to derive the master equation from the Schrödinger equation, for a quantum mechanical system endowed with an infinite number of degrees of freedom, such as a quantum field. The idea is to consider the limit

$$\lambda \to 0 \text{ keeping } \tilde{t} = \lambda^2 t \text{ finite} \quad (\lambda - \text{independent constant}), \tag{1.1}$$

where λ is the coupling constant and t time. One then looks at the evolution of the quantum system as a function of the rescaled time $\tilde{t}$. This is called Van Hove's "$\lambda^2 t$" limit and provided an interesting solution to some long-standing problems in quantum mechanics and quantum field theory, such as a rigorous justification of the Fermi "golden" rule [2,3] and of the Weisskopf–Wigner approximation [4–6].

Van Hove's prescription avoided the rigorous consequences of the quantum mechanical evolution law, which is governed by strictly unitary operators and predicts that the decay of an unstable quantum system cannot be purely exponential, being quadratic for very short times [7–13] and given by a power law for very long times [14–19]. These features of quantum evolution are so well known that they are discussed even in textbooks of quantum mechanics [20,21] and quantum field theory [22]. The temporal behavior of quantum systems is reviewed in [23].

One should notice that the problem of deviations from exponential decay was considered an academic one until very recently. The renewed interest in the short-time nonexponential behavior was motivated by a nice proposal by Cook [24], the subsequent experiment performed by Itano et al. [25] and the debate that followed [26–49]. One must notice, however, that Cook's idea and the subsequent papers did not deal with bona fide unstable systems. The latter require a quantum field-theoretical analysis and the careful treatment of cut-offs and divergent quantities [50–57]. It is also worth emphasizing that no deviations from the exponential behavior for an unstable system were observed until 1997, when Raizen's group detected non-exponential leakage through a potential barrier [58].

In this paper we shall look at Van Hove's limit from the perspective of the complex energy plane. We shall consider some particular cases, concentrating our attention on the exponential decay law and the irreversible features [59,60] of the evolution. There is interesting related work in the literature, in particular in connection with quantum dynamical semigroups [61] and the so-called "stochastic limit" in quantum theory [62,63].

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2. A simple example: the N-level atom

We start our analysis by considering a simple nonrelativistic model: an N-level atom in interaction with the electromagnetic field [64]. This example will help us to pin down some salient features of the $\lambda^2 t$ limit. The Hamiltonian is

$$H = H_0 + \lambda V, \tag{2.2}$$

with ($\hbar = c = 1$)

$$H_0 \equiv \sum_{\nu} \omega_\nu b_\nu^\dagger b_\nu + \sum_{\beta} \int_0^\infty \mathrm{d}\omega\, \omega a_{\omega\beta}^\dagger a_{\omega\beta}, \tag{2.3}$$

$$V = \sum_{\mu,\nu} \sum_{\beta} \int_0^\infty \mathrm{d}\omega [\varphi_\beta^{\mu\nu}(\omega) b_\mu^\dagger b_\nu a_{\omega\beta}^\dagger + \varphi_\beta^{\mu\nu *}(\omega) b_\nu^\dagger b_\mu a_{\omega\beta}], \tag{2.4}$$

where ν runs over all the atomic states, $b_\nu^\dagger, b_\nu$ are the annihilation and creation operators of the atomic level ν, obeying anticommutation relations

$$\{b_k, b_\ell^\dagger\} = \delta_{k\ell}, \tag{2.5}$$

and $a_{\omega\beta}^\dagger, a_{\omega\beta}$ are the annihilation and creation operators of the electromagnetic field, satisfying commutation relations

$$[a_{\omega\beta}, a_{\omega'\beta'}^\dagger] = \delta(\omega - \omega')\delta_{\beta\beta'}, \tag{2.6}$$

where ω is energy and β stands for other (discrete) quantum numbers (e.g. $\beta = (j, m, \epsilon)$, where j is the total angular momentum (orbital + spin) of the photon, m its magnetic quantum number and ϵ defines the photon parity $P = (-1)^{j+1+\epsilon}$). The general features of the form factors are well known for a wide class of physical systems [65,66] and some particular cases of the above Hamiltonian have been widely investigated in the literature [67–72].

Assume one can prepare, say at time $t = 0$, the system in the initial state $|\mu; 0\rangle$ (atom in state μ and no photons). The problem of state preparation is a subtle one that will be carefully discussed later. The initial state is an eigenstate of the unperturbed Hamiltonian H_0 and the evolution is governed by the unitary operator

$$U(t) = \exp(-\mathrm{i}Ht) = \frac{\mathrm{i}}{2\pi} \int_C \mathrm{d}E \frac{\mathrm{e}^{-\mathrm{i}Et}}{E - H}, \tag{2.7}$$

where the path C is a straight horizontal line just above the real axis. By defining the resolvents ($\mathrm{Im}\, E > 0$)

$$S(E) \equiv \langle \mu; 0| \frac{1}{E - H_0} |\mu; 0\rangle = \frac{1}{E - \omega_\mu}, \qquad S'(E) \equiv \langle \mu; 0| \frac{1}{E - H} |\mu; 0\rangle, \tag{2.8}$$

Dyson's resummation reads

$$S'(E) = S(E) + \lambda^2 S(E)\Sigma(E)S(E) + \lambda^4 S(E)\Sigma(E)S(E)\Sigma(E)S(E) + \cdots, \tag{2.9}$$

where $\Sigma(E) = \langle \mu; 0| V(E - H_0)^{-1} V |\mu; 0\rangle$ is the one-particle irreducible self-energy function, which can be evaluated by the expansion

$$\Sigma(E) = \Sigma^{(2)}(E) + \lambda^2 \Sigma^{(4)}(E) + \cdots, \tag{2.10}$$

with

$$\Sigma^{(2)}(E) \equiv \sum_{\nu,\beta} \int_0^\infty \mathrm{d}\omega \frac{|\varphi_\beta^{\nu\mu}(\omega)|^2}{E - \omega_\nu - \omega}. \tag{2.11}$$

Both $\Sigma^{(2)}$ and $\Sigma^{(4)}$ are shown as Feynman diagrams in Fig. 1. In the complex E-plane $\Sigma(E)$ and $\Sigma^{(2)}(E)$ have a branch cut running from the ground-state energy to ∞ and no singularity on the first Riemann sheet. Summing the series (2.9) one obtains

$$S'(E) = \frac{1}{S(E)^{-1} - \lambda^2 \Sigma(E)} = \frac{1}{E - \omega_\mu - \lambda^2 \Sigma(E)}. \tag{2.12}$$

We define the "survival" or nondecay amplitude and probability at time t (interaction picture):

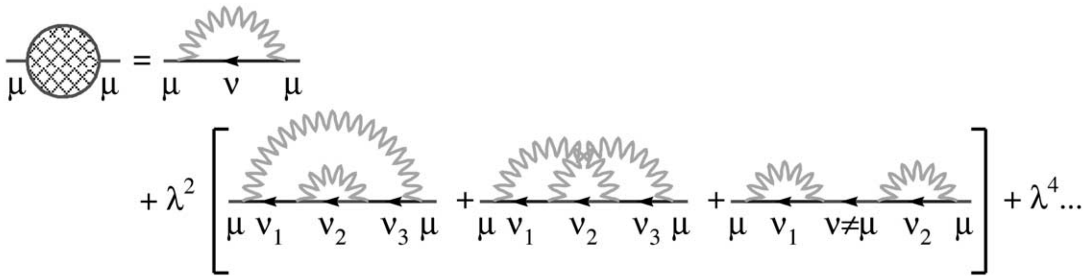

Fig. 1. Graphic representation of (2.10): $\Sigma^{(2)}$ and $\Sigma^{(4)}$ are in the first and second line, respectively.

$$\mathscr{A}(t) = \langle\mu;0|\mathrm{e}^{\mathrm{i}H_0t}U(t)|\mu;0\rangle, \tag{2.13}$$

$$P(t) = |\langle\mu;0|\mathrm{e}^{\mathrm{i}H_0t}U(t)|\mu;0\rangle|^2. \tag{2.14}$$

Incidentally, notice that the survival probability at short times behaves quadratically:

$$P(t) = 1 - t^2/\tau_Z^2 + \cdots, \qquad \tau_Z \equiv (\lambda^2\langle\mu;0|V^2|\mu;0\rangle)^{-1/2}. \tag{2.15}$$

The quantity τ_Z is the "Zeno time": it is the convexity of $P(t)$ in the origin. The nonexponential behavior at short times, besides its fundamental interest, entails the quantum Zeno effect [7–13]. Notice that the expansion (2.15) is formal: we implicitly require that the second moment of the interaction Hamiltonian exists – a delicate assumption in quantum field theory [50–54].

The survival amplitude can be expressed as

$$\mathscr{A}(t) = \frac{\mathrm{i}}{2\pi}\int_C \mathrm{d}E\,\mathrm{e}^{-\mathrm{i}Et}S'(E+\omega_\mu) = \frac{\mathrm{i}}{2\pi}\int_C \mathrm{d}E\frac{\mathrm{e}^{-\mathrm{i}Et}}{E-\lambda^2\Sigma(E+\omega_\mu)}. \tag{2.16}$$

In Van Hove's limit one looks at the evolution of the system over time intervals of order $t=\tilde{t}/\lambda^2$ ($\tilde{t}$ independent of λ), in the limit of small λ. Let us see how this procedure works in the complex-energy plane. To this end, by rescaling time $\tilde{t}\equiv\lambda^2 t$, we can write

$$\mathscr{A}\left(\frac{\tilde{t}}{\lambda^2}\right) = \frac{\mathrm{i}}{2\pi}\int_C \mathrm{d}\tilde{E}\frac{\mathrm{e}^{-\mathrm{i}\tilde{E}\tilde{t}}}{\tilde{E}-\Sigma(\lambda^2\tilde{E}+\omega_\mu)}, \tag{2.17}$$

where we are naturally led to introduce the rescaled energy $\tilde{E}\equiv E/\lambda^2$. Taking the Van Hove limit we get

$$\Sigma(\lambda^2\tilde{E}+\omega_\mu)\overset{\lambda\to 0}{\to}\Sigma^{(2)}(\lambda^2\tilde{E}+\omega_\mu)\Big|_{\lambda=0} = \Sigma^{(2)}(\omega_\mu+\mathrm{i}0^+) = \sum_{\nu,\beta}\int_0^\infty \mathrm{d}\omega\frac{|\varphi_\beta^{\nu\mu}(\omega)|^2}{\omega_\mu-\omega_\nu-\omega+\mathrm{i}0^+} \equiv \Delta(\omega_\mu) - \frac{\mathrm{i}}{2}\Gamma(\omega_\mu), \tag{2.18}$$

where

$$\Delta(\omega_\mu) \equiv \mathscr{P}\sum_{\nu,\beta}\int_0^\infty \mathrm{d}\omega\frac{|\varphi_\beta^{\nu\mu}(\omega)|^2}{\omega_\mu-\omega_\nu-\omega+\mathrm{i}0^+}, \tag{2.19}$$

$$\Gamma(\omega_\mu) \equiv 2\pi\sum_{\nu,\beta}|\varphi_\beta^{\nu\mu}(\omega)|^2, \tag{2.20}$$

the term $+\mathrm{i}0^+$ being due to the fact that $\mathrm{Im}\,\tilde{E}>0$. The propagator becomes

$$\tilde{S}'(\tilde{E}) \equiv \lim_{\lambda\to 0}\frac{1}{\tilde{E}-\Sigma^{(2)}(\lambda^2\tilde{E}+\omega_\mu)+\mathrm{O}(\lambda^2)} = \frac{1}{\tilde{E}-\Sigma^{(2)}(\omega_\mu+\mathrm{i}0^+)} \tag{2.21}$$

and the survival probability reads

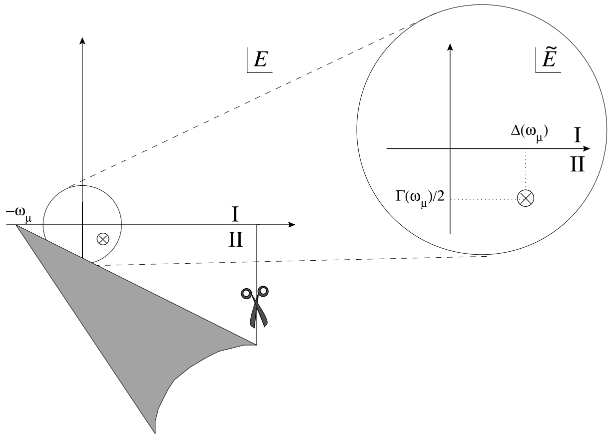

Fig. 2. Singularities of the propagator (2.16) in the complex-E plane. The first Riemann sheet (I) is singularity free. The logarithmic cut is due to $\Sigma^{(2)}(E)$ and the pole is located on the second Riemann sheet (II). The Van Hove rescaling procedure acts as a "magnifying glass" in the complex energy plane. After rescaling, the pole has coordinates (2.19), (2.20) in the complex-$\tilde{E}$ plane, without higher-order corrections in the coupling constant.

$$\tilde{\mathscr{A}}(\tilde{t}) \equiv \lim_{\lambda\to 0} \mathscr{A}\left(\frac{\tilde{t}}{\lambda^2}\right) = \frac{\mathrm{i}}{2\pi}\int_C \mathrm{d}\tilde{E}\, \mathrm{e}^{-\mathrm{i}\tilde{E}\tilde{t}}\tilde{S}'(\tilde{E}) = \mathrm{e}^{-[\mathrm{i}\Delta(\omega_\mu)+\Gamma(\omega_\mu)/2]t}, \tag{2.22}$$

which yields a purely exponential decay (Weisskopf–Wigner approximation and Fermi golden rule). In Fig. 2 we endeavoured to clarify the role played by the time–energy rescaling in the complex-E plane.

A few comments are in order. In the present model, the Van Hove limit works in two "steps". First, it constrains the evolution in a Tamm–Dancoff sector [73,74]: the system can only "explore" those states that are directly related to the initial state μ by the interaction V: the "excitation number" $\mathscr{N}_\mu \equiv b_\mu^\dagger b_\mu + \sum_{\beta,\omega} a_{\omega\beta}^\dagger a_{\omega\beta}$ becomes a conserved quantity and, as a consequence, the self-energy function consists only of a second-order contribution that can be evaluated exactly. Second, it reduces this second-order contribution, which depends on energy as in (2.11), to a constant (its value in the energy ω_μ of the initial state), like in (2.18). Hence the analytical properties of the propagator, which had branch-cut singularities, reduce to those of a single complex pole, whose imaginary part (responsible for exponential decay) yields the Fermi golden rule, evaluated at second order of perturbation theory.

Notice that it is the latter step (and not the former one) which is strictly necessary to obtain a dissipative behavior: indeed, substitution of the pole value in the total self-energy function yields exponential decay, including, as is well known, higher-order corrections to the Fermi golden rule. On the other hand, the first step is very important when one is interested in computing the leading order corrections to the exponential behavior. To this purpose one can solve the problem in a restricted Tamm–Dancoff sector of the total Hilbert space (i.e., in an eigenspace of $\mathscr{N}_\mu$ – in our case, $\mathscr{N}_\mu = 1$) and exactly evaluate the evolution of the system with its deviations from exponential law.

3. A more general framework

Let us generalize the analysis of the previous section. Consider the Hamiltonian

$$H = H_0 + \lambda V \tag{3.1}$$

and suppose that one can prepare an initial state $|a\rangle$ with the following properties:

$$H_0|a\rangle = E_a|a\rangle, \qquad \langle a|V|a\rangle = 0, \qquad \langle a|a\rangle = 1. \tag{3.2}$$

The survival amplitude of state $|a\rangle$ reads

$$\mathscr{A}(t) \equiv \langle a|\mathrm{e}^{\mathrm{i}H_0 t}U(t)|a\rangle = \frac{\mathrm{i}}{2\pi}\int_C \mathrm{d}E\, \mathrm{e}^{-\mathrm{i}Et}S'(E+E_a) = \frac{\mathrm{i}}{2\pi}\int_C \mathrm{d}E \frac{\mathrm{e}^{-\mathrm{i}Et}}{E-\lambda^2\Sigma(E+E_a)}, \tag{3.3}$$

where $S'(E) \equiv \langle a|(E-H)^{-1}|a\rangle$ and $\Sigma(E)$ is the one-particle irreducible self-energy function, which can be expressed by the perturbation expansion

$$\lambda^2\Sigma(E) = \lambda^2\Sigma^{(2)}(E) + \lambda^4\Sigma^{(4)}(E) + \cdots \tag{3.4}$$

The second-order contribution has the general form

$$\Sigma^{(2)}(E) \equiv \langle a|VP_{\rm d}\frac{1}{E-H_0}P_{\rm d}V|a\rangle = \sum_{n\neq a}|\langle a|V|n\rangle|^2\frac{1}{E-E_n} = \int_0^\infty \frac{{\rm d}E'}{2\pi}\frac{\Gamma(E')}{E-E'}, \tag{3.5}$$

where $P_{\rm d} = 1 - |a\rangle\langle a|$ is the projector over the decayed states, $\{|n\rangle\}$ is a complete set of eigenstates of H_0 ($H_0|n\rangle = E_n|n\rangle$ and we set $E_0 = 0$) and

$$\Gamma(E) \equiv 2\pi\sum_{n\neq a}|\langle a|V|n\rangle|^2\delta(E-E_n). \tag{3.6}$$

Notice that $\Gamma(E) \geqslant 0$ for $E > 0$ and is zero otherwise. In the Van Hove limit we get

$$\tilde{\mathscr{A}}(\tilde t) \equiv \lim_{\lambda\to 0}\mathscr{A}\left(\frac{\tilde t}{\lambda^2}\right) = \frac{\rm i}{2\pi}\int_C {\rm d}\tilde E\, {\rm e}^{-{\rm i}\tilde E\tilde t}\tilde S'(\tilde E), \tag{3.7}$$

where the resulting propagator in the rescaled energy $\tilde E = E/\lambda^2$ reads

$$\tilde S'(\tilde E) = \frac{1}{\tilde E - \Sigma^{(2)}(E_a + {\rm i}0^+)}, \tag{3.8}$$

where we used

$$\Sigma(\lambda^2\tilde E + E_a) \stackrel{\lambda\to 0}{\to} \Sigma^{(2)}(\lambda^2\tilde E + E_a)\Big|_{\lambda=0} = \Sigma^{(2)}(E_a + {\rm i}0^+) \tag{3.9}$$

(Weisskopf–Wigner approximation and Fermi golden rule).

Let us compute the leading order corrections to the exponential behavior, in particular at short times. Just above the positive real axis we can write

$$\Sigma^{(2)}(E + {\rm i}0^+) = \Delta(E) - \frac{\rm i}{2}\Gamma(E), \tag{3.10}$$

where

$$\Delta(E) = \mathscr{P}\int_0^\infty \frac{{\rm d}E'}{2\pi}\frac{\Gamma(E')}{E-E'}. \tag{3.11}$$

We assume that $\Gamma(E)$ is summable in $(0, +\infty)$, so that, for some $\eta > 1$,

$$\Gamma(E) \propto E^{\eta-1} \quad \text{for } E \to 0. \tag{3.12}$$

It is then straightforward to obtain

$$\tau_{\rm Z} = \frac{1}{\lambda}\left[\int_0^\infty \frac{{\rm d}E}{2\pi}\Gamma(E)\right]^{-1/2}, \tag{3.13}$$

$$\tau_{\rm E} = \frac{1}{\lambda^2\Gamma(E_a)}, \tag{3.14}$$

which are the Zeno time and the lifetime, respectively. When time is rescaled according to Van Hove, the Zeno region vanishes

$$\tilde\tau_{\rm Z} \equiv \lambda^2\tau_{\rm Z} = \lambda\left[\int_0^\infty \frac{{\rm d}E}{2\pi}\Gamma(E)\right]^{-1/2} = {\rm O}(\lambda), \tag{3.15}$$

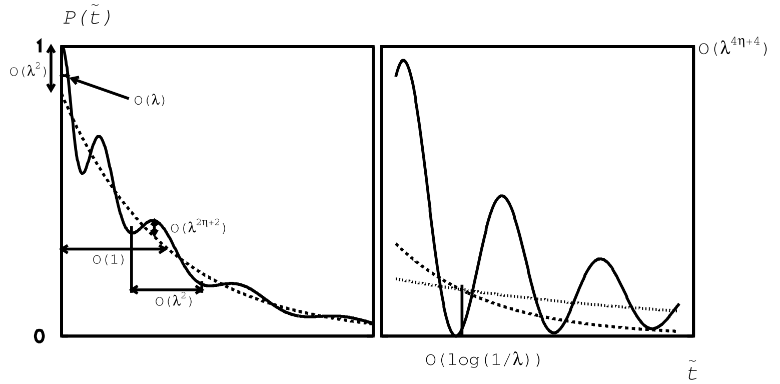

Fig. 3. Essential features (not to scale!) of the survival probability as a function of the rescaled time $\tilde{t}$. The Zeno time is $O(\lambda)$, the lifetime $O(1)$, during the whole evolution there are oscillations of amplitude $O(\lambda^{2\eta+2})$ and the transition to a power law occurs after a time $O(\log(1/\lambda))$ [see (3.15) and (3.16)]. The normalization factor becomes unity as $1 - O(\lambda^2)$. The dashed line is the exponential and the dotted line the power law.

and the lifetime reads

$$\tilde{\tau}_{\mathrm{E}} \equiv \lambda^2 \tau_{\mathrm{E}} = \frac{1}{\Gamma(E_a)}. \tag{3.16}$$

It goes without saying that the evolution must then be described in terms of the rescaled time $\tilde{t} = \lambda^2 t$.

The details of the evolution were thoroughly investigated in [64] in terms of the coupling constant. We only show in Fig. 3 the most salient features of the survival probability. In the Van Hove limit the coupling constant vanishes ($\lambda \to 0$) and several things happen at once: the initial quadratic (quantum Zeno) region vanishes, the oscillations are "squeezed" out and the power law is "pushed" to infinity: only a clean exponential law is left at all times, with the right normalization factors. All this is not surprising, being implied by the Weisskopf–Wigner approximation. However, the concomitance of these features is so remarkable that one cannot but wonder at the effectiveness of this limiting procedure.

In atomic and molecular physics one gets very small deviations from the exponential law. For this reason, we displayed in Fig. 3 the survival probability by greatly exaggerating its most salient features.

4. Relativistic quantum field theory

We look now at a more complicated system. Consider the decay of a massive scalar particle Φ of mass M into two identical massive scalar particles ϕ of mass m [see Fig. 4(a)]. We shall work in a completely relativistic framework. The Lagrangian density of our model reads

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{1}{2}M^2\Phi^2 + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{2}\mu\Phi\phi^2 + \mathscr{L}_{\mathrm{CT}}. \tag{4.1}$$

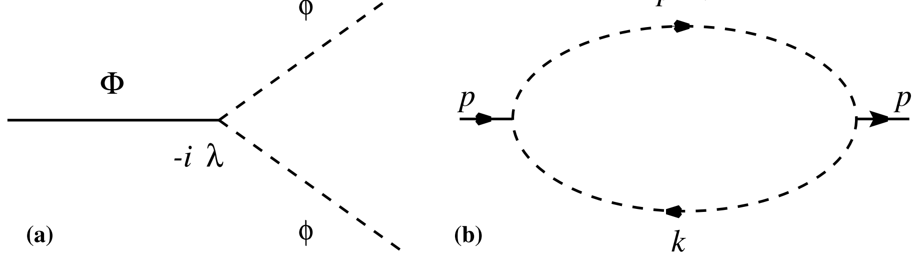

Fig. 4. (a) Decay of Φ into $\phi + \phi$. (b) Loop contribution to $\Sigma^{(2)}(p^2)$.

The Lagrangian $\mathscr{L}_{\rm CT}$ contains the counterterms absorbing the infinite but unobservable shifts between the bare parameters (M_0, m_0, λ_0) and the physical ones (M, m, λ):

$$\mathscr{L}_{\rm CT} = \frac{1}{2}\delta_Z(\partial_\mu \Phi)^2 - \frac{1}{2}\delta_M \Phi^2 + \frac{1}{2}\delta_z(\partial_\mu \phi)^2 - \frac{1}{2}\delta_m \phi^2 - \frac{\delta_\lambda}{2}\mu\Phi\phi^2, \tag{4.2}$$

with

$$\delta_Z = Z - 1, \quad \delta_z = z - 1, \quad \delta_M = M_0^2 Z - M^2, \quad \delta_m = m_0^2 z - m^2, \quad \delta_\lambda = \lambda_0 z Z^{1/2} - \lambda, \tag{4.3}$$

where Z and z are the field-strength renormalization constants ($\Phi_0 = Z^{1/2}\Phi$ and $\phi_0 = z^{1/2}\phi$) and μ is a constant with dimensions of mass.

The full two-point function

$$G(p) \equiv \int d^4x\, e^{ipx} \langle \Omega | T\Phi(x)\Phi(0) | \Omega \rangle \tag{4.4}$$

is given by Dyson's resummation of the geometric series:

$$G(p) = \frac{i}{p^2 - M^2 + i0^+} + \frac{i}{p^2 - M^2 + i0^+}(-i\Sigma(p^2))\frac{i}{p^2 - M^2 + i0^+} + \cdots = \frac{i}{p^2 - M^2 - \Sigma(p^2) + i0^+}, \tag{4.5}$$

where $\Sigma(p^2)$ is the one-particle irreducible self-energy function

$$\Sigma(p^2) = \lambda^2 \Sigma^{(2)}(p^2) + \lambda^4 \Sigma^{(4)}(p^2) + \cdots \tag{4.6}$$

By using Feynman rules it is straightforward to write down the contribution of the loop diagram in Fig. 4(b):

$$-i\Sigma^{(2)}_{\rm loop}(p^2) \equiv \frac{(-i\mu)^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2 + i0^+}\frac{i}{(p+k)^2 - m^2 + i0^+}. \tag{4.7}$$

We compute $\Sigma^{(2)}_{\rm loop}$ by using dimensional regularization. Introducing a Feynman parameter ξ and shifting the integration variable ($q = k + xp$) we get

$$\begin{aligned}
-i\Sigma^{(2)}_{\rm loop}(p^2) &= \frac{(-i\mu)^2}{2}\int_0^1 d\xi \int \frac{d^Dk}{(2\pi)^D}\frac{i^2}{[k^2 + 2\xi kp + \xi p^2 - m^2]^2} \\
&= \frac{(-i\mu)^2}{2}\int_0^1 d\xi \int \frac{d^Dq}{(2\pi)^D}\frac{i^2}{[q^2 + \xi(1-\xi)p^2 - m^2]^2}.
\end{aligned} \tag{4.8}$$

Performing a Wick rotation to Euclidean space ($q^0_{\rm E} = -iq^0$) and evaluating the momentum integral we obtain

$$\begin{aligned}
\Sigma^{(2)}_{\rm loop}(p^2) &= -\frac{\mu^2}{2}\int_0^1 d\xi \int \frac{d^Dq_{\rm E}}{(2\pi)^D}\frac{1}{[q_{\rm E}^2 - \xi(1-\xi)p^2 + m^2]^2} \\
&= -\frac{\mu^2}{2(4\pi)^2}\Gamma\left(2 - \frac{D}{2}\right)\int_0^1 d\xi \left(\frac{m^2 - \xi(1-\xi)p^2}{4\pi}\right)^{\frac{D}{2}-2},
\end{aligned} \tag{4.9}$$

which diverges as $2/(4-D)$ for $D \to 4$:

$$\Sigma^{(2)}_{\rm loop}(p^2) \sim -\frac{\mu^2}{2(4\pi)^2}\int_0^1 d\xi \left[\frac{2}{4-D} - \gamma - \log\left(\frac{m^2 - \xi(1-\xi)p^2}{4\pi}\right)\right], \tag{4.10}$$

where γ is the Euler–Mascheroni constant. Let us define the particle mass M by the condition

$$\mathrm{Re}\Sigma(M^2) = 0, \tag{4.11}$$

so that the propagator has the following behavior:

$$G(p) \sim \frac{i\bar{Z}}{p^2 - M^2 - i\bar{Z}\mathrm{Im}\Sigma(M^2)}, \quad \text{for } p^2 \to M^2, \tag{4.12}$$

with $\bar{Z}^{-1} = 1 - \Sigma'(M^2)$. Notice that $\bar{Z}$ is a finite field-strength normalization constant. Imposing the renormalization condition (4.11) one gets

$$\Sigma^{(2)}(p^2) = \Sigma^{(2)}_{\text{loop}}(p^2) - \mathrm{Re}\Sigma^{(2)}_{\text{loop}}(M^2) = \frac{\mu^2}{2(4\pi)^2}\left[\int_0^1 \mathrm{d}\xi \log\left(\frac{m^2 - \xi(1-\xi)p^2}{m^2}\right) - \mathscr{C}\right], \tag{4.13}$$

where

$$\mathscr{C} \equiv \int_0^1 \mathrm{d}\xi \log\left|\frac{m^2 - \xi(1-\xi)M^2}{m^2}\right|. \tag{4.14}$$

This is equivalent to setting $\delta_M = -\lambda^2 \mathrm{Re}\Sigma^{(2)}_{\text{loop}}(M^2)$ and $\delta_Z = 0$ in Eq. (4.3). The function $\Sigma^{(2)}(s)$ is analytic in the cut s plane and the discontinuity across the cut is $(x > 4m^2)$

$$\Sigma^{(2)}(x + \mathrm{i}0^+) - \Sigma^{(2)}(x - \mathrm{i}0^+) = -2\pi\mathrm{i}\frac{\mu^2}{2(4\pi)^2}\sqrt{1 - \frac{4m^2}{x}}. \tag{4.15}$$

Hence $\Sigma^{(2)}(s)$ can be represented by a dispersion relation. It is indeed straightforward to obtain

$$\Sigma^{(2)}(s) = \frac{\mu^2}{2(4\pi)^2}\left[s\int_{4m^2}^{\infty} \mathrm{d}s' \frac{\rho(s')}{s'(s-s')} - \mathscr{C}\right], \tag{4.16}$$

with

$$\rho(s) = \sqrt{1 - \frac{4m^2}{s}}. \tag{4.17}$$

Therefore the propagator (4.4)

$$G(s) = \frac{\mathrm{i}}{s - M^2 - \Sigma(s)} \tag{4.18}$$

has a simple pole s_{pole} near M^2 in the second Riemann sheet and for s close to s_{pole} one gets

$$G(s) \sim \frac{\mathrm{i}\mathscr{Z}}{s - s_{\text{pole}}}, \tag{4.19}$$

where

$$\mathscr{Z} = \frac{1}{1 - \Sigma'_{\text{II}}(s_{\text{pole}})} = 1 + \lambda^2 \Sigma^{(2)\prime}(M^2 + \mathrm{i}0^+) + \mathrm{O}(\lambda^4) \tag{4.20}$$

and

$$s_{\text{pole}} = M^2 + \Sigma_{\text{II}}(s_{\text{pole}}) = M^2 + \lambda^2\Sigma^{(2)}(M^2 + \mathrm{i}0^+) + \mathrm{O}(\lambda^4) = M^2 - \mathrm{i}\lambda^2 M\Gamma(M^2) + \mathrm{O}(\lambda^4), \tag{4.21}$$

with

$$\Gamma(s) \equiv \frac{\mu^2}{32\pi M}\rho(s). \tag{4.22}$$

The time evolution of the correlation function (4.4) reads

$$\mathscr{A}(t) \equiv G(t, \boldsymbol{p}) = \int \frac{\mathrm{d}E}{2\pi}\mathrm{e}^{-\mathrm{i}Et}G(p) = \mathrm{e}^{-\mathrm{i}E_p t}\int \frac{\mathrm{d}E}{2\pi}\mathrm{e}^{-\mathrm{i}Et}\frac{\mathrm{i}}{E(2E_p + E) - \Sigma(M^2 + E(2E_p + E))}, \tag{4.23}$$

where $E_p = \sqrt{\boldsymbol{p}^2 + M^2}$ is the energy of the particle Φ. By introducing the rescaled time $t = \tilde{t}/\lambda^2$ and energy $E = \lambda^2\tilde{E}$, Eq. (4.23) becomes

$$\mathscr{A}\left(\frac{\tilde{t}}{\lambda^2}\right) = \mathrm{e}^{-\mathrm{i}(E_p/\lambda^2)\tilde{t}}\int \frac{\mathrm{d}\tilde{E}}{2\pi}\mathrm{e}^{-\mathrm{i}\tilde{E}\tilde{t}}\frac{\mathrm{i}}{\tilde{E}(2E_p + \lambda^2\tilde{E}) - (1/\lambda^2)\Sigma(M^2 + \lambda^2\tilde{E}(2E_p + \lambda^2\tilde{E}))} \tag{4.24}$$

and taking Van Hove's limit, we obtain

$$\tilde{\mathscr{A}}(\tilde{t}) = \lim_{\lambda\to 0}\mathrm{e}^{+\mathrm{i}(E_p/\lambda^2)\tilde{t}}\mathscr{A}\left(\frac{\tilde{t}}{\lambda^2}\right) = \int \frac{\mathrm{d}\tilde{E}}{2\pi}\mathrm{e}^{-\mathrm{i}\tilde{E}\tilde{t}}\tilde{G}(\tilde{E}) \tag{4.25}$$

with the limiting propagator

$$\tilde{G}(\tilde{E}) = \lim_{\lambda\to 0} \frac{\mathrm{i}}{\tilde{E}(2E_p + \lambda^2\tilde{E}) - (1/\lambda^2)\Sigma(M^2 + \lambda^2\tilde{E}(2E_p + \lambda^2\tilde{E}))} = \frac{\mathrm{i}}{2E_p\tilde{E} - \Sigma^{(2)}(M^2 + \mathrm{i}0^+)}$$
$$= \frac{1}{2E_p}\frac{\mathrm{i}}{\tilde{E} + \mathrm{i}(M/E_p)(\Gamma/2)}. \tag{4.26}$$

Therefore the time evolution (4.25) becomes

$$\tilde{\mathscr{A}}(\tilde{t}) = \frac{1}{2E_p}\exp\left(-\frac{M}{E_p}\frac{\Gamma}{2}\tilde{t}\right) = \frac{1}{2E_p}\exp\left(-\frac{\tilde{t}}{2\tau_p}\right), \tag{4.27}$$

and the particle decays exponentially with a mean lifetime

$$\tau_p = \frac{E_p}{M}\Gamma^{-1} = \frac{\Gamma^{-1}}{\sqrt{1-v^2}}, \tag{4.28}$$

which has the proper relativistic time dilatation factor. This result is remarkably simple, for the model considered.

5. Conclusions and comments

The time evolution obtained in Van Hove's limit is always purely exponential: the quantum dynamics is governed by a master equation and by dynamical semigroups. However, it is obvious that the very procedure of time rescaling hides, in some sense, the problem of state preparation.

In all the decay processes considered in this paper, an initial pure state is considered at time "$t = 0$". What is the meaning of $t = 0$? This question is often dismissed, in particular in quantum field theory, where all "relevant" physical quantities are constructed from the S-matrix. There are however interesting examples in which the issue of state preparation is discussed, both in the context of semigroups [75] and scattering processes [76,77]. It is difficult not to wonder at the concept of initial time, in particular when one considers fundamental processes like particle creation in quantum field theory. Think again of the relativistic model analyzed in the previous section, as well as of other examples recently considered in the literature [51–54,78]. Any classical picture of a decay process is necessarily mind-boggling. The preparation of an initial wave function (or initial state of a quantum field) is an inherently quantum mechanical process, certainly not an easy one to conceive.

This problem is difficult to tackle. Nico Van Kampen, after refereeing one of our papers, put forward the following interesting and thought-provoking comment [79]: "As to your suggestion for preparing an initial pure state, there is no objection to it, from the mathematical viewpoint. But we are doing physics. Your construction is of the same caliber as the construction in statistical mechanics of those time reversed states whose entropies increase. The answer there too was that they are permissible from the mathematical point of view, but that they are tremendously improbable. Remember also the work by Wheeler and Feynman, which argued that there can be no coherence in incoming waves owing to the absorption property of the universe. My feeling is that there is no real difficulty or paradox, but only the task to formulate precisely what is intuitively clear."

We agree. Although an initial pure state like those considered in this paper are *mathematically* easy to conceive, their *physical* construction is prohibitive. Does this mean that nonexponential decays in atomic or elementary particle physics are extremely improbable to observe because "unstable" quantum systems are practically always created in some sort of mixed states, whose statistical features justify time coarse-graining procedures like Van Hove's? This is an interesting question, which goes to the very core of the notion of irreversibility. A physical answer is needed.

References

[1] Van Hove L. Physica 1955;21:517.
[2] Fermi E. Rev Mod Phys 1932;4:87.
[3] Fermi E. Nuclear physics. Chicago: University of Chicago; 1950. p. 136, 148 [see also in: Segré E, editor. Notes on quantum mechanics. A course given at the University of Chicago in 1954. Lecture 23. Chicago: University of Chicago; 1960].
[4] Weisskopf V, Wigner EP. Z Phys 1930;63:54.
[5] Weisskopf V, Wigner EP. Z Phys 1930;65:18.
[6] Breit G, Wigner EP. Phys Rev 1936;49:519.

[7] Beskow A, Nilsson J. Ark Fys 1967;34:561.
[8] Khalfin LA. Zh Eksp Teor Fiz Pisma Red 1968;8:106 [JETP Lett 1968;8:65].
[9] Fonda L, Ghirardi GC, Rimini A, Weber T. Nuovo Cimento A 1973;15:689.
[10] Fonda L, Ghirardi GC, Rimini A, Weber T. Nuovo Cimento A 1973;18:805.
[11] Misra B, Sudarshan ECG. J Math Phys 1977;18:756.
[12] Peres A. Am J Phys 1980;48:931.
[13] Peres A. Ann Phys 1980;129:33.
[14] Mandelstam L, Tamm I. J Phys 1945;9:249.
[15] Fock V, Krylov N. J Phys 1947;11:112.
[16] Hellund EJ. Phys Rev 1953;89:919.
[17] Namiki M, Mugibayashi N. Prog Theor Phys 1953;10:474.
[18] Khalfin LA. Dokl Acad Nauk USSR 1957;115:277 [Sov Phys Dokl 1957;2:340].
[19] Khalfin LA. Zh Eksp Teor Fiz 1958;33:1371 [Sov Phys JETP 1958;6:1053].
[20] Sakurai JJ. Modern quantum mechanics. Reading (MA): Addison-Wesley; 1994.
[21] Messiah A. Quantum mechanics. New York: Interscience; 1961.
[22] Brown LS. Quantum field theory. Bristol: Cambridge University Press; 1992.
[23] Nakazato H, Namiki M, Pascazio S. Int J Mod Phys B 1996;10:247.
[24] Cook RJ. Phys Scr T 1988;21:49.
[25] Itano WH, Heinzen DJ, Bollinger JJ, Wineland DJ. Phys Rev A 1990;41:2295.
[26] Petrosky T, Tasaki S, Prigogine I. Phys Lett A 1990;151:109.
[27] Petrosky T, Tasaki S, Prigogine I. Physica A 1991;170:306.
[28] Peres A, Ron A. Phys Rev A 1990;42:5720.
[29] Ballentine LE. Phys Rev A 1991;43:5165.
[30] Itano WH, Heinzen DJ, Bollinger JJ, Wineland DJ. Phys Rev A 1991;43:5168.
[31] Frerichs V, Schenzle A. In: Black TD, Nieto MM, Pilloff HS, Scully MO, Sinclair RM, editors. Foundations of quantum mechanics. Singapore: World Scientific; 1992.
[32] Inagaki S, Namiki M, Tajiri T. Phys Lett A 1992;166:5.
[33] Home D, Whitaker MAB. J Phys A 1992;25:657.
[34] Home D, Whitaker MAB. Phys Lett A 1993;173:327.
[35] Pascazio S, Namiki M, Badurek G, Rauch H. Phys Lett A 1993;179:155.
[36] Blanchard Ph, Jadczyk A. Phys Lett A 1993;183:272.
[37] Altenmuller TP, Schenzle A. Phys Rev A 1994;49:2016.
[38] Pascazio S, Namiki M. Phys Rev A 1994;50:4582.
[39] Schulman LS, Ranfagni A, Mugnai D. Phys Scr 1994;49:536.
[40] Berry M. In: Greenberger DM, Zeilinger A, editors. Fundamental problems in quantum theory. Ann NY Acad Sci 1995;755:303.
[41] Nakazato H, Namiki M, Pascazio S, Rauch H. Phys Lett A 1995;199:27.
[42] Beige A, Hegerfeldt G. Phys Rev A 1996;53:53.
[43] Plenio MB, Knight PL, Thompson RC. Opt Commun 1996;123:278.
[44] Luis A, Periña J. Phys Rev Lett 1996;76:4340.
[45] Nakazato H, Namiki M, Pascazio S, Rauch H. Phys Lett A 1996;217:203.
[46] Schulman LS. J Phys A 1997;30:L293.
[47] Schulman LS. Phys Rev A 1998;57:1509.
[48] Thun K, Periña J. Phys Lett A 1998;249:363.
[49] Hradil Z, Nakazato H, Namiki M, Pascazio S, Rauch H. Phys Lett A 1998;239:333.
[50] Facchi P, Pascazio S. Phys Lett A 1998;241:139.
[51] Bernardini C, Maiani L, Testa M. Phys Rev Lett 1993;71:2687.
[52] Maiani L, Testa M. Ann Phys NY 1998;263:353.
[53] Joichi I, Matsumoto Sh, Yoshimura M. Phys Rev D 1998;58:043507, 045004.
[54] Alvarez-Estrada RF, Sánchez-Gómez JL. Phys Lett A 1999;253:252.
[55] Kofman AG, Kurizki G. Acta Phys Slovaca 1999;49:541.
[56] Kofman AG, Kurizki G. Phys Rev A 1996;54:R3750.
[57] Harel G, Kofman AG, Kozhekin A, Kurizki G. Opt Express 1998;2:355.
[58] Wilkinson SR, et al. Nature 1997;387:575.
[59] van Kampen NG. Stochastic processes in physics and chemistry. Amsterdam: North-Holland; 1992.
[60] Gardiner CW. Handbook of stochastic methods. Berlin: Springer; 1990.
[61] Bohm A, Doebner HD, Kielanowski P, editors. Irreversibility and causality. Semigroups and rigged Hilbert spaces. In: Lecture notes in physics. Berlin: Springer; 1998.
[62] Accardi L, Kozyrev SV, Volovich IV. Phys Rev A 1997;56:2557.
[63] Accardi L, Lu YG, Volovich IV. Quantum theory and its stochastic limit. London: Springer Verlag [in press].
[64] Facchi P, Pascazio S. Physica A 1999;271:133.
[65] Heitler W. Proc Cambridge Philos Soc 1936;32:112.

[66] Akhiezer AI, Berestetskii VB. Quantum electrodynamics. New York: Interscience; 1965.
[67] Knight PL, Milonni PW. Phys Lett A 1976;56:275.
[68] Davidovich L, Nussenzveig HM. In: Barut AO, editor. Foundations of radiation theory and quantum electrodynamics. New York: Plenum; 1980. p. 83.
[69] Hillery M. Phys Rev A 1981;24:933.
[70] Seke J, Herfort W. Phys Rev A 1989;40:1926.
[71] Seke J. Phys Rev A 1992;45:542.
[72] Enaki NA. Zh Eksp Teor Fiz 1996;109:1130 [Sov Phys JETP 1996;82:607].
[73] Tamm I. J Phys Moscow 1945;9:449.
[74] Dancoff S. Phys Rev 1950;78:382.
[75] Bohm A. Phys Rev A 1999;60:861.
[76] Petrosky T, Ordonez G, Prigogine I. Liouville-space formulation of dressed quantum unstable states. Preprint, 1999.
[77] Petrosky T, Ordonez G, Prigogine I. Liouville-space formulation of quantum transitions. Preprint, 1999.
[78] Antoniou I, Gadella M, Prigogine I, Pronko GP. J Math Phys 1998;39:2995.
[79] van Kampen N. Private communication.