Self-stabilized Fractality of Seacoasts through Erosion

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Let us start with a “classic” …

“How long is the Coastline of Great Britain, Statistical Self-similarity and Fractional Dimension”, B. Mandelbrot, Science, 155, 636 (1967)

This is the start of the fractal history, but … no model for fractal coast formation!
“As a matter of some urgency, researchers concerned with coastal evolution should consider the alternative models, even if there are few supporting data. The ideas of stochastic development, … and criticality, all deserves investigation.”
Some examples

http://www.ngdc.noaa.gov/mgg/shorelines/shorelines.html

Cantabria (Spain)  

North-west Sardinia:
fractal dimension $\sim 1.33$
The Model

- The model wants to describe how a stationary fractal geometry can arise spontaneously from the **mutual self-stabilization** of the coast morphology and the sea eroding power.

- In rocky coasts two main processes: (1) mechanical erosion (**FAST**), (2) chemical weakening and others (**SLOW**).

Erosion dynamics

- Erosion increases the geometry irregularity of the coastline which in its turn creates a stronger damping ➔

⇒ self-stabilization
Modelling Erosion Dynamics

The sea is considered to constitute a resonator dissipating power on the coastal boundaries

\[ P = \text{average excitation power of the waves} \]

\[ F = P \cdot Q = \text{“force” acting on the unit length} \]

\[ Q = \text{morphology dependent quality factor} \]

There are different independent sources of wave damping: bulk + boundary (=coast)

\[
\frac{1}{Q} = \frac{1}{Q_{\text{bulk}}} + \frac{1}{Q_{\text{coast}}}
\]
Bulk dissipation is more important than the one on the coast:

\[ Q_{coast} \gg Q_{bulk} \]

From experimental studies on fractal acoustical cavities one knows that:
Dissipated power \( \sim \) boundary length

\[ Q_{coast}(t) \propto 1/L_p(t) = (\text{coast length})^{-1} \]

Therefore we can write:

\[ F(t) = \frac{P}{1 + g \frac{L_p(t)}{L_0}} \]

With \( g \ll 1 \) giving relative importance of the coast dissipation with respect to the bulk viscous one.
x = random and uncorellated local lithologies $\in [0,1]$

$r(t) = \text{local resistance to erosion} = \text{"lithology"} + \text{"mechanical stability"}$

First nn connectivity sea-coast

$n(t) = \text{contact sides with the sea}$

$$r(t) = \begin{cases} x^n(t) & \text{for } n(t) = 1, 2 \\ 0 & \text{for } n(t) = 3, 4 \end{cases}$$

Coast sites with $r(t) < F(t)$ are eroded and the coastline with $r$, $L_p$ and $F$ are updated.
Erosion dynamics proceeds increasing progressively the roughness of the coast and the number and the typical size of detached islands, then it stops spontaneously when all the coast sites are too hard to be eroded.

At this point the coastline is fractal up to a scale $\sigma$ scaling with $g$, with fractal dimension $D_f = 1.33 \sim 4/3$, and the force $F$ is close to the percolation threshold.
The force per unit length is initially damped rapidly by the increase of coast length, then it slows down in the “critical” percolation regime and finally stops in the critical situation.

$F(t)$ plays the role of a depth depending occupation probability and “g” the role of a spatial gradient of this probability.

Measured fractal dimension is $D_f=1.33\approx 4/3$ which is the fractal dimension of the accessible perimeter of the critical percolation cluster.
Universality and Percolation

The model is in the universality class of Gradient Percolation

Occupation probability:
\[ p(x) = 1 - \frac{x}{L_g} \text{ with } |\nabla p| = \frac{1}{L_g} \]

The separation surface has a depth
\[ \sigma \sim |\nabla p|^{-4/7} = L_g^{4/7} \]

With fractal dimension
\[ D_f = \frac{7}{4} \text{ reduced to } D_f = \frac{4}{3} \text{ by cutting filaments (G.A. effect)} \]

In our model the parametr \( g \) plays the role of \( |\nabla p| \):
\[ F[t(x)] \rightarrow |dF/dx| = (g/L_0) \cdot |dL_p/dx| \sim g \]
In order to mimic the slow chemical weakening, at the end of the fast erosion process, the “lithologies” of the coast sites are decreased

\[ X \rightarrow (1-\varepsilon)X \text{ with } \varepsilon << 1 \]

In this way some sites become weaker than the sea force \( F(t) \) and the dynamics starts again.

Because of \( F \sim p_c \) this creates avalanches of erosion and the geometry remains statistically the same (same \( \sigma \) and \( D_f \)).
Fractality of World Coastlines

From “A Global Self-consistent, Hierarchical, High-resolution Shoreline Database” (P. Wessel):
http://www.soest.hawaii.edu/wessel/gshhs/gshhs.html

Analysis by A. Baldassarri and M. Montuori
The relation between our model and Percolation can be shown also by writing a functional Langevin equation \textit{à la Landau} for an activity field $\rho(x,t)$.

This has been done for a similar model of corrosion of thin films of disordered solids [A. Gabrielli, M. A. Munoz, B. Sapoval, Phys. Rev. E, vol.64, 016108 (2001)].

$s(x,t) = \text{local density of earth that can be still eroded}$

$c(x,t) = \text{local density of eroded earth (occupied by sea)}$

$\rho(x,t) = s(x,t) \cdot c(x,t) = \text{local density of active sites ("active field" of the erosion surface)}$
It is possible to show that one can write a Langevin equation of the type:

$$\partial_t \rho(x, t) = \mu(t)\rho(x, t) - \alpha \rho(x, t) \int_0^t dt' \rho(x, t') + \nabla^2 \rho(x, t) + \sqrt{\rho(x, t)} \eta(x, t)$$

With constant $\mu > 0$ this is the field theory of Dynamical Percolation.

In our case $\mu(t) > 0$ is a F-dependent “mass” decreasing spontaneously to zero.

↓

Self-organized Dynamical Percolation

Spontaneous Absorbing Phase Transition
References

Fractal coast model:

http://greco.phys2.uniroma1.it/twiki/bin/view/Pil/CoastalSystem

Corrosion model and field theory:
A. Gabrielli, A. Baldassarri, and B. Sapoval, Phys. Rev. E, 62, 3103 (2000);

Fractal drums: