Chapter 10

Potentials and Fields

10.1 The Potential Formulation

10.1.1 Scalar and Vector Potentials

In this chapter we ask how the sources (ρ and $\mathbf{J}$) generate electric and magnetic fields; in other words, we seek the *general* solution to Maxwell's equations,

$$\left.\begin{array}{llll} \text{(i)} & \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho, & \text{(iii)} & \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}, \\ \text{(ii)} & \nabla \cdot \mathbf{B} = 0, & \text{(iv)} & \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}. \end{array}\right\} \tag{10.1}$$

Given $\rho(\mathbf{r}, t)$ and $\mathbf{J}(\mathbf{r}, t)$, what are the fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$? In the static case Coulomb's law and the Biot-Savart law provide the answer. What we're looking for, then, is the generalization of those laws to time-dependent configurations.

This is not an easy problem, and it pays to begin by representing the fields in terms of potentials. In electrostatics $\nabla \times \mathbf{E} = 0$ allowed us to write $\mathbf{E}$ as the gradient of a scalar potential: $\mathbf{E} = -\nabla V$. In electro*dynamics* this is no longer possible, because the curl of $\mathbf{E}$ is nonzero. But $\mathbf{B}$ remains divergenceless, so we can still write

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A},} \tag{10.2}$$

as in magnetostatics. Putting this into Faraday's law (iii) yields

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}),$$

or

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0.$$

Here is a quantity, unlike $\mathbf{E}$ alone, whose curl *does* vanish; it can therefore be written as the gradient of a scalar:

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V.$$

In terms of V and $\mathbf{A}$, then,

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}.} \tag{10.3}$$

This reduces to the old form, of course, when $\mathbf{A}$ is constant.

The potential representation (Eqs. 10.2 and 10.3) automatically fulfills the two homogeneous Maxwell equations, (ii) and (iii). How about Gauss's law (i) and the Ampère/Maxwell law (iv)? Putting Eq. 10.3 into (i), we find that

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{1}{\epsilon_0}\rho; \tag{10.4}$$

this replaces Poisson's equation (to which it reduces in the static case). Putting Eqs. 10.2 and 10.3 into (iv) yields

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J} - \mu_0 \epsilon_0 \nabla \left(\frac{\partial V}{\partial t}\right) - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2},$$

or, using the vector identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$, and rearranging the terms a bit:

$$\left(\nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) - \nabla \left(\nabla \cdot \mathbf{A} + \mu_0 \epsilon_0 \frac{\partial V}{\partial t}\right) = -\mu_0 \mathbf{J}. \tag{10.5}$$

Equations 10.4 and 10.5 contain all the information in Maxwell's equations.

Example 10.1 NO

Find the charge and current distributions that would give rise to the potentials

$$V = 0, \quad \mathbf{A} = \begin{cases} \dfrac{\mu_0 k}{4c}(ct - |x|)^2 \,\hat{\mathbf{z}}, & \text{for } |x| < ct, \\ 0, & \text{for } |x| > ct, \end{cases}$$

where k is a constant, and $c = 1/\sqrt{\epsilon_0 \mu_0}$.

Solution: First we'll determine the electric and magnetic fields, using Eqs. 10.2 and 10.3:

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 k}{2}(ct - |x|)\,\hat{\mathbf{z}},$$

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{\mu_0 k}{4c}\frac{\partial}{\partial x}(ct - |x|)^2\,\hat{\mathbf{y}} = \pm\frac{\mu_0 k}{2c}(ct - |x|)\,\hat{\mathbf{y}},$$

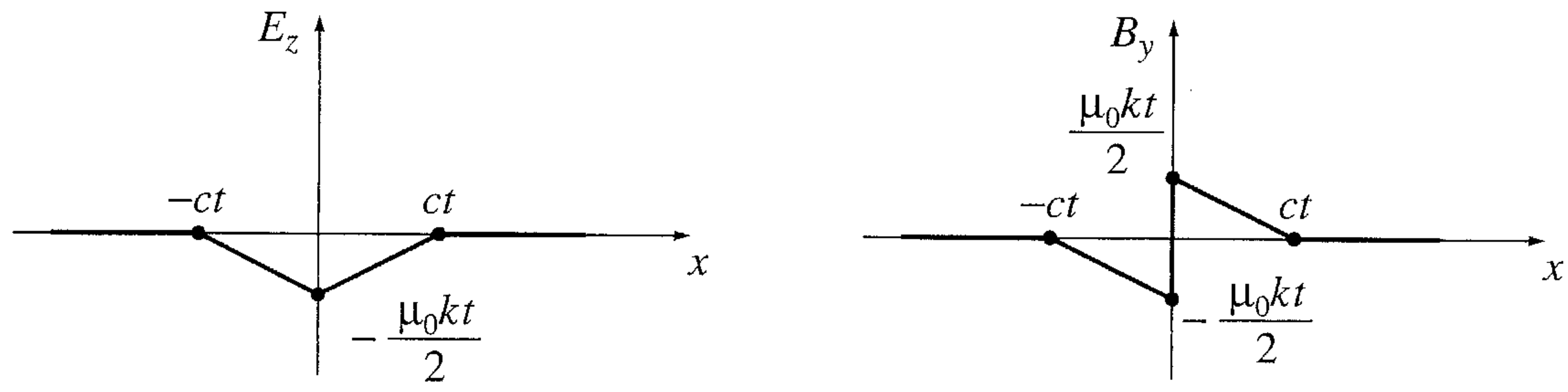

Figure 10.1

(plus for $x > 0$, minus for $x < 0$). These are for $|x| < ct$; when $|x| > ct$, $\mathbf{E} = \mathbf{B} = 0$ (Fig.10.1). Calculating every derivative in sight, I find

$$\nabla \cdot \mathbf{E} = 0; \quad \nabla \cdot \mathbf{B} = 0; \quad \nabla \times \mathbf{E} = \mp\frac{\mu_0 k}{2}\hat{\mathbf{y}}; \quad \nabla \times \mathbf{B} = -\frac{\mu_0 k}{2c}\hat{\mathbf{z}};$$

$$\frac{\partial \mathbf{E}}{\partial t} = -\frac{\mu_0 kc}{2}\hat{\mathbf{z}}; \quad \frac{\partial \mathbf{B}}{\partial t} = \pm\frac{\mu_0 k}{2}\hat{\mathbf{y}}.$$

As you can easily check, Maxwell's equations are all satisfied, with ρ and $\mathbf{J}$ both *zero*. Notice, however, that $\mathbf{B}$ has a discontinuity at $x = 0$, and this signals the presence of a surface current $\mathbf{K}$ in the yz plane; boundary condition (iv) in Eq. 7.63 gives

$$kt\,\hat{\mathbf{y}} = \mathbf{K} \times \hat{\mathbf{x}},$$

and hence

$$\mathbf{K} = kt\,\hat{\mathbf{z}}.$$

Evidently we have here a uniform surface current flowing in the z direction over the plane $x = 0$, which starts up at $t = 0$, and increases in proportion to t. Notice that the news travels out (in both directions) at the speed of light: for points $|x| > ct$ the message (that current is now flowing) has not yet arrived, so the fields are zero.

Problem 10.1 Show that the differential equations for V and $\mathbf{A}$ (Eqs. 10.4 and 10.5) can be written in the more symmetrical form

$$\left.\begin{aligned} \Box^2 V + \frac{\partial L}{\partial t} &= -\frac{1}{\epsilon_0}\rho, \\ \Box^2 \mathbf{A} - \nabla L &= -\mu_0 \mathbf{J}, \end{aligned}\right\} \qquad (10.6)$$

where

$$\Box^2 \equiv \nabla^2 - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2} \quad \text{and} \quad L \equiv \nabla \cdot \mathbf{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}.$$

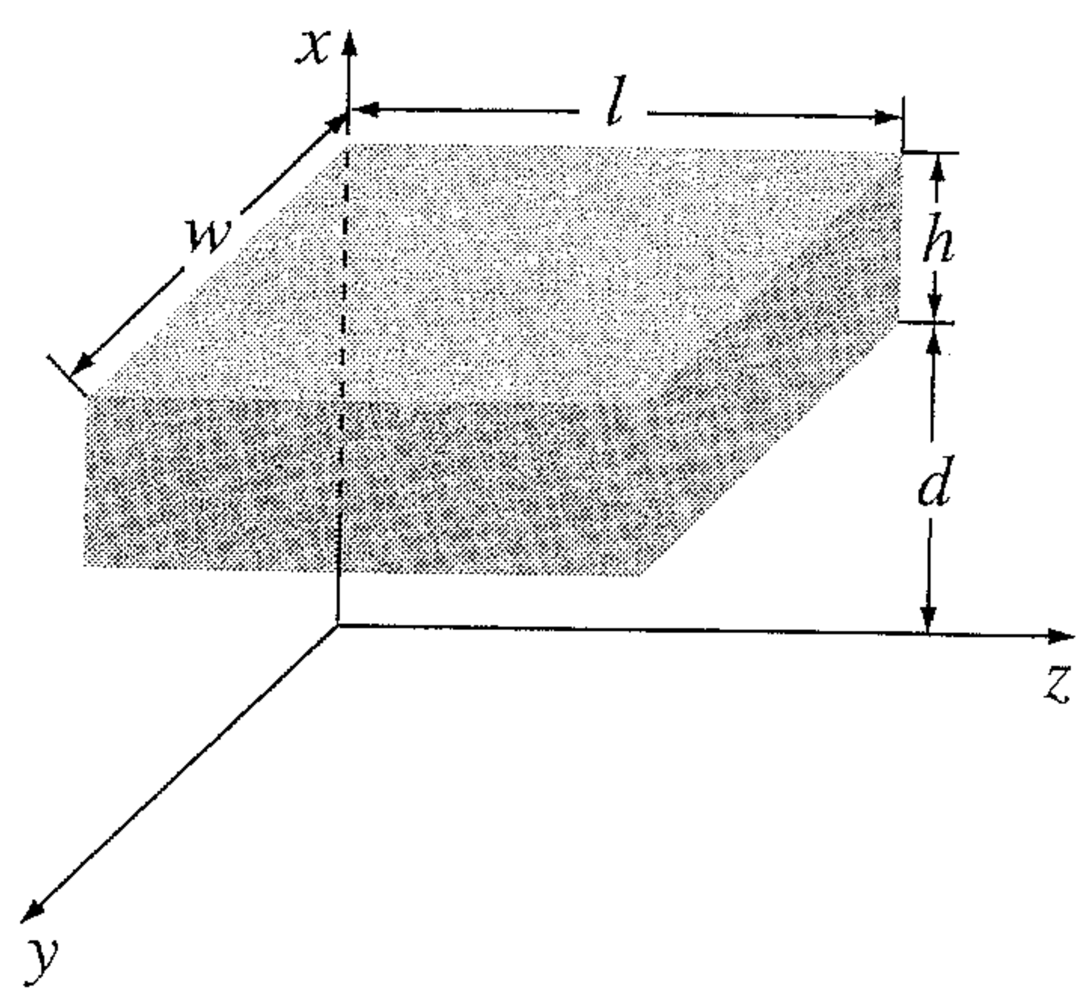

Figure 10.2

Problem 10.2 For the configuration in Ex. 10.1, consider a rectangular box of length l, width w, and height h, situated a distance d above the yz plane (Fig. 10.2).

(a) Find the energy in the box at time $t_1 = d/c$, and at $t_2 = (d+h)/c$.

(b) Find the Poynting vector, and determine the energy per unit time flowing into the box during the interval $t_1 < t < t_2$.

(c) Integrate the result in (b) from t_1 to t_2 and confirm that the increase in energy (part (a)) equals the net influx.

10.1.2 Gauge Transformations

Equations 10.4 and 10.5 are *ugly*, and you might be inclined at this stage to abandon the potential formulation altogether. However, we *have* succeeded in reducing six problems—finding **E** and **B** (three components each)—down to four: V (one component) and **A** (three more). Moreover, Eqs. 10.2 and 10.3 do not uniquely define the potentials; we are free to impose extra conditions on V and **A**, as long as nothing happens to **E** and **B**. Let's work out precisely what this **gauge freedom** entails. Suppose we have two sets of potentials, $(V, \mathbf{A})$ and $(V', \mathbf{A}')$, which correspond to the *same* electric and magnetic fields. By how much can they differ? Write

$$\mathbf{A}' = \mathbf{A} + \boldsymbol{\alpha} \quad \text{and} \quad V' = V + \beta.$$

Since the two **A**'s give the same **B**, their curls must be equal, and hence

$$\nabla \times \boldsymbol{\alpha} = 0.$$

We can therefore write $\boldsymbol{\alpha}$ as the gradient of some scalar:

$$\boldsymbol{\alpha} = \nabla \lambda.$$

The two potentials also give the same **E**, so

$$\nabla\beta + \frac{\partial\boldsymbol{\alpha}}{\partial t} = 0,$$

or

$$\nabla\left(\beta + \frac{\partial\lambda}{\partial t}\right) = 0.$$

The term in parentheses is therefore independent of position (it could, however, depend on time); call it $k(t)$:

$$\beta = -\frac{\partial\lambda}{\partial t} + k(t).$$

Actually, we might as well absorb $k(t)$ into λ, defining a new λ by adding $\int_0^t k(t')dt'$ to the old one. This will not affect the gradient of λ; it just adds $k(t)$ to $\partial\lambda/\partial t$. It follows that

$$\left.\begin{aligned} \mathbf{A}' &= \mathbf{A} + \nabla\lambda, \\ V' &= V - \frac{\partial\lambda}{\partial t}. \end{aligned}\right\} \tag{10.7}$$

Conclusion: For any old scalar function λ, we can with impunity add $\nabla\lambda$ to **A**, provided we simultaneously subtract $\partial\lambda/\partial t$ from V. None of this will affect the physical quantities **E** and **B**. Such changes in V and **A** are called **gauge transformations**. They can be exploited to adjust the divergence of **A**, with a view to simplifying the "ugly" equations 10.4 and 10.5. In magnetostatics, it was best to choose $\nabla \cdot \mathbf{A} = 0$ (Eq. 5.61); in electrodynamics the situation is not so clear cut, and the most convenient gauge depends to some extent on the problem at hand. There are many famous gauges in the literature; I'll show you the two most popular ones.

NO

Problem 10.3 Find the fields, and the charge and current distributions, corresponding to

$$V(\mathbf{r}, t) = 0, \quad \mathbf{A}(\mathbf{r}, t) = -\frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\,\hat{\mathbf{r}}.$$

Problem 10.4 Suppose $V = 0$ and $\mathbf{A} = A_0 \sin(kx - \omega t)\,\hat{\mathbf{y}}$, where A_0, ω, and k are constants. Find **E** and **B**, and check that they satisfy Maxwell's equations in vacuum. What condition must you impose on ω and k?

Problem 10.5 Use the gauge function $\lambda = -(1/4\pi\epsilon_0)(qt/r)$ to transform the potentials in Prob. 10.3, and comment on the result.

10.1.3 Coulomb Gauge and Lorentz Gauge

The Coulomb Gauge. As in magnetostatics, we pick

$$\nabla \cdot \mathbf{A} = 0. \tag{10.8}$$

With this, Eq. 10.4 becomes

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho. \tag{10.9}$$

This is Poisson's equation, and we already know how to solve it: setting $V = 0$ at infinity,

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t)}{\mathscr{r}}\, d\tau'. \tag{10.10}$$

Don't be fooled, though—unlike electrostatics, V by itself doesn't tell you $\mathbf{E}$; you have to know $\mathbf{A}$ as well (Eq. 10.3).

There is a peculiar thing about the scalar potential in the Coulomb gauge: it is determined by the distribution of charge *right now*. If I move an electron in my laboratory, the potential V on the moon immediately records this change. That sounds particularly odd in the light of special relativity, which allows no message to travel faster than the speed of light. The point is that V *by itself* is not a physically measurable quantity—all the man in the moon can measure is $\mathbf{E}$, and that involves $\mathbf{A}$ as well. Somehow it is built into the vector potential, in the Coulomb gauge, that whereas V instantaneously reflects all changes in ρ, the combination $-\nabla V - (\partial\mathbf{A}/\partial t)$ does *not*; $\mathbf{E}$ will change only after sufficient time has elapsed for the "news" to arrive.[1]

The *advantage* of the Coulomb gauge is that the *scalar* potential is particularly simple to calculate; the *disadvantage* (apart from the acausal appearance of V) is that $\mathbf{A}$ is particularly *difficult* to calculate. The differential equation for $\mathbf{A}$ (10.5) in the Coulomb gauge reads

$$\nabla^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J} + \mu_0\epsilon_0\nabla\left(\frac{\partial V}{\partial t}\right). \tag{10.11}$$

The Lorentz gauge. In the Lorentz gauge we pick

$$\boxed{\nabla \cdot \mathbf{A} = -\mu_0\epsilon_0\frac{\partial V}{\partial t}.} \tag{10.12}$$

This is designed to eliminate the middle term in Eq. 10.5 (in the language of Prob. 10.1, it sets $L = 0$). With this

$$\nabla^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J}. \tag{10.13}$$

Meanwhile, the differential equation for V, (10.4), becomes

$$\nabla^2 V - \mu_0\epsilon_0\frac{\partial^2 V}{\partial t^2} = -\frac{1}{\epsilon_0}\rho. \tag{10.14}$$

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[1] See O. L. Brill and B. Goodman. *Am. J. Phys.* **35**, 832 (1967).

The virtue of the Lorentz gauge is that it treats V and $\mathbf{A}$ on an equal footing: the same differential operator

$$\boxed{\nabla^2 - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2} \equiv \Box^2,} \tag{10.15}$$

(called the **d'Alembertian**) occurs in both equations:

$$\boxed{\begin{aligned} &\text{(i)} \quad \Box^2 V = -\frac{1}{\epsilon_0}\rho, \\ &\text{(ii)} \quad \Box^2 \mathbf{A} = -\mu_0\mathbf{J}. \end{aligned}} \tag{10.16}$$

This democratic treatment of V and $\mathbf{A}$ is particularly nice in the context of special relativity, where the d'Alembertian is the natural generalization of the Laplacian, and Eqs. 10.16 can be regarded as four-dimensional versions of Poisson's equation. (In this same spirit the wave equation, for propagation speed c, $\Box^2 f = 0$, might be regarded as the four-dimensional version of Laplace's equation.) In the Lorentz gauge V and $\mathbf{A}$ satisfy the **inhomogeneous wave equation**, with a "source" term (in place of zero) on the right. From now on I shall use the Lorentz gauge exclusively, and the whole of electrodynamics reduces to the problem of *solving the inhomogeneous wave equation for specified sources*. That's my project for the next section.

Problem 10.6 Which of the potentials in Ex. 10.1, Prob. 10.3, and Prob. 10.4 are in the Coulomb gauge? Which are in the Lorentz gauge? (Notice that these gauges are not mutually exclusive.)

Problem 10.7 In Chapter 5, I showed that it is always possible to pick a vector potential whose divergence is zero (Coulomb gauge). Show that it is always possible to choose $\nabla \cdot \mathbf{A} = -\mu_0\epsilon_0(\partial V/\partial t)$, as required for the Lorentz gauge, assuming you know how to solve equations of the form 10.16. Is it always possible to pick $V = 0$? How about $\mathbf{A} = 0$?

10.2 Continuous Distributions

10.2.1 Retarded Potentials

In the static case, Eqs. 10.16 reduce to (four copies of) Poisson's equation,

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho, \quad \nabla^2 \mathbf{A} = -\mu_0\mathbf{J},$$

with the familiar solutions

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}}\,d\tau', \qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\,d\tau', \tag{10.17}$$

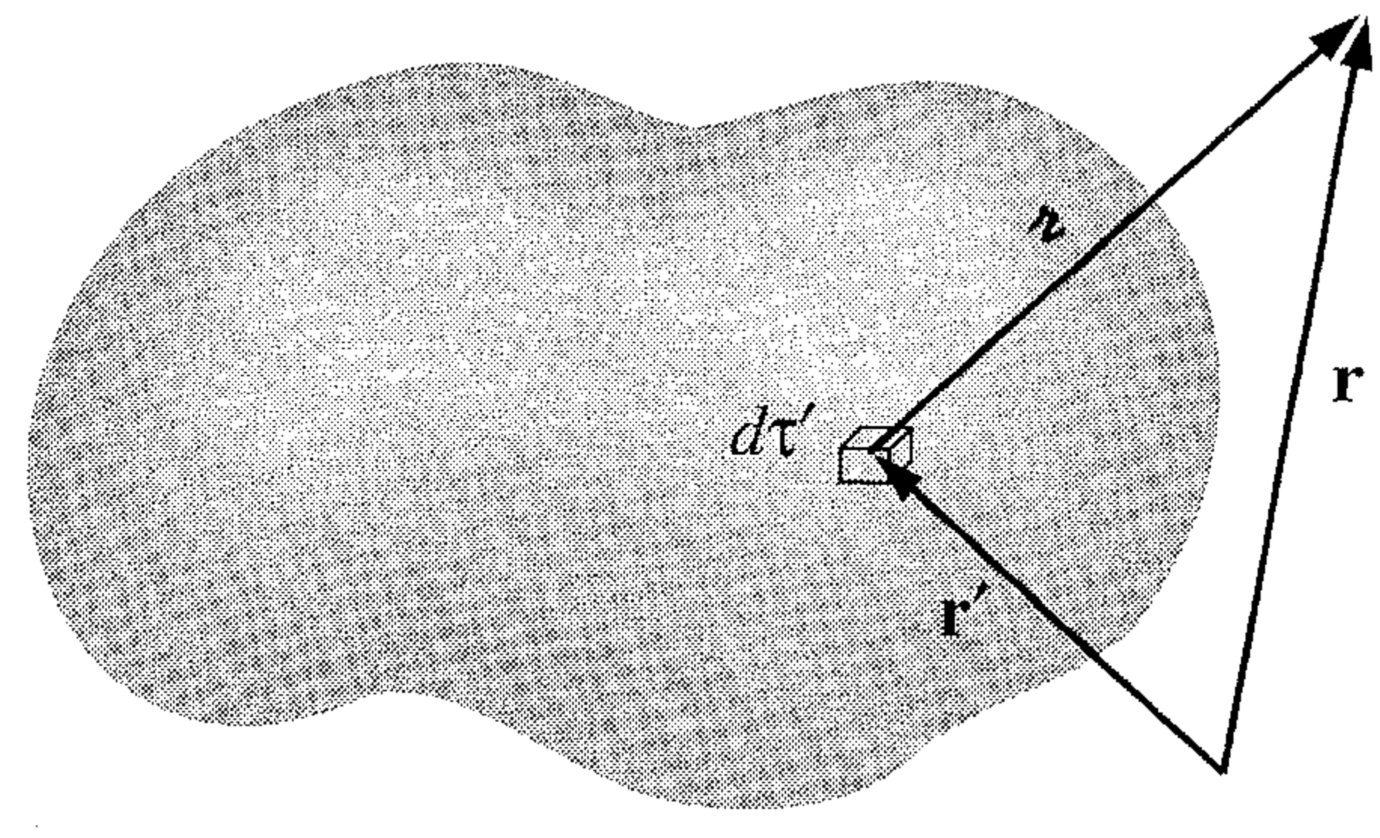

Figure 10.3

where $\mathscr{r}$, as always, is the distance from the source point $\mathbf{r}'$ to the field point $\mathbf{r}$ (Fig. 10.3). Now, electromagnetic "news" travels at the speed of light. In the *nonstatic* case, therefore, it's not the status of the source *right now* that matters, but rather its condition at some earlier time t_r (called the **retarded time**) when the "message" left. Since this message must travel a distance $\mathscr{r}$, the delay is $\mathscr{r}/c$:

$$\boxed{t_r \equiv t - \frac{\mathscr{r}}{c}.} \tag{10.18}$$

The natural generalization of Eq. 10.17 for nonstatic sources is therefore

$$\boxed{V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}}\,d\tau', \qquad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}', t_r)}{\mathscr{r}}\,d\tau'.} \tag{10.19}$$

Here $\rho(\mathbf{r}', t_r)$ is the charge density that prevailed at point $\mathbf{r}'$ at the retarded time t_r. Because the integrands are evaluated at the retarded time, these are called **retarded potentials**. (I speak of "the" retarded time, but of course the most distant parts of the charge distribution have earlier retarded times than nearby ones. It's just like the night sky: The light we see now left each star at the retarded time corresponding to that star's distance from the earth.) Note that the retarded potentials reduce properly to Eq. 10.17 in the static case, for which ρ and $\mathbf{J}$ are independent of time.

Well, that all sounds *reasonable*—and surprisingly simple. But are we sure it's *right*? I didn't actually *derive* these formulas for V and $\mathbf{A}$; all I did was invoke a heuristic argument ("electromagnetic news travels at the speed of light") to make them seem *plausible*. To *prove* them, I must show that they satisfy the inhomogeneous wave equation (10.16) and meet the Lorentz condition (10.12). In case you think I'm being fussy, let me warn you that if you apply the same argument to the *fields* you'll get entirely the *wrong* answer:

$$\mathbf{E}(\mathbf{r}, t) \neq \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,d\tau', \quad \mathbf{B}(\mathbf{r}, t) \neq \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}', t_r)\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau',$$

as you would expect if the same "logic" worked for Coulomb's law and the Biot-Savart law. Let's stop and check, then, that the retarded scalar potential satisfies Eq. 10.16; essentially the same argument would serve for the vector potential.[2] I shall leave it for you (Prob. 10.8) to check that the retarded potentials obey the Lorentz condition.

In calculating the Laplacian of $V(\mathbf{r}, t)$, the crucial point to notice is that the integrand (in Eq. 10.19) depends on $\mathbf{r}$ in *two* places: *explicitly*, in the denominator ($\imath = |\mathbf{r} - \mathbf{r}'|$), and *implicitly*, through $t_r = t - \imath/c$, in the numerator. Thus

$$\nabla V = \frac{1}{4\pi\epsilon_0}\int\left[(\nabla\rho)\frac{1}{\imath} + \rho\nabla\left(\frac{1}{\imath}\right)\right]d\tau', \tag{10.20}$$

and

$$\nabla\rho = \dot{\rho}\nabla t_r = -\frac{1}{c}\dot{\rho}\nabla\imath \tag{10.21}$$

(the dot denotes differentiation with respect to time).[3] Now $\nabla\imath = \hat{\boldsymbol{\imath}}$ and $\nabla(1/\imath) = -\hat{\boldsymbol{\imath}}/\imath^2$ (Prob. 1.13), so

$$\nabla V = \frac{1}{4\pi\epsilon_0}\int\left[-\frac{\dot{\rho}}{c}\frac{\hat{\boldsymbol{\imath}}}{\imath} - \rho\frac{\hat{\boldsymbol{\imath}}}{\imath^2}\right]d\tau'. \tag{10.22}$$

Taking the divergence,

$$\nabla^2 V = \frac{1}{4\pi\epsilon_0}\int\left\{-\frac{1}{c}\left[\frac{\hat{\boldsymbol{\imath}}}{\imath}\cdot(\nabla\dot{\rho}) + \dot{\rho}\nabla\cdot\left(\frac{\hat{\boldsymbol{\imath}}}{\imath}\right)\right] - \left[\frac{\hat{\boldsymbol{\imath}}}{\imath^2}\cdot(\nabla\rho) + \rho\nabla\cdot\left(\frac{\hat{\boldsymbol{\imath}}}{\imath^2}\right)\right]\right\}d\tau'.$$

But

$$\nabla\dot{\rho} = -\frac{1}{c}\ddot{\rho}\nabla\imath = -\frac{1}{c}\ddot{\rho}\,\hat{\boldsymbol{\imath}},$$

as in Eq. 10.21, and

$$\nabla\cdot\left(\frac{\hat{\boldsymbol{\imath}}}{\imath}\right) = \frac{1}{\imath^2}$$

(Prob. 1.62), whereas

$$\nabla\cdot\left(\frac{\hat{\boldsymbol{\imath}}}{\imath^2}\right) = 4\pi\delta^3(\boldsymbol{\imath})$$

(Eq. 1.100). So

$$\nabla^2 V = \frac{1}{4\pi\epsilon_0}\int\left[\frac{1}{c^2}\frac{\ddot{\rho}}{\imath} - 4\pi\rho\delta^3(\boldsymbol{\imath})\right]d\tau' = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} - \frac{1}{\epsilon_0}\rho(\mathbf{r}, t),$$

confirming that the retarded potential (10.19) satisfies the inhomogeneous wave equation (10.16). qed

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[2] I'll give you the straightforward but cumbersome proof; for a clever indirect argument see M. A. Heald and J. B. Marion, *Classical Electromagnetic Radiation*, 3d ed., Sect. 8.1 (Orlando, FL: Saunders (1995)).

[3] Note that $\partial/\partial t_r = \partial/\partial t$, since $t_r = t - \imath/c$ and $\imath$ is independent of t.

Incidentally, this proof applies equally well to the **advanced potentials**,

$$V_a(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t_a)}{\mathscr{r}}\,d\tau', \qquad \mathbf{A}_a(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}', t_a)}{\mathscr{r}}\,d\tau', \tag{10.23}$$

in which the charge and the current densities are evaluated at the **advanced time**

$$t_a \equiv t + \frac{\mathscr{r}}{c}. \tag{10.24}$$

A few signs are changed, but the final result is unaffected. Although the advanced potentials are entirely consistent with Maxwell's equations, they violate the most sacred tenet in all of physics: the principle of **causality**. They suggest that the potentials *now* depend on what the charge and the current distribution *will* be at some time in the future—the effect, in other words, precedes the cause. Although the advanced potentials are of some theoretical interest, they have no direct physical significance.[4]

Example 10.2

An infinite straight wire carries the current

$$I(t) = \begin{cases} 0, & \text{for } t \le 0, \\ I_0, & \text{for } t > 0. \end{cases}$$

That is, a constant current I_0 is turned on abruptly at $t = 0$. Find the resulting electric and magnetic fields.

Solution: The wire is presumably electrically neutral, so the scalar potential is zero. Let the wire lie along the z axis (Fig. 10.4); the retarded vector potential at point P is

$$\mathbf{A}(s, t) = \frac{\mu_0}{4\pi}\,\hat{\mathbf{z}}\int_{-\infty}^{\infty}\frac{I(t_r)}{\mathscr{r}}\,dz.$$

For $t < s/c$, the "news" has not yet reached P, and the potential is zero. For $t > s/c$, only the segment

$$|z| \le \sqrt{(ct)^2 - s^2} \tag{10.25}$$

contributes (outside this range t_r is negative, so $I(t_r) = 0$); thus

$$\begin{aligned} \mathbf{A}(s, t) &= \left(\frac{\mu_0 I_0}{4\pi}\,\hat{\mathbf{z}}\right) 2\int_0^{\sqrt{(ct)^2-s^2}} \frac{dz}{\sqrt{s^2+z^2}} \\ &= \frac{\mu_0 I_0}{2\pi}\,\hat{\mathbf{z}}\,\ln\left(\sqrt{s^2+z^2}+z\right)\Big|_0^{\sqrt{(ct)^2-s^2}} = \frac{\mu_0 I_0}{2\pi}\ln\left(\frac{ct+\sqrt{(ct)^2-s^2}}{s}\right)\hat{\mathbf{z}}. \end{aligned}$$

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[4] Because the d'Alembertian involves t^2 (as opposed to t), the theory itself is **time-reversal invariant**, and does not distinguish "past" from "future." Time asymmetry is introduced when we select the retarded potentials in preference to the advanced ones, reflecting the (not unreasonable!) belief that electromagnetic influences propagate forward, not backward, in time.

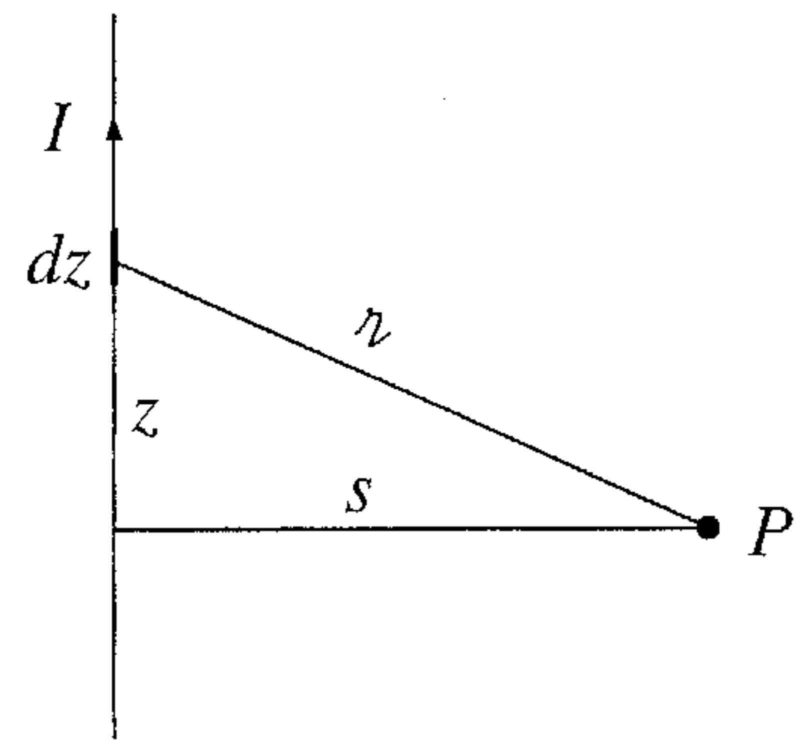

Figure 10.4

The electric field is

$$\mathbf{E}(s,t) = -\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 I_0 c}{2\pi\sqrt{(ct)^2 - s^2}}\,\hat{\mathbf{z}},$$

and the magnetic field is

$$\mathbf{B}(s,t) = \nabla \times \mathbf{A} = -\frac{\partial A_z}{\partial s}\,\hat{\boldsymbol{\phi}} = \frac{\mu_0 I_0}{2\pi s}\frac{ct}{\sqrt{(ct)^2 - s^2}}\,\hat{\boldsymbol{\phi}}.$$

Notice that as $t \to \infty$ we recover the static case: $\mathbf{E} = 0$, $\mathbf{B} = (\mu_0 I_0 / 2\pi s)\,\hat{\boldsymbol{\phi}}$.

! **Problem 10.8** Confirm that the retarded potentials satisfy the Lorentz gauge condition. [*Hint:* First show that

$$\nabla \cdot \left(\frac{\mathbf{J}}{\mathscr{r}}\right) = \frac{1}{\mathscr{r}}(\nabla \cdot \mathbf{J}) + \frac{1}{\mathscr{r}}(\nabla' \cdot \mathbf{J}) - \nabla' \cdot \left(\frac{\mathbf{J}}{\mathscr{r}}\right),$$

where ∇ denotes derivatives with respect to $\mathbf{r}$, and ∇' denotes derivatives with respect to $\mathbf{r}'$. Next, noting that $\mathbf{J}(\mathbf{r}', t - \mathscr{r}/c)$ depends on $\mathbf{r}'$ both explicitly and through $\mathscr{r}$, whereas it depends on $\mathbf{r}$ only through $\mathscr{r}$, confirm that

$$\nabla \cdot \mathbf{J} = -\frac{1}{c}\dot{\mathbf{J}} \cdot (\nabla \mathscr{r}), \quad \nabla' \cdot \mathbf{J} = -\dot{\rho} - \frac{1}{c}\dot{\mathbf{J}} \cdot (\nabla' \mathscr{r}).$$

Use this to calculate the divergence of $\mathbf{A}$ (Eq. 10.19).]

! **Problem 10.9**

(a) Suppose the wire in Ex. 10.2 carries a linearly increasing current

$$I(t) = kt,$$

for $t > 0$. Find the electric and magnetic fields generated.

(b) Do the same for the case of a sudden burst of current:

$$I(t) = q_0 \delta(t).$$

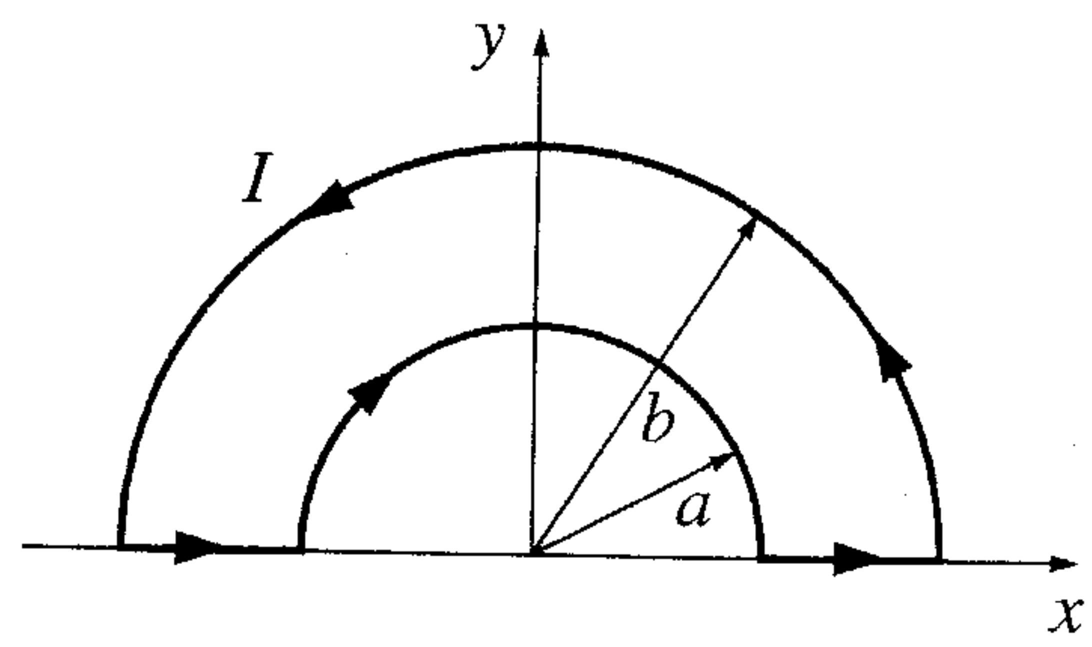

Figure 10.5

Problem 10.10 A piece of wire bent into a loop, as shown in Fig. 10.5, carries a current that increases linearly with time:

$$I(t) = kt.$$

Calculate the retarded vector potential **A** at the center. Find the electric field at the center. Why does this (neutral) wire produce an *electric* field? (Why can't you determine the *magnetic* field from this expression for **A**?)

10.2.2 Jefimenko's Equations

Given the retarded potentials

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}}\, d\tau', \qquad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}', t_r)}{\mathscr{r}}\, d\tau', \tag{10.26}$$

it is, in principle, a straightforward matter to determine the fields:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}. \tag{10.27}$$

But the details are not entirely trivial because, as I mentioned earlier, the integrands depend on **r** both explicitly, through $\mathscr{r} = |\mathbf{r} - \mathbf{r}'|$ in the denominator, and implicitly, through the retarded time $t_r = t - \mathscr{r}/c$ in the argument of the numerator.

I already calculated the gradient of V (Eq. 10.22); the time derivative of **A** is easy:

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{\mu_0}{4\pi}\int \frac{\dot{\mathbf{J}}}{\mathscr{r}}\, d\tau'. \tag{10.28}$$

Putting them together (and using $c^2 = 1/\mu_0\epsilon_0$):

$$\boxed{\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int \left[\frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}} + \frac{\dot{\rho}(\mathbf{r}', t_r)}{c\mathscr{r}}\hat{\boldsymbol{\mathscr{r}}} - \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{c^2\mathscr{r}}\right] d\tau'.} \tag{10.29}$$

This is the time-dependent generalization of Coulomb's law, to which it reduces in the static case (where the second and third terms drop out and the first term loses its dependence on t_r).

As for **B**, the curl of **A** contains two terms:

$$\nabla \times \mathbf{A} = \frac{\mu_0}{4\pi} \int \left[\frac{1}{\mathscr{r}} (\nabla \times \mathbf{J}) - \mathbf{J} \times \nabla \left(\frac{1}{\mathscr{r}} \right) \right] d\tau'.$$

Now

$$(\nabla \times \mathbf{J})_x = \frac{\partial J_z}{\partial y} - \frac{\partial J_y}{\partial z},$$

and

$$\frac{\partial J_z}{\partial y} = \dot{J}_z \frac{\partial t_r}{\partial y} = -\frac{1}{c} \dot{J}_z \frac{\partial \mathscr{r}}{\partial y},$$

so

$$(\nabla \times \mathbf{J})_x = -\frac{1}{c} \left(\dot{J}_z \frac{\partial \mathscr{r}}{\partial y} - \dot{J}_y \frac{\partial \mathscr{r}}{\partial z} \right) = \frac{1}{c} \left[\dot{\mathbf{J}} \times (\nabla \mathscr{r}) \right]_x .$$

But $\nabla \mathscr{r} = \hat{\boldsymbol{\mathscr{r}}}$ (Prob. 1.13), so

$$\nabla \times \mathbf{J} = \frac{1}{c} \dot{\mathbf{J}} \times \hat{\boldsymbol{\mathscr{r}}}. \tag{10.30}$$

Meanwhile $\nabla(1/\mathscr{r}) = -\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$ (again, Prob. 1.13), and hence

$$\boxed{\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \left[\frac{\mathbf{J}(\mathbf{r}', t_r)}{\mathscr{r}^2} + \frac{\dot{\mathbf{J}}(\mathbf{r}', t_r)}{c\mathscr{r}} \right] \times \hat{\boldsymbol{\mathscr{r}}} \, d\tau'.} \tag{10.31}$$

This is the time-dependent generalization of the Biot-Savart law, to which it reduces in the static case.

Equations 10.29 and 10.31 are the (causal) solutions to Maxwell's equations. For some reason they do not seem to have been published until quite recently—the earliest explicit statement of which I am aware was by Oleg Jefimenko, in 1966.[5] In practice **Jefimenko's equations** are of limited utility, since it is typically easier to calculate the retarded potentials and differentiate them, rather than going directly to the fields. Nevertheless, they provide a satisfying sense of closure to the theory. They also help to clarify an observation I made in the previous section: To get to the retarded *potentials*, all you do is replace t by t_r in the electrostatic and magnetostatic formulas, but in the case of the *fields* not only is time replaced by retarded time, but completely new terms (involving derivatives of ρ and **J**) appear. And they provide surprisingly strong support for the quasistatic approximation (see Prob. 10.12).

[5] O. D. Jefimenko, *Electricity and Magnetism*, Sect. 15.7 (New York: Appleton-Century-Crofts, 1996). Closely related expressions appear in W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, Sect. 14.3 (Reading, MA: Addison-Wesley, 1962). See K. T. McDonald, *Am. J. Phys.* **65**, 1074 (1997) for illuminating commentary and references.

Problem 10.11 Suppose $\mathbf{J}(\mathbf{r})$ is constant in time, so (Prob. 7.55) $\rho(\mathbf{r}, t) = \rho(\mathbf{r}, 0) + \dot{\rho}(\mathbf{r}, 0)t$. Show that

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t)}{\mathscr{r}^2} \hat{\boldsymbol{\mathscr{r}}}\, d\tau';$$

that is, Coulomb's law holds, with the charge density evaluated at the *non-retarded* time.

Problem 10.12 Suppose the current density changes slowly enough that we can (to good approximation) ignore all higher derivatives in the Taylor expansion

$$\mathbf{J}(t_r) = \mathbf{J}(t) + (t_r - t)\dot{\mathbf{J}}(t) + \ldots$$

(for clarity, I suppress the $\mathbf{r}$-dependence, which is not at issue). Show that a fortuitous cancellation in Eq. 10.31 yields

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t) \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau'.$$

That is: the Biot-Savart law holds, with $\mathbf{J}$ evaluated at the *non-retarded* time. This means that the quasistatic approximation is actually much *better* than we had any right to expect: the *two* errors involved (neglecting retardation and dropping the second term in Eq. 10.31) *cancel* one another, to first order.

10.3 Point Charges

10.3.1 Liénard-Wiechert Potentials

My next project is to calculate the (retarded) potentials, $V(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$, of a point charge q that is moving on a specified trajectory

$$\mathbf{w}(t) \equiv \text{position of } q \text{ at time } t. \tag{10.32}$$

The retarded time is determined implicitly by the equation

$$|\mathbf{r} - \mathbf{w}(t_r)| = c(t - t_r), \tag{10.33}$$

for the left side is the distance the "news" must travel, and $(t - t_r)$ is the time it takes to make the trip (Fig. 10.6). I shall call $\mathbf{w}(t_r)$ the **retarded position** of the charge; $\boldsymbol{\mathscr{r}}$ is the vector from the retarded position to the field point $\mathbf{r}$:

$$\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{w}(t_r). \tag{10.34}$$

It is important to note that at most *one* point on the trajectory is "in communication" with $\mathbf{r}$ at any particular time t. For suppose there were *two* such points, with retarded times t_1 and t_2:

$$\mathscr{r}_1 = c(t - t_1) \quad \text{and} \quad \mathscr{r}_2 = c(t - t_2).$$

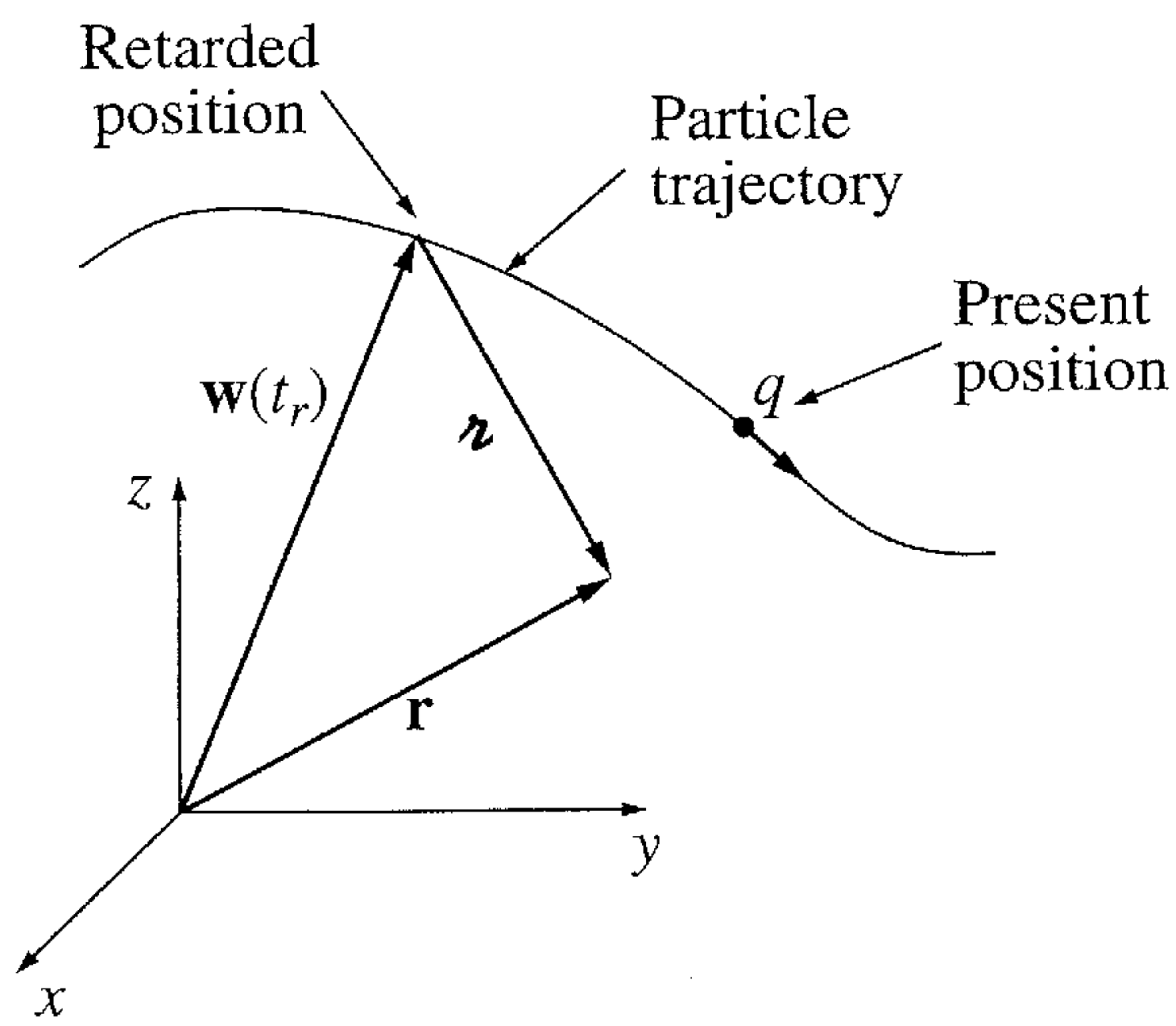

Figure 10.6

Then $\mathscr{r}_1 - \mathscr{r}_2 = c(t_2 - t_1)$, so the average velocity of the particle in the direction of $\mathbf{r}$ would have to be c—and that's not counting whatever velocity the charge might have in *other* directions. Since no charged particle can travel at the speed of light, it follows that only *one retarded point contributes to the potentials, at any given moment.*[6]

Now, a naïve reading of the formula

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t_r)}{\mathscr{r}}\, d\tau' \tag{10.35}$$

might suggest to you that the retarded potential of a point charge is simply

$$\frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}}$$

(the same as in the static case, only with the understanding that $\mathscr{r}$ is the distance to the *retarded* position of the charge). But this is wrong, for a very subtle reason: It is true that for a point source the denominator $\mathscr{r}$ comes outside the integral,[7] but what remains,

$$\int \rho(\mathbf{r}', t_r)\, d\tau', \tag{10.36}$$

is *not* equal to the charge of the particle. To calculate the total charge of a configuration you must integrate ρ over the entire distribution at *one instant of time*, but here the retardation, $t_r = t - \mathscr{r}/c$, obliges us to evaluate ρ at *different times* for different parts of the configuration. If the source is moving, this will give a distorted picture of the total charge. You might

[6]For the same reason, an observer at $\mathbf{r}$ *sees* the particle in only one place at a time. By contrast, it is possible to *hear* an object in two places at once. Consider a bear who growls at you and then runs toward you at the speed of sound and growls again; you hear both growls at the same time, coming from two different locations, but there's only one bear.

[7]There is, however, an implicit change in its functional dependence: *Before* the integration, $\mathscr{r} = |\mathbf{r} - \mathbf{r}'|$ is a function of $\mathbf{r}$ and $\mathbf{r}'$; *after* the integration (which fixes $\mathbf{r}' = \mathbf{w}(t_r)$) $\mathscr{r} = |\mathbf{r} - \mathbf{w}(t_r)|$ is (like t_r) a function of $\mathbf{r}$ and t.

think that this problem would disappear for *point* charges, but it doesn't. In Maxwell's electrodynamics, formulated as it is in terms of charge and current *densities*, a point charge must be regarded as the limit of an extended charge, when the size goes to zero. And for an extended particle, no matter how small, the retardation in Eq. 10.36 throws in a factor $(1 - \hat{\boldsymbol{\imath}} \cdot \mathbf{v}/c)^{-1}$, where $\mathbf{v}$ is the velocity of the charge at the retarded time:

$$\int \rho(\mathbf{r}', t_r)\, d\tau' = \frac{q}{1 - \hat{\boldsymbol{\imath}} \cdot \mathbf{v}/c}. \tag{10.37}$$

Proof: This is a purely *geometrical* effect, and it may help to tell the story in a less abstract context. You will not have noticed it, for obvious reasons, but the fact is that a train coming towards you looks a little longer than it really is, because the light you receive from the caboose left earlier than the light you receive simultaneously from the engine, and at that earlier time the train was farther away (Fig. 10.7). In the interval it takes light from the caboose to travel the extra distance L', the train itself moves a distance $L' - L$:

$$\frac{L'}{c} = \frac{L' - L}{v}, \quad \text{or} \quad L' = \frac{L}{1 - v/c}.$$

So approaching trains appear *longer*, by a factor $(1 - v/c)^{-1}$. By contrast, a train going *away* from you looks *shorter*,[8] by a factor $(1+v/c)^{-1}$. In general, if the train's velocity makes an angle θ with your line of sight,[9] the extra distance light from the caboose must cover is $L' \cos\theta$ (Fig. 10.8). In the time $L' \cos\theta/c$, then, the train moves a distance $(L' - L)$:

$$\frac{L' \cos\theta}{c} = \frac{L' - L}{v}, \quad \text{or} \quad L' = \frac{L}{1 - v\cos\theta/c}.$$

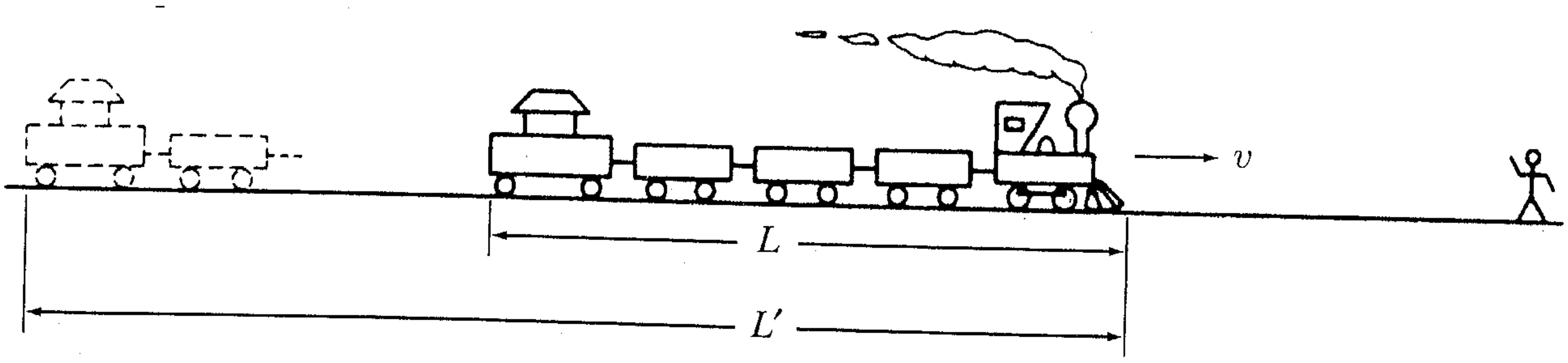

Figure 10.7

[8]Please note that this has nothing whatever to do with special relativity or Lorentz contraction—L is the length of the *moving* train, and its *rest* length is not at issue. The argument *is* somewhat reminiscent of the Doppler effect.

[9]I assume the train is far enough away or (more to the point) *short* enough so that rays from the caboose and engine can be considered parallel.

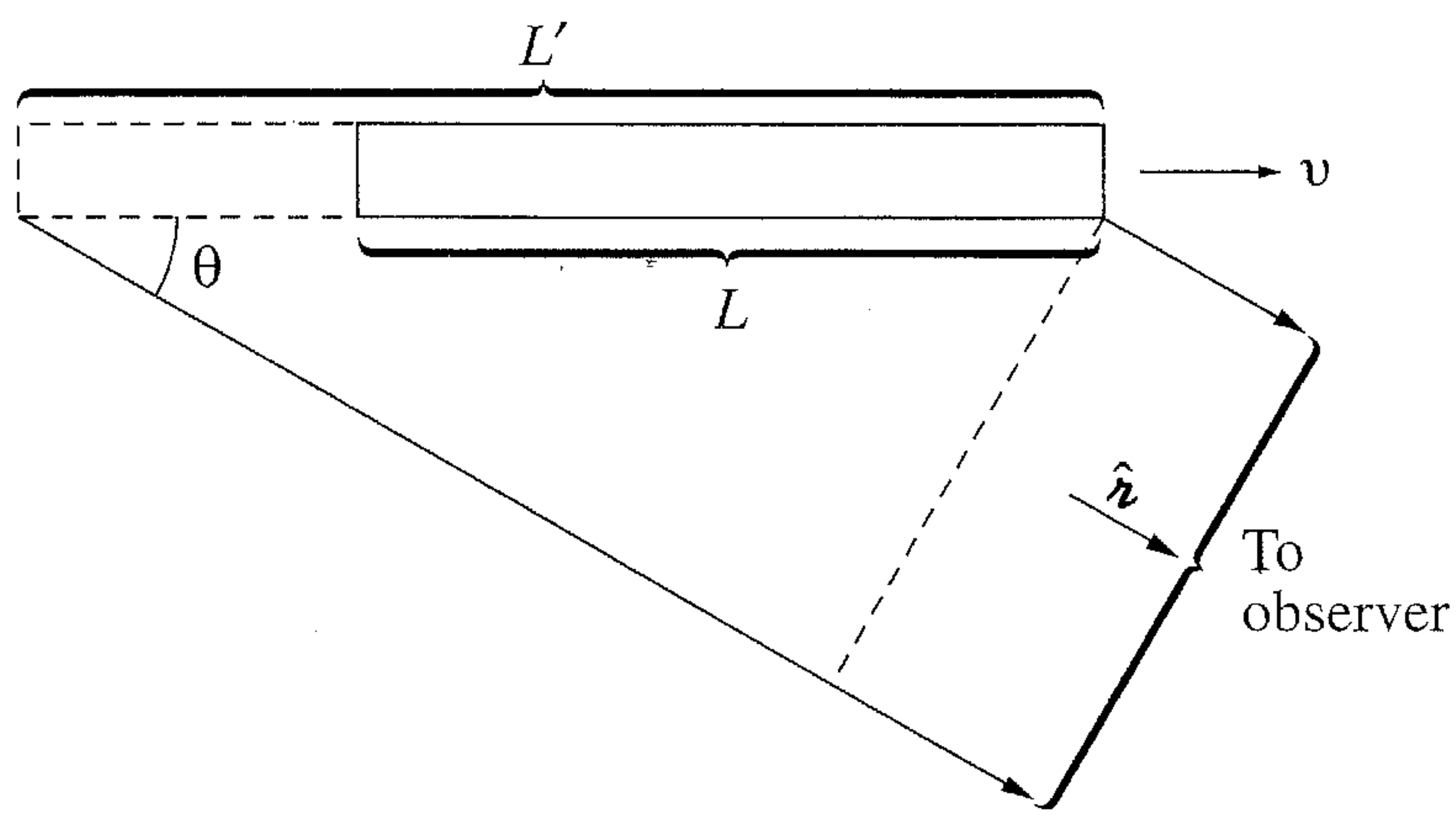

Figure 10.8

Notice that this effect does *not* distort the dimensions perpendicular to the motion (the height and width of the train). Never mind that the light from the far side is delayed in reaching you (relative to light from the near side)—since there's no *motion* in that direction, they'll still look the same distance apart. The apparent *volume* τ' of the train, then, is related to the *actual* volume τ by

$$\tau' = \frac{\tau}{1 - \hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{v}/c}, \tag{10.38}$$

where $\hat{\boldsymbol{\mathscr{r}}}$ is a unit vector from the train to the observer.

In case the connection between moving trains and retarded potentials escapes you, the point is this: Whenever you do an integral of the type 10.37, in which the integrand is evaluated at the retarded time, the effective volume is modified by the factor in Eq. 10.38, just as the apparent volume of the train was—and for the same reason. Because this correction factor makes no reference to the size of the particle, it is every bit as significant for a point charge as for an extended charge. qed

It follows, then, that

$$\boxed{V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})},} \tag{10.39}$$

where $\mathbf{v}$ is the velocity of the charge at the retarded time, and $\boldsymbol{\mathscr{r}}$ is the vector from the retarded position to the field point $\mathbf{r}$. Meanwhile, since the current density of a rigid object is $\rho\mathbf{v}$ (Eq. 5.26), we also have

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\rho(\mathbf{r}', t_r)\mathbf{v}(t_r)}{\mathscr{r}} \, d\tau' = \frac{\mu_0}{4\pi} \frac{\mathbf{v}}{\mathscr{r}} \int \rho(\mathbf{r}', t_r) \, d\tau',$$

or

$$\boxed{\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \frac{qc\mathbf{v}}{(\imath c - \boldsymbol{\imath} \cdot \mathbf{v})} = \frac{\mathbf{v}}{c^2} V(\mathbf{r}, t).} \tag{10.40}$$

Equations 10.39 and 10.40 are the famous **Liénard-Wiechert potentials** for a moving point charge.[10]

Example 10.3

Find the potentials of a point charge moving with constant velocity.

Solution: For convenience, let's say the particle passes through the origin at time $t = 0$, so that

$$\mathbf{w}(t) = \mathbf{v}t.$$

We first compute the retarded time, using Eq. 10.33:

$$|\mathbf{r} - \mathbf{v}t_r| = c(t - t_r),$$

or, squaring:

$$r^2 - 2\mathbf{r} \cdot \mathbf{v}t_r + v^2 t_r^2 = c^2(t^2 - 2tt_r + t_r^2).$$

Solving for t_r by the quadratic formula, I find that

$$t_r = \frac{(c^2 t - \mathbf{r} \cdot \mathbf{v}) \pm \sqrt{(c^2 t - \mathbf{r} \cdot \mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}}{c^2 - v^2}. \tag{10.41}$$

To fix the sign, consider the limit $v = 0$:

$$t_r = t \pm \frac{r}{c}.$$

In this case the charge is at rest at the origin, and the retarded time should be $(t - r/c)$; evidently we want the *minus* sign.

Now, from Eqs. 10.33 and 10.34,

$$\imath = c(t - t_r), \quad \text{and} \quad \hat{\boldsymbol{\imath}} = \frac{\mathbf{r} - \mathbf{v}t_r}{c(t - t_r)},$$

so

$$\begin{aligned}
\imath(1 - \hat{\boldsymbol{\imath}} \cdot \mathbf{v}/c) &= c(t - t_r)\left[1 - \frac{\mathbf{v}}{c} \cdot \frac{(\mathbf{r} - \mathbf{v}t_r)}{c(t - t_r)}\right] = c(t - t_r) - \frac{\mathbf{v} \cdot \mathbf{r}}{c} - \frac{v^2}{c} t_r \\
&= \frac{1}{c}[(c^2 t - \mathbf{r} \cdot \mathbf{v}) - (c^2 - v^2)t_r] \\
&= \frac{1}{c}\sqrt{(c^2 t - \mathbf{r} \cdot \mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}
\end{aligned}$$

[10] There are many ways to obtain the Liénard-Wiechert potentials. I have tried to emphasize the *geometrical* origin of the factor $(1 - \hat{\boldsymbol{\imath}} \cdot \mathbf{v}/c)^{-1}$; for illuminating commentary see W. K. H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, 2d ed., pp. 342-3 (Reading, MA: Addison-Wesley, 1962). A more rigorous derivation is provided by J. R. Reitz, F. J. Milford, and R. W. Christy, *Foundations of Electromagnetic Theory*, 3d ed., Sect. 21.1 (Reading, MA: Addison-Wesley, 1979), or M. A. Heald and J. B. Marion, *Classical Electromagnetic Radiation*, 3d ed., Sect. 8.3 (Orlando, FL: Saunders, 1995).

(I used Eq. 10.41, with the minus sign, in the last step). Therefore,

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{\sqrt{(c^2t - \mathbf{r}\cdot\mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}}, \tag{10.42}$$

and (Eq. 10.40)

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \frac{qc\mathbf{v}}{\sqrt{(c^2t - \mathbf{r}\cdot\mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2t^2)}}. \tag{10.43}$$

Problem 10.13 A particle of charge q moves in a circle of radius a at constant angular velocity ω. (Assume that the circle lies in the xy plane, centered at the origin, and at time $t = 0$ the charge is at $(a, 0)$, on the positive x axis.) Find the Liénard-Wiechert potentials for points on the z axis.

• **Problem 10.14** Show that the scalar potential of a point charge moving with constant velocity (Eq. 10.42) can be written equivalently as

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{q}{R\sqrt{1 - v^2\sin^2\theta/c^2}}, \tag{10.44}$$

where $\mathbf{R} \equiv \mathbf{r} - \mathbf{v}t$ is the vector from the *present (!)* position of the particle to the field point $\mathbf{r}$, and θ is the angle between $\mathbf{R}$ and $\mathbf{v}$ (Fig. 10.9). Evidently for nonrelativistic velocities $(v^2 \ll c^2)$,

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{q}{R}.$$

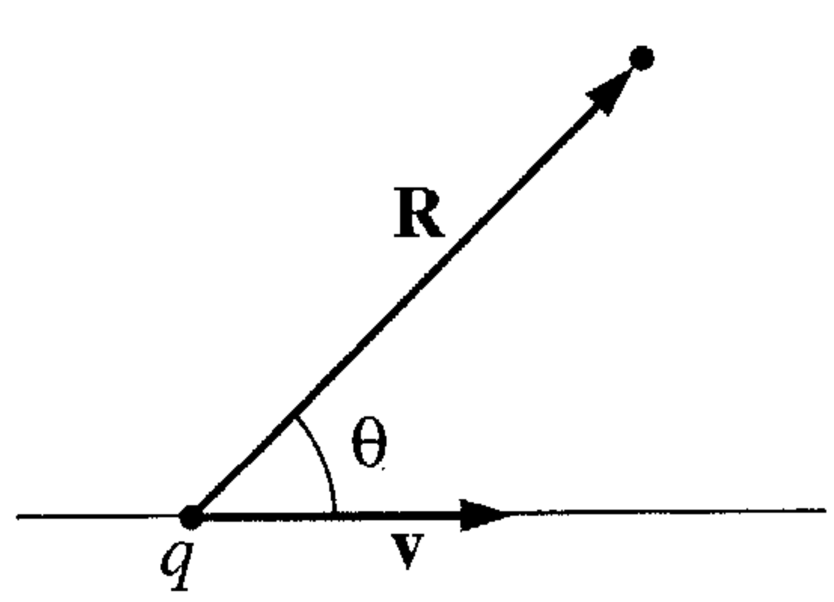

Figure 10.9

Problem 10.15 I showed that *at most one* point on the particle trajectory communicates with $\mathbf{r}$ at any given time. In some cases there may be *no* such point (an observer at $\mathbf{r}$ would not see the particle—in the colorful language of General Relativity it is "beyond the **horizon**"). As an example, consider a particle in **hyperbolic motion** along the x axis:

$$\mathbf{w}(t) = \sqrt{b^2 + (ct)^2}\,\hat{\mathbf{x}} \qquad (-\infty < t < \infty). \tag{10.45}$$

(In Special Relativity this is the trajectory of a particle subject to a constant force $F = mc^2/b$.) Sketch the graph of w versus t. At four or five representative points on the curve, draw the trajectory of a light signal emitted by the particle at that point—both in the plus x direction and in the minus x direction. What region on your graph corresponds to points and times (x, t) from which the particle cannot be seen? At what time does someone at point x first see the particle? (Prior to this the potential at x is evidently zero.) Is it possible for a particle, once seen, to *disappear* from view?

! **Problem 10.16** Determine the Liénard-Wiechert potentials for a charge in hyperbolic motion (Eq. 10.45). Assume the point $\mathbf{r}$ is on the x axis and to the right of the charge.

10.3.2 The Fields of a Moving Point Charge

We are now in a position to calculate the electric and magnetic fields of a point charge in arbitrary motion, using the Liénard-Wiechert potentials:[11]

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})}, \qquad \mathbf{A}(\mathbf{r}, t) = \frac{\mathbf{v}}{c^2} V(\mathbf{r}, t), \tag{10.46}$$

and the equations for $\mathbf{E}$ and $\mathbf{B}$:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}.$$

The differentiation is tricky, however, because

$$\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{w}(t_r) \quad \text{and} \quad \mathbf{v} = \dot{\mathbf{w}}(t_r) \tag{10.47}$$

are both evaluated at the retarded time, and t_r—defined implicitly by the equation

$$|\mathbf{r} - \mathbf{w}(t_r)| = c(t - t_r) \tag{10.48}$$

—is *itself* a function of $\mathbf{r}$ and t.[12] So hang on: the next two pages are rough going ... but the answer is worth the effort.

Let's begin with the gradient of V:

$$\nabla V = \frac{qc}{4\pi\epsilon_0} \frac{-1}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})^2} \nabla(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v}). \tag{10.49}$$

---

[11] You can get the fields directly from Jefimenko's equations, but it's not easy. See, for example, M. A. Heald and J. B. Marion, *Classical Electromagnetic Radiation*, 3d ed., Sect. 8.4 (Orlando, FL: Saunders, 1995).

[12] The following calculation is done by the most direct, "brute force" method. For a more clever and efficient approach see J. D. Jackson, *Classical Electrodynamics*, 3d ed., Sect. 14.1 (New York: John Wiley, 1999).

Since $\mathscr{r} = c(t - t_r)$,

$$\nabla \mathscr{r} = -c \nabla t_r. \tag{10.50}$$

As for the second term, product rule 4 gives

$$\nabla(\boldsymbol{\mathscr{r}} \cdot \mathbf{v}) = (\boldsymbol{\mathscr{r}} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\boldsymbol{\mathscr{r}} + \boldsymbol{\mathscr{r}} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \boldsymbol{\mathscr{r}}). \tag{10.51}$$

Evaluating these terms one at a time:

$$\begin{aligned}(\boldsymbol{\mathscr{r}} \cdot \nabla)\mathbf{v} &= \left(\mathscr{r}_x \frac{\partial}{\partial x} + \mathscr{r}_y \frac{\partial}{\partial y} + \mathscr{r}_z \frac{\partial}{\partial z}\right)\mathbf{v}(t_r) \\ &= \mathscr{r}_x \frac{d\mathbf{v}}{dt_r}\frac{\partial t_r}{\partial x} + \mathscr{r}_y \frac{d\mathbf{v}}{dt_r}\frac{\partial t_r}{\partial y} + \mathscr{r}_z \frac{d\mathbf{v}}{dt_r}\frac{\partial t_r}{\partial z} \\ &= \mathbf{a}(\boldsymbol{\mathscr{r}} \cdot \nabla t_r),\end{aligned} \tag{10.52}$$

where $\mathbf{a} \equiv \dot{\mathbf{v}}$ is the *acceleration* of the particle at the retarded time. Now

$$(\mathbf{v} \cdot \nabla)\boldsymbol{\mathscr{r}} = (\mathbf{v} \cdot \nabla)\mathbf{r} - (\mathbf{v} \cdot \nabla)\mathbf{w}, \tag{10.53}$$

and

$$\begin{aligned}(\mathbf{v} \cdot \nabla)\mathbf{r} &= \left(v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z}\right)(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) \\ &= v_x\,\hat{\mathbf{x}} + v_y\,\hat{\mathbf{y}} + v_z\,\hat{\mathbf{z}} = \mathbf{v},\end{aligned} \tag{10.54}$$

while

$$(\mathbf{v} \cdot \nabla)\mathbf{w} = \mathbf{v}(\mathbf{v} \cdot \nabla t_r)$$

(same reasoning as Eq. 10.52). Moving on to the third term in Eq. 10.51,

$$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}}$$

$$= \left(\frac{dv_z}{dt_r}\frac{\partial t_r}{\partial y} - \frac{dv_y}{dt_r}\frac{\partial t_r}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{dv_x}{dt_r}\frac{\partial t_r}{\partial z} - \frac{dv_z}{dt_r}\frac{\partial t_r}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{dv_y}{dt_r}\frac{\partial t_r}{\partial x} - \frac{dv_x}{dt_r}\frac{\partial t_r}{\partial y}\right)\hat{\mathbf{z}}$$

$$= -\mathbf{a} \times \nabla t_r. \tag{10.55}$$

Finally,

$$\nabla \times \boldsymbol{\mathscr{r}} = \nabla \times \mathbf{r} - \nabla \times \mathbf{w}, \tag{10.56}$$

but $\nabla \times \mathbf{r} = 0$, while, by the same argument as Eq. 10.55,

$$\nabla \times \mathbf{w} = -\mathbf{v} \times \nabla t_r. \tag{10.57}$$

Putting all this back into Eq. 10.51, and using the "BAC-CAB" rule to reduce the triple cross products,

$$\begin{aligned}\nabla(\boldsymbol{\mathscr{r}}\cdot\mathbf{v}) &= \mathbf{a}(\boldsymbol{\mathscr{r}}\cdot\nabla t_r)+\mathbf{v}-\mathbf{v}(\mathbf{v}\cdot\nabla t_r)-\boldsymbol{\mathscr{r}}\times(\mathbf{a}\times\nabla t_r)+\mathbf{v}\times(\mathbf{v}\times\nabla t_r)\\ &= \mathbf{v}+(\boldsymbol{\mathscr{r}}\cdot\mathbf{a}-v^2)\nabla t_r.\end{aligned} \tag{10.58}$$

Collecting Eqs. 10.50 and 10.58 together, we have

$$\nabla V=\frac{qc}{4\pi\epsilon_0}\frac{1}{(\mathscr{r}c-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})^2}\left[\mathbf{v}+(c^2-v^2+\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\nabla t_r\right]. \tag{10.59}$$

To complete the calculation, we need to know ∇t_r. This can be found by taking the gradient of the defining equation (10.48)—which we have already done in Eq. 10.50—and expanding out $\nabla\mathscr{r}$:

$$\begin{aligned}-c\nabla t_r &= \nabla\mathscr{r}=\nabla\sqrt{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}}}=\frac{1}{2\sqrt{\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}}}}\nabla(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{\mathscr{r}})\\ &= \frac{1}{\mathscr{r}}[(\boldsymbol{\mathscr{r}}\cdot\nabla)\boldsymbol{\mathscr{r}}+\boldsymbol{\mathscr{r}}\times(\nabla\times\boldsymbol{\mathscr{r}})].\end{aligned} \tag{10.60}$$

But

$$(\boldsymbol{\mathscr{r}}\cdot\nabla)\boldsymbol{\mathscr{r}}=\boldsymbol{\mathscr{r}}-\mathbf{v}(\boldsymbol{\mathscr{r}}\cdot\nabla t_r)$$

(same idea as Eq. 10.53), while (from Eq. 10.56 and 10.57)

$$\nabla\times\boldsymbol{\mathscr{r}}=(\mathbf{v}\times\nabla t_r).$$

Thus

$$-c\nabla t_r=\frac{1}{\mathscr{r}}[\boldsymbol{\mathscr{r}}-\mathbf{v}(\boldsymbol{\mathscr{r}}\cdot\nabla t_r)+\boldsymbol{\mathscr{r}}\times(\mathbf{v}\times\nabla t_r)]=\frac{1}{\mathscr{r}}[\boldsymbol{\mathscr{r}}-(\boldsymbol{\mathscr{r}}\cdot\mathbf{v})\nabla t_r],$$

and hence

$$\nabla t_r=\frac{-\boldsymbol{\mathscr{r}}}{\mathscr{r}c-\boldsymbol{\mathscr{r}}\cdot\mathbf{v}}. \tag{10.61}$$

Incorporating this result into Eq. 10.59, I conclude that

$$\nabla V=\frac{1}{4\pi\epsilon_0}\frac{qc}{(\mathscr{r}c-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})^3}\left[(\mathscr{r}c-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})\mathbf{v}-(c^2-v^2+\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\boldsymbol{\mathscr{r}}\right]. \tag{10.62}$$

A similar calculation, which I shall leave for you (Prob. 10.17), yields

$$\begin{aligned}\frac{\partial\mathbf{A}}{\partial t} &= \frac{1}{4\pi\epsilon_0}\frac{qc}{(\mathscr{r}c-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})^3}\left[(\mathscr{r}c-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})(-\mathbf{v}+\mathscr{r}\mathbf{a}/c)\right.\\ &+ \left.\frac{\mathscr{r}}{c}(c^2-v^2+\boldsymbol{\mathscr{r}}\cdot\mathbf{a})\mathbf{v}\right].\end{aligned} \tag{10.63}$$

Combining these results, and introducing the vector

$$\mathbf{u} \equiv c\hat{\boldsymbol{\imath}} - \mathbf{v}, \tag{10.64}$$

I find

$$\boxed{\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\imath}{(\boldsymbol{\imath} \cdot \mathbf{u})^3}[(c^2 - v^2)\mathbf{u} + \boldsymbol{\imath} \times (\mathbf{u} \times \mathbf{a})].} \tag{10.65}$$

Meanwhile,

$$\nabla \times \mathbf{A} = \frac{1}{c^2}\nabla \times (V\mathbf{v}) = \frac{1}{c^2}[V(\nabla \times \mathbf{v}) - \mathbf{v} \times (\nabla V)].$$

We have already calculated $\nabla \times \mathbf{v}$ (Eq. 10.55) and ∇V (Eq. 10.62). Putting these together,

$$\nabla \times \mathbf{A} = -\frac{1}{c}\frac{q}{4\pi\epsilon_0}\frac{1}{(\mathbf{u} \cdot \boldsymbol{\imath})^3}\boldsymbol{\imath} \times [(c^2 - v^2)\mathbf{v} + (\boldsymbol{\imath} \cdot \mathbf{a})\mathbf{v} + (\boldsymbol{\imath} \cdot \mathbf{u})\mathbf{a}].$$

The quantity in brackets is strikingly similar to the one in Eq. 10.65, which can be written, using the BAC-CAB rule, as $[(c^2 - v^2)\mathbf{u} + (\boldsymbol{\imath} \cdot \mathbf{a})\mathbf{u} - (\boldsymbol{\imath} \cdot \mathbf{u})\mathbf{a}]$; the main difference is that we have $\mathbf{v}$'s instead of $\mathbf{u}$'s in the first two terms. In fact, since it's all crossed into $\boldsymbol{\imath}$ *anyway*, we can with impunity *change* these $\mathbf{v}$'s into $-\mathbf{u}$'s; the extra term proportional to $\hat{\boldsymbol{\imath}}$ disappears in the cross product. It follows that

$$\boxed{\mathbf{B}(\mathbf{r}, t) = \frac{1}{c}\hat{\boldsymbol{\imath}} \times \mathbf{E}(\mathbf{r}, t).} \tag{10.66}$$

Evidently *the magnetic field of a point charge is always perpendicular to the electric field, and to the vector from the retarded point.*

The first term in $\mathbf{E}$ (the one involving $(c^2 - v^2)\mathbf{u}$) falls off as the inverse *square* of the distance from the particle. If the velocity and acceleration are both zero, this term alone survives and reduces to the old electrostatic result

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{\imath^2}\hat{\boldsymbol{\imath}}.$$

For this reason, the first term in $\mathbf{E}$ is sometimes called the **generalized Coulomb field**. (Because it does not depend on the acceleration, it is also known as the **velocity field**.) The second term (the one involving $\boldsymbol{\imath} \times (\mathbf{u} \times \mathbf{a})$) falls off as the inverse *first* power of $\imath$ and is therefore dominant at large distances. As we shall see in Chapter 11, it is this term that is responsible for electromagnetic radiation; accordingly, it is called the **radiation field**—or, since it is proportional to a, the **acceleration field**. The same terminology applies to the magnetic field.

Back in Chapter 2, I commented that if we could only write down the formula for the force one charge exerts on another, we would be done with electrodynamics, in principle. That, together with the superposition principle, would tell us the force exerted on a test charge Q by any configuration whatsoever. Well ... here we are: Eqs. 10.65 and 10.66 give us the fields, and the Lorentz force law determines the resulting force:

$$\begin{aligned}\mathbf{F} &= \frac{qQ}{4\pi\epsilon_0}\frac{\boldsymbol{\mathscr{r}}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\Big\{[(c^2-v^2)\mathbf{u}+\boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})] \\ &+ \frac{\mathbf{V}}{c}\times\left[\hat{\boldsymbol{\mathscr{r}}}\times[(c^2-v^2)\mathbf{u}+\boldsymbol{\mathscr{r}}\times(\mathbf{u}\times\mathbf{a})]\right]\Big\}, \qquad (10.67)\end{aligned}$$

where $\mathbf{V}$ is the velocity of Q, and $\boldsymbol{\mathscr{r}}$, $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{a}$ are all evaluated at the retarded time. The entire theory of classical electrodynamics is contained in that equation ... but you see why I preferred to start out with Coulomb's law.

Example 10.4

Calculate the electric and magnetic fields of a point charge moving with constant velocity.

Solution: Putting $\mathbf{a}=0$ in Eq. 10.65,

$$\mathbf{E}=\frac{q}{4\pi\epsilon_0}\frac{(c^2-v^2)\mathscr{r}}{(\boldsymbol{\mathscr{r}}\cdot\mathbf{u})^3}\mathbf{u}.$$

In this case, using $\mathbf{w}=\mathbf{v}t$,

$$\mathscr{r}\mathbf{u}=c\boldsymbol{\mathscr{r}}-\mathscr{r}\mathbf{v}=c(\mathbf{r}-\mathbf{v}t_r)-c(t-t_r)\mathbf{v}=c(\mathbf{r}-\mathbf{v}t).$$

In Ex. 10.3 we found that

$$\mathscr{r}c-\boldsymbol{\mathscr{r}}\cdot\mathbf{v}=\boldsymbol{\mathscr{r}}\cdot\mathbf{u}=\sqrt{(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)}.$$

In Prob. 10.14, you showed that this radical could be written as

$$Rc\sqrt{1-v^2\sin^2\theta/c^2},$$

where

$$\mathbf{R}\equiv\mathbf{r}-\mathbf{v}t$$

is the vector from the *present* location of the particle to $\mathbf{r}$, and θ is the angle between $\mathbf{R}$ and $\mathbf{v}$ (Fig. 10.9). Thus

$$\boxed{\mathbf{E}(\mathbf{r},t)=\frac{q}{4\pi\epsilon_0}\frac{1-v^2/c^2}{\left(1-v^2\sin^2\theta/c^2\right)^{3/2}}\frac{\hat{\mathbf{R}}}{R^2}.} \qquad (10.68)$$

Notice that $\mathbf{E}$ points along the line from the *present* position of the particle. This is an *extraordinary* coincidence, since the "message" came from the *retarded* position. Because of the $\sin^2\theta$ in the denominator, the field of a fast-moving charge is flattened out like a pancake in the direction perpendicular to the motion (Fig. 10.10). In the forward and backward directions $\mathbf{E}$ is *reduced* by a factor $(1-v^2/c^2)$ relative to the field of a charge at rest; in the perpendicular direction it is *enhanced* by a factor $1/\sqrt{1-v^2/c^2}$.

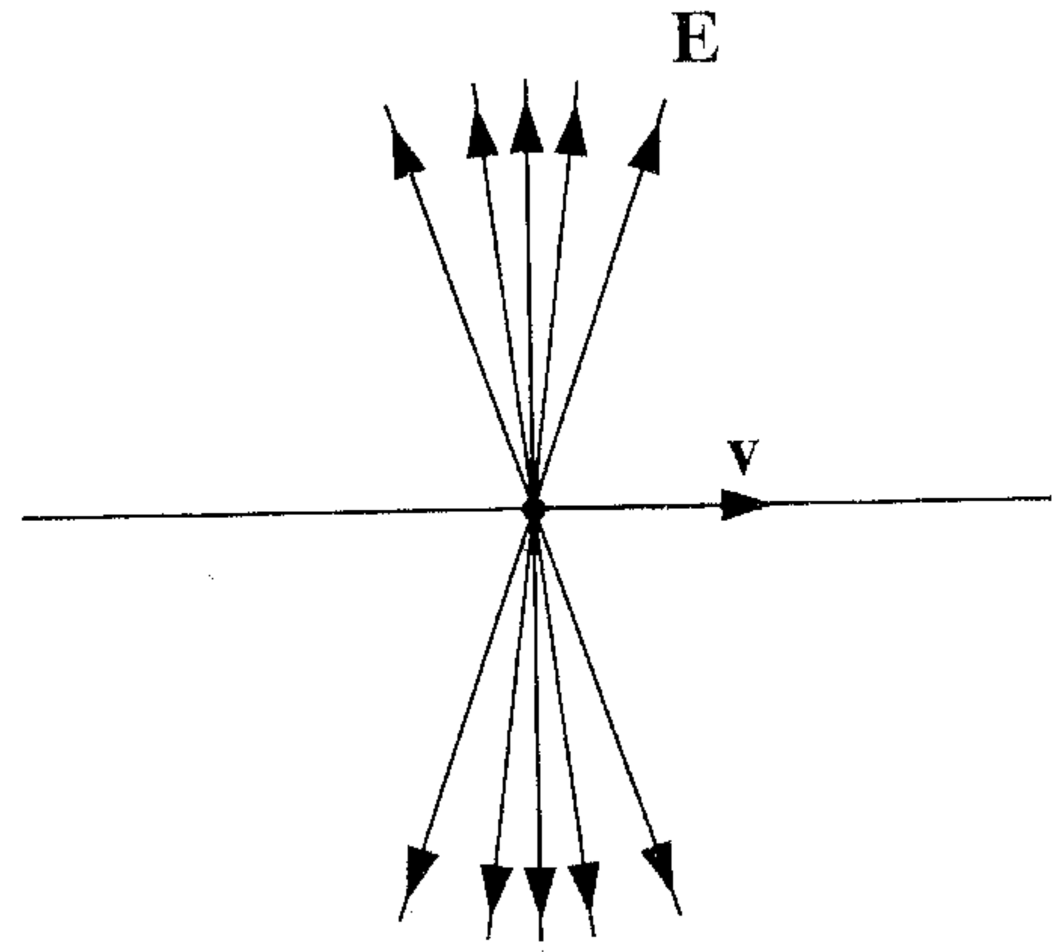

Figure 10.10

As for **B**, we have

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{\mathbf{r} - \mathbf{v}t_r}{\mathscr{r}} = \frac{(\mathbf{r} - \mathbf{v}t) + (t - t_r)\mathbf{v}}{\mathscr{r}} = \frac{\mathbf{R}}{\mathscr{r}} + \frac{\mathbf{v}}{c},$$

and therefore

$$\boxed{\mathbf{B} = \frac{1}{c}(\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}) = \frac{1}{c^2}(\mathbf{v} \times \mathbf{E}).} \tag{10.69}$$

Lines of **B** *circle around* the charge, as shown in Fig. 10.11.

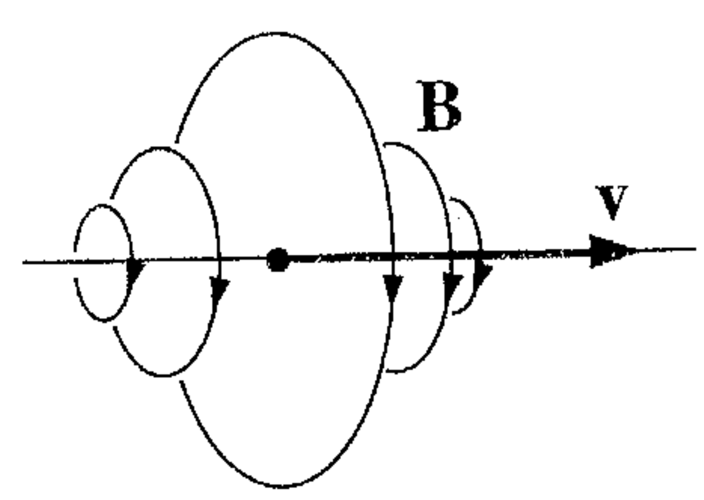

Figure 10.11

The fields of a point charge moving at constant velocity (Eqs. 10.68 and 10.69) were first obtained by Oliver Heaviside in 1888.[13] When $v^2 \ll c^2$ they reduce to

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\hat{\mathbf{R}}; \qquad \mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\frac{q}{R^2}(\mathbf{v} \times \hat{\mathbf{R}}). \tag{10.70}$$

The first is essentially Coulomb's law, and the latter is the "Biot-Savart law for a point charge" I warned you about in Chapter 5 (Eq. 5.40).

[13] For history and references, see O. J. Jefimenko, *Am. J. Phys.* **62**, 79 (1994).

11.2 Point Charges

11.2.1 Power Radiated by a Point Charge

In Chapter 10 we derived the fields of a point charge q in arbitrary motion (Eqs. 10.65 and 10.66):

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3}[(c^2 - v^2)\mathbf{u} + \boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})], \tag{11.62}$$

where $\mathbf{u} = c\hat{\boldsymbol{\mathscr{r}}} - \mathbf{v}$, and

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E}(\mathbf{r}, t). \tag{11.63}$$

The first term in Eq. 11.62 is called the **velocity field**, and the second one (with the triple cross-product) is called the **acceleration field**.

The Poynting vector is

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) = \frac{1}{\mu_0 c}[\mathbf{E} \times (\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{E})] = \frac{1}{\mu_0 c}[E^2\hat{\boldsymbol{\mathscr{r}}} - (\hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{E})\mathbf{E}]. \tag{11.64}$$

However, not all of this energy flux constitutes *radiation*; some of it is just field energy carried along by the particle as it moves. The *radiated* energy is the stuff that, in effect, *detaches* itself from the charge and propagates off to infinity. (It's like flies breeding on a garbage truck: Some of them hover around the truck as it makes its rounds; others fly away and never come back.) To calculate the total power radiated by the particle at time t_r, we draw a huge sphere of radius $\mathscr{r}$ (Fig. 11.11), centered at the position of the particle (at time t_r), wait the appropriate interval

$$t - t_r = \frac{\mathscr{r}}{c} \tag{11.65}$$

for the radiation to reach the sphere, and at that moment integrate the Poynting vector over the surface.[6] I have used the notation t_r because, in fact, this *is* the retarded time for all points on the sphere at time t.

Now, the area of the sphere is proportional to $\mathscr{r}^2$, so any term in $\mathbf{S}$ that goes like $1/\mathscr{r}^2$ will yield a finite answer, but terms like $1/\mathscr{r}^3$ or $1/\mathscr{r}^4$ will contribute nothing in the limit $\mathscr{r} \to \infty$. For this reason only the *acceleration* fields represent true radiation (hence their other name, **radiation fields**):

$$\mathbf{E}_{\text{rad}} = \frac{q}{4\pi\epsilon_0} \frac{\mathscr{r}}{(\boldsymbol{\mathscr{r}} \cdot \mathbf{u})^3}[\boldsymbol{\mathscr{r}} \times (\mathbf{u} \times \mathbf{a})]. \tag{11.66}$$

[6]Note the subtle change in strategy here: In Sect. 11.1 we worked from a fixed point (the origin), but here it is more appropriate to use the (moving) location of the charge. The implications of this change in perspective will become clearer in a moment.

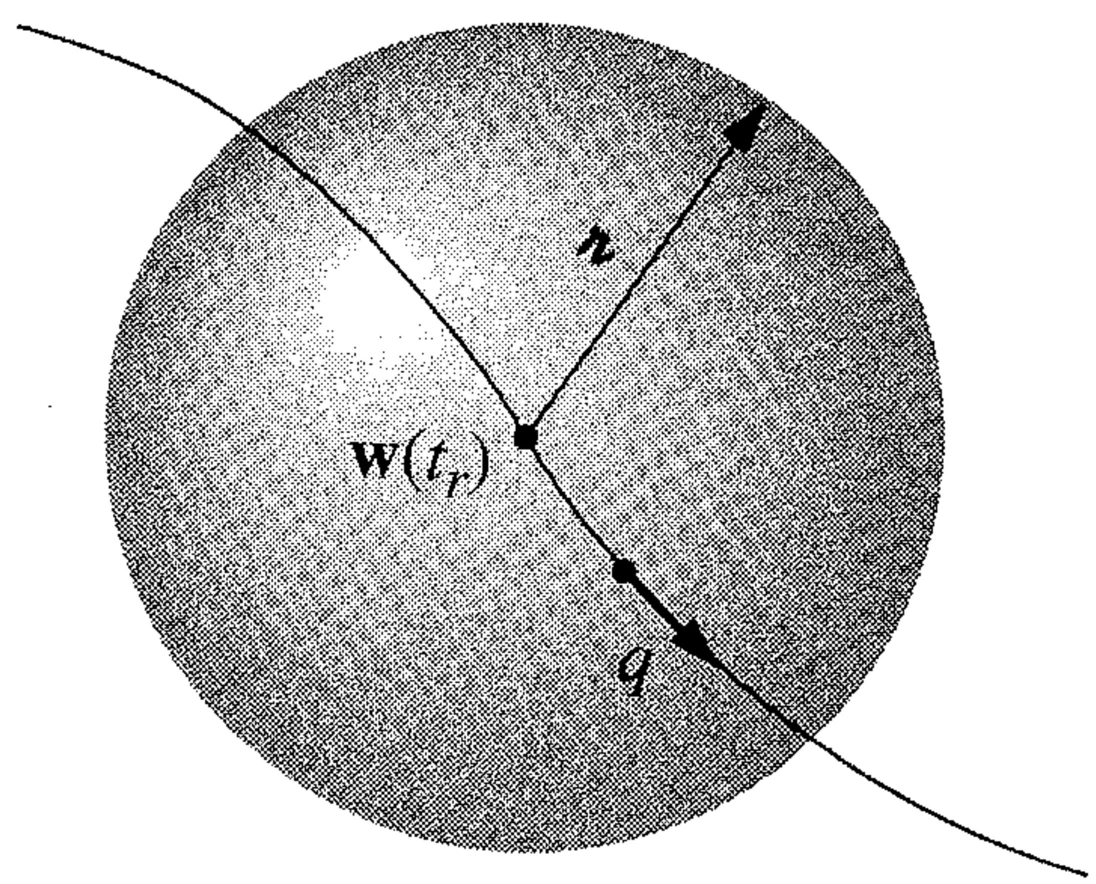

Figure 11.11

The velocity fields carry energy, to be sure, and as the charge moves this energy is dragged along—but it's not *radiation.* (It's like the flies that stay with the garbage truck.) Now $\mathbf{E}_{\text{rad}}$ is perpendicular to $\hat{\mathscr{r}}$, so the second term in Eq. 11.64 vanishes:

$$\mathbf{S}_{\text{rad}} = \frac{1}{\mu_0 c} E_{\text{rad}}^2 \,\hat{\mathscr{r}}. \tag{11.67}$$

If the charge is instantaneously at *rest* (at time t_r), then $\mathbf{u} = c\hat{\mathscr{r}}$, and

$$\mathbf{E}_{\text{rad}} = \frac{q}{4\pi\epsilon_0 c^2 \mathscr{r}}[\hat{\mathscr{r}} \times (\hat{\mathscr{r}} \times \mathbf{a})] = \frac{\mu_0 q}{4\pi \mathscr{r}}[(\hat{\mathscr{r}} \cdot \mathbf{a})\,\hat{\mathscr{r}} - \mathbf{a}]. \tag{11.68}$$

In that case

$$\mathbf{S}_{\text{rad}} = \frac{1}{\mu_0 c}\left(\frac{\mu_0 q}{4\pi \mathscr{r}}\right)^2 [a^2 - (\hat{\mathscr{r}} \cdot \mathbf{a})^2]\,\hat{\mathscr{r}} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\left(\frac{\sin^2\theta}{\mathscr{r}^2}\right)\hat{\mathscr{r}}, \tag{11.69}$$

where θ is the angle between $\hat{\mathscr{r}}$ and $\mathbf{a}$. No power is radiated in the forward or backward direction—rather, it is emitted in a donut about the direction of instantaneous acceleration (Fig. 11.12).

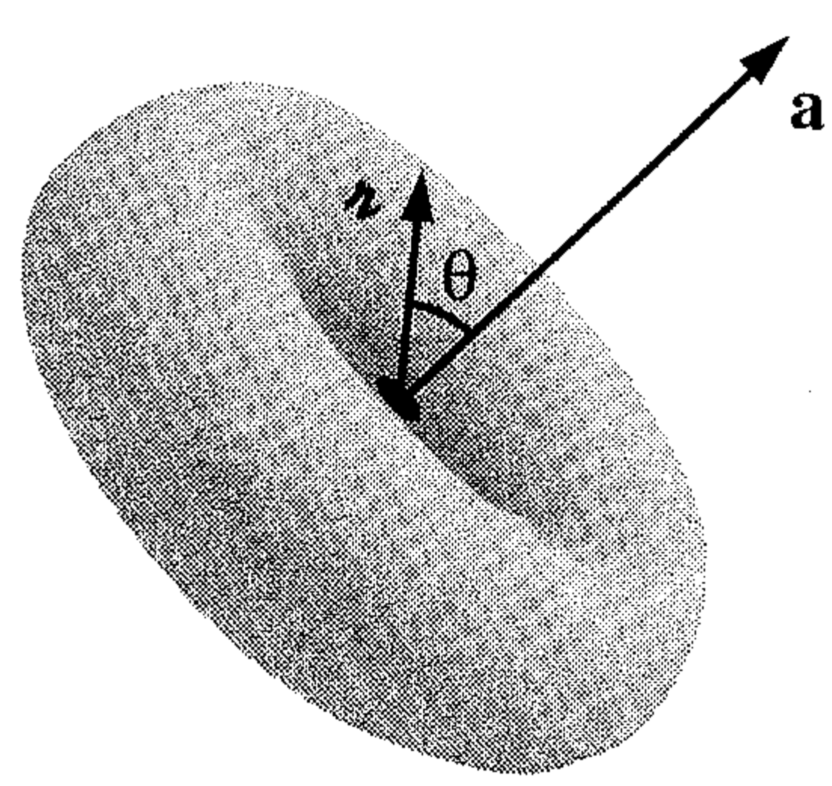

Figure 11.12

The total power radiated is evidently

$$P = \oint \mathbf{S}_{\text{rad}} \cdot d\mathbf{a} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \int \frac{\sin^2\theta}{\mathscr{r}^2} \mathscr{r}^2 \sin\theta \, d\theta \, d\phi,$$

or

$$\boxed{P = \frac{\mu_0 q^2 a^2}{6\pi c}.} \tag{11.70}$$

This, again, is the **Larmor formula**, which we obtained earlier by another route (Eq. 11.61).

Although I *derived* them on the assumption that $v = 0$, Eqs. 11.69 and 11.70 actually hold to good approximation as long as $v \ll c$. An exact treatment of the case $v \neq 0$ is more difficult,[7] both for the obvious reason that $\mathbf{E}_{\text{rad}}$ is more complicated, and also for the more subtle reason that $\mathbf{S}_{\text{rad}}$, the rate at which energy passes through the sphere, is *not* the same as the rate at which energy left the particle. Suppose someone is firing a stream of bullets out the window of a moving car (Fig.11.13). The rate N_t at which the bullets strike a stationary target is not the same as the rate N_g at which they left the gun, because of the motion of the car. In fact, you can easily check that $N_g = (1 - v/c)N_t$, if the car is moving towards the target, and

$$N_g = \left(1 - \frac{\hat{\mathscr{r}} \cdot \mathbf{v}}{c}\right) N_t$$

for arbitrary directions (here $\mathbf{v}$ is the velocity of the car, c is that of the bullets—relative to the ground—and $\hat{\mathscr{r}}$ is a unit vector from car to target). In our case, if dW/dt is the rate at which energy passes through the sphere at radius $\mathscr{r}$, then the rate at which energy left the charge was

$$\frac{dW}{dt_r} = \frac{dW/dt}{\partial t_r / \partial t} = \left(\frac{\boldsymbol{\mathscr{r}} \cdot \mathbf{u}}{\mathscr{r} c}\right) \frac{dW}{dt}. \tag{11.71}$$

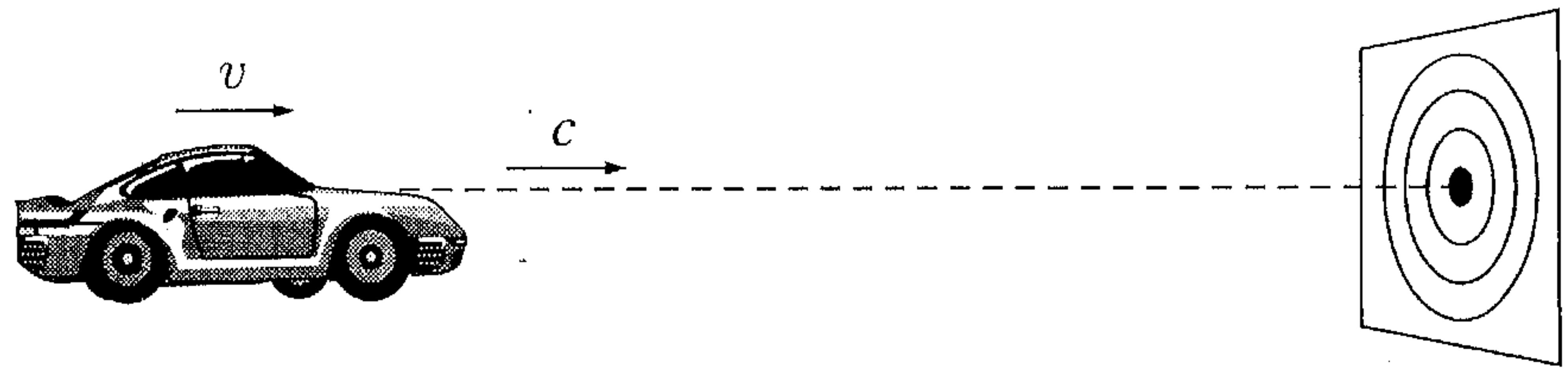

Figure 11.13

[7] In the context of special relativity, the condition $v = 0$ simply represents an astute choice of reference system, with no essential loss of generality. If you can decide how P transforms, you can *deduce* the general (Liénard) result from the $v = 0$ (Larmor) formula (see Prob. 12.69).

(I used Eq. 10.71 to express $\partial t_r/\partial t$.) But

$$\frac{\boldsymbol{\mathscr{r}} \cdot \mathbf{u}}{\mathscr{r} c} = 1 - \frac{\hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{v}}{c},$$

which is precisely the ratio of N_g to N_t; it's a purely geometrical factor (the same as in the Doppler effect).

The power radiated by the particle into a patch of area $\mathscr{r}^2 \sin\theta\, d\theta\, d\phi = \mathscr{r}^2\, d\Omega$ on the sphere is therefore given by

$$\frac{dP}{d\Omega} = \left(\frac{\boldsymbol{\mathscr{r}} \cdot \mathbf{u}}{\mathscr{r} c}\right) \frac{1}{\mu_0 c} E_{\text{rad}}^2 \mathscr{r}^2 = \frac{q^2}{16\pi^2 \epsilon_0} \frac{|\hat{\boldsymbol{\mathscr{r}}} \times (\mathbf{u} \times \mathbf{a})|^2}{(\hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{u})^5}, \tag{11.72}$$

where $d\Omega = \sin\theta\, d\theta\, d\phi$ is the **solid angle** into which this power is radiated. Integrating over θ and ϕ to get the total power radiated is no picnic, and for once I shall simply quote the answer:

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi c}\left(a^2 - \left|\frac{\mathbf{v} \times \mathbf{a}}{c}\right|^2\right), \tag{11.73}$$

where $\gamma \equiv 1/\sqrt{1 - v^2/c^2}$. This is **Liénard's generalization** of the Larmor formula (to which it reduces when $v \ll c$). The factor γ^6 means that the radiated power increases enormously as the particle velocity approaches the speed of light.

Example 11.3

Suppose $\mathbf{v}$ and $\mathbf{a}$ are instantaneously collinear (at time t_r), as, for example, in straight-line motion. Find the angular distribution of the radiation (Eq. 11.72) and the total power emitted.

Solution: In this case $(\mathbf{u} \times \mathbf{a}) = c(\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{a})$, so

$$\frac{dP}{d\Omega} = \frac{q^2 c^2}{16\pi^2 \epsilon_0} \frac{|\hat{\boldsymbol{\mathscr{r}}} \times (\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{a})|^2}{(c - \hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{v})^5}.$$

Now

$$\hat{\boldsymbol{\mathscr{r}}} \times (\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{a}) = (\hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{a})\hat{\boldsymbol{\mathscr{r}}} - \mathbf{a}, \quad \text{so } |\hat{\boldsymbol{\mathscr{r}}} \times (\hat{\boldsymbol{\mathscr{r}}} \times \mathbf{a})|^2 = a^2 - (\hat{\boldsymbol{\mathscr{r}}} \cdot \mathbf{a})^2.$$

In particular, if we let the z axis point along $\mathbf{v}$, then

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \frac{\sin^2\theta}{(1 - \beta\cos\theta)^5}, \tag{11.74}$$

where $\beta \equiv v/c$. This is consistent, of course, with Eq. 11.69, in the case $v = 0$. However, for very large v ($\beta \approx 1$) the donut of radiation (Fig. 11.12) is stretched out and pushed forward by the factor $(1 - \beta\cos\theta)^{-5}$, as indicated in Fig. 11.14. Although there is still no radiation in *precisely* the forward direction, most of it is concentrated within an increasingly narrow cone *about* the forward direction (see Prob. 11.15).

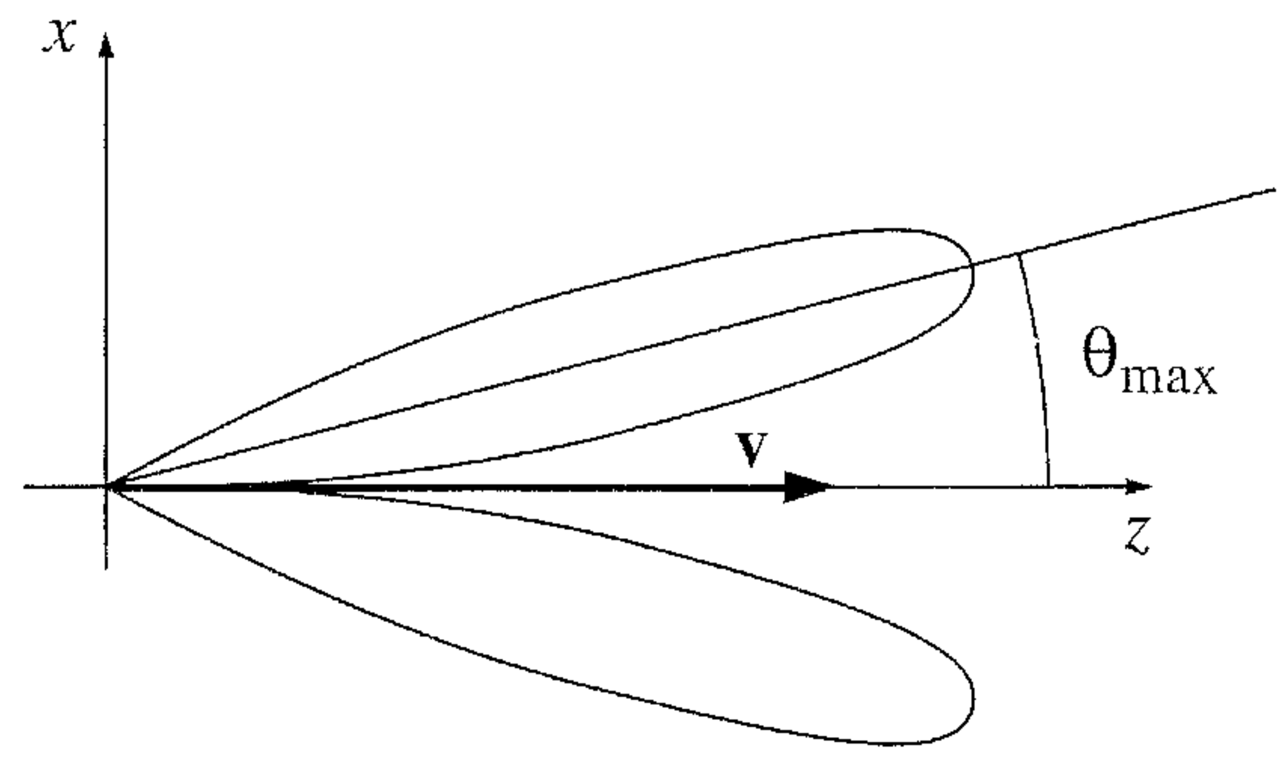

Figure 11.14

The *total* power emitted is found by integrating Eq. 11.74 over all angles:

$$P = \int \frac{dP}{d\Omega}\, d\Omega = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \int \frac{\sin^2\theta}{(1-\beta\cos\theta)^5} \sin\theta \, d\theta \, d\phi.$$

The ϕ integral is 2π; the θ integral is simplified by the substitution $x \equiv \cos\theta$:

$$P = \frac{\mu_0 q^2 a^2}{8\pi c} \int_{-1}^{+1} \frac{(1-x^2)}{(1-\beta x)^5}\, dx.$$

Integration by parts yields $\frac{4}{3}(1-\beta^2)^{-3}$, and I conclude that

$$P = \frac{\mu_0 q^2 a^2 \gamma^6}{6\pi c}. \tag{11.75}$$

This result is consistent with the Liénard formula (Eq. 11.73), for the case of collinear **v** and **a**. Notice that the angular distribution of the radiation is the same whether the particle is *accelerating* or *decelerating*; it only depends on the *square* of a, and is concentrated in the forward direction (with respect to the velocity) in either case. When a high speed electron hits a metal target it rapidly decelerates, giving off what is called **bremsstrahlung**, or "braking radiation." What I have described in this example is essentially the classical theory of bremsstrahlung.

Problem 11.13

(a) Suppose an electron decelerated at a constant rate a from some initial velocity v_0 down to zero. What fraction of its initial kinetic energy is lost to radiation? (The rest is absorbed by whatever mechanism keeps the acceleration constant.) Assume $v_0 \ll c$ so that the Larmor formula can be used.

(b) To get a sense of the numbers involved, suppose the initial velocity is thermal (around 10^5 m/s) and the distance the electron goes is 30 Å. What can you conclude about radiation losses for the electrons in an ordinary conductor?

Problem 11.14 In Bohr's theory of hydrogen, the electron in its ground state was supposed to travel in a circle of radius 5×10^{-11}m, held in orbit by the Coulomb attraction of the proton.