ANALYTICAL STUDY
OF THE CRITICAL BEHAVIOUR
OF 3D U(1) LGT AT FINITE TEMPERATURE

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OUTLINE

1. BKT phase transition in spin and gauge models
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3. Overview of the results and problems
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6. Effective vortex model at finite temperatures
7. Conclusions

I. BKT phase transition in spin and gauge models

➜ *V. L. Berezinsky, Sov. Phys. JETP 32 (1971) 493.*

➜ *J. Kosterlitz, D. Thouless, J.Phys. C6 (1973) 1181.*

BKT phase transition takes place in a number of spin and gauge models

- $2D$ spin models: XY, $Z(N > 4)$, Coulomb-gas like systems, SOS model.

- $3D$ gauge theories at finite temperature: $U(1)$ and possibly $Z(N)$ and $SU(N)$ for N large enough.

Characteristic features of this phase transition are

- no spontaneous symmetry breaking occurs
- free energy and all its derivatives are analytic functions of temperature
- correlation length diverges exponentially in the vicinity of critical point
- correlation function decreases with power law at low temperatures

- Difficulties in studying the BKT phase transition

- Phenomenological RG by Kosterlitz and Thouless

- Monte-Carlo simulations

II. Compact $3D\ U(1)$ model on the lattice

$3d\ U(1)$ gauge theory on the anisotropic lattice is defined as

$$Z \equiv Z(\Lambda; \beta_t, \beta_s) = \int_0^{2\pi} \prod_l \frac{d\omega_l}{2\pi} \ \exp S[\omega] \ ,$$

where S is the Wilson action

$$S[\omega] = \beta_s \sum_{p_s} \cos \omega(p_s) + \beta_t \sum_{p_t} \cos \omega(p_t)$$

Z is evaluated on configurations which satisfy periodic BC

$$\omega_n(x) \ = \ \omega_n(x + N_t) \ , \ a_t N_t = \beta = 1/T$$

Dual representation of the anisotropic model

→ *T. Banks, J. Kogut, R. Myerson, Nucl.Phys. B121 (1977) 493.*

$$Z = \sum_{r(x)=-\infty}^{\infty} \prod_{x} \prod_{n=0}^{2} I_{r(x)-r(x+e_n)}(\beta_n) \ ,$$

$I_r(z)$ - the modified Bessel function.

The Villain version of $3D \ U(1)$ in dual representation

$$Z = \sum_{r(x)=-\infty}^{\infty} \exp\left[- \sum_{x} \sum_{n=0}^{2} \frac{1}{2\beta_n} (r(x) - r(x+e_n))^2 \right] .$$

III. Overview of the results and problems

➜ *A. Polyakov, Nucl.Phys. B120 (1977) 429*

➜ *M. Göpfert, G. Mack, Commun.Math.Phys. 81 (1981) 97*

String tension and mass gap in the Euclidean limit $\beta_t = \beta_s = \beta$,

$$a^2\sigma(j=1) \;\geq\; \frac{8}{\sqrt{2\pi^2\beta}}\,\exp\left[-\frac{1}{2}\pi^2\beta G_0\right] \;,$$

$$am \;=\; (8\pi^2\beta)^{1/2}\,\exp\left[-\frac{1}{2}\pi^2\beta G_0\right] \;.$$

$G_0 \approx 0.5054$ is zero-distance Green function.

Theory at finite temperature

➜ *N. Parga, Phys.Lett. B107 (1981) 442*

At high temperatures the system becomes two-dimensional. Monopoles of the $U(1)$ gauge theory become vortices of the $2d$ system. The partition function turns out to coincide (in the leading order of the high-temperature expansion) with the $2d$ XY model in the Villain representation. The effective coupling of the XY model reads

$$\beta_{eff} = 1/(g^2\beta) ,$$

while the effective activity of the vortices is

$$y(\beta) \;=\; \exp\left[-\frac{\pi^2}{2\beta_{eff}}(1+\beta^2/3)\right]$$

The XY model is known to have the Berezinskii-Kosterlitz-Thouless (BKT) phase transition of the infinite order which occurs for the Villain model at

$$\beta_{eff} \approx 2/\pi .$$

Svetitsky-Yaffe conjecture

➜ *B. Svetitsky, L. Yaffe, Nucl.Phys B210 (1982) 423*

- The finite-temperature phase transition in the $3d$ $U(1)$ LGT belongs to the universality class of the $2d$ XY model.

- The global $U(1)$ symmetry cannot be broken spontaneously because of the Mermin-Wagner theorem. There is no local order parameter.

- The correlation function of the Polyakov loops (which become spins of the XY model) decreases with the power law at $\beta \geq \beta_c$

$$P_1(R) \asymp \frac{1}{R^{\eta(T)}} \, , \qquad \eta(\mathbf{T_c}) = \mathbf{1/4}$$

- For $\beta < \beta_c$, $t = \beta_c/\beta - 1$

$$P_1(R) \asymp \exp\left[-R/\xi(t)\right] \, , \; \xi \sim e^{bt^{-\nu}} \, , \; \nu = \mathbf{1/2}$$

Some numerical results

➜ *P. Coddington, A. Hey, A. Middleton, J. Townsend, Phys.Lett. B175 (1986) 64*

Check of the universality conjecture on the lattices $L^2 \times N_t$ with $L = 16, 32$ and $N_t = 4, 6, 8$.

- The varying of η with β and the behaviour of the susceptibility are indicative for BKT type of the phase transition.

- The critical index is almost three times of that predicted for the XY model, $\eta \approx \mathbf{0.78}$.

Problems

- Universality problem. So far there is no numerical indications that critical indices of $3d\,U(1)$ LGT coincides with those of the $2d\,XY$ model. Analytical calculations have been performed with the leading term of the high-temperature expansion.

- Monte-Carlo simulations. In finite-temperature simulations the scaling was not reached. The problem can be in the finite-size effects. In the XY model, due to logarithmic corrections, in order to reliably determine critical indices one should use the FSS technics and/or simulate the model on large thermodynamic lattices, i.e. $L \gg \xi$.

- Construction of the continuum limit at high temperatures. In the zero-temperature model the continuum limit is constructed as a limit $a \to 0$ such to maintain the mass gap constant.

IV. Limiting values of anisotropic couplings

1. The limit $\beta_t = 0$

The model reduces to a product of non-interacting two-dimensional gauge models.

$$Z(\beta_t = 0, \beta_s) \;=\; \left[\sum_{r=-\infty}^{\infty} I_r^{L^2}(\beta_s) \right]^{N_t}$$

The model is in the confined phase at all values of β_s

$$\sigma \;=\; \ln \frac{I_0(\beta_s)}{I_j(\beta_s)} \,.$$

2. The limit $\beta_s = 0$

In this limit the $U(1)$ model reduces to the XY-like model

$$Z(\beta_t, \beta_s = 0) \;=\; \int_0^{2\pi} \prod_x \frac{d\omega_x}{2\pi} \prod_{x,n} \left[\sum_{r=-\infty}^{\infty} \; I_r^{N_t}(\beta_t) \;\exp\left[ir(\omega_x - \omega_{x+e_n}) \right] \right] .$$

$e^{ir\omega_x}$ - the Polyakov loop in the representation r.

For $N_t = 1$ it coincides with the $2d\ XY$ model

$$Z(\beta_t, \beta_s = 0, N_t = 1) \;=\; \int_0^{2\pi} \prod_x \frac{d\omega_x}{2\pi} \;\exp\left[\beta_t \sum_{x,n} \cos(\omega_x - \omega_{x+e_n}) \right]$$

In this case the dynamics of the system is governed by the XY model with the inverse temperature β_t. For $N_t \geq 2$ the model is of the XY-type, i.e. it describes interaction between nearest neighbours spins (Polyakov loops) and possesses the global $U(1)$ symmetry.

3. BKT phase transition at $\beta_t \gg 1$

$$Z(\beta_t \gg 1, \beta_s = 0) \; = \sum_{r(x)=-\infty}^{\infty} \exp\left[-\frac{1}{2}\tilde{\beta} \sum_x \sum_{n=1}^{2} (r(x) - r(x+e_n))^2 \right] .$$

This is the Villain version of the XY model in the dual formulation with an effective coupling $\tilde{\beta}$ given by

$$\tilde{\beta} \; = \; N_t/\beta_t \; = \; g^2/T \; .$$

This shows that the region $\beta_s = 0$, $\beta_t \gg 1$ is described by the XY model, in particular

$$\beta_{cr} \; = \; 2/(\pi g^2) \; , \;\; \eta(T_c) \; = \; 1/4$$

V. Perturbative calculations at high temperatures

1. Correlation function of the Polyakov loops

Two-point correlation function in the XY model

$$\Gamma_{XY}(R) \;=\; 1 \;-\; \frac{g^2}{2}\,D(R) \;+\; \frac{g^4}{8}\,D(R)\,[D(R)-1]$$

$$D(R) = G_0 - G_R \asymp (1/\pi)\ln \mid R \mid$$

Perturbative β function of the XY model vanishes at weak coupling

$$\beta(g) \;=\; 0$$

The Polyakov loop correlation

$$P_j(C) \;=\; 1 - g^2\,C_1 + g^4\,C_2 \;+\; \mathcal{O}(g^6)$$

$$C_1 \;=\; \frac{1}{2}\,j^2\,\beta\,D(R)\,,$$

$$C_2 \;=\; \frac{1}{8}\,j^4\,\beta^2\,D^2(R) \;-\; \frac{1}{4}\,j^2\,a_t\beta\,D(R)\,\left(1-\beta_t^{-1}D_{n_1}\right)$$

From last formulae one can extract the potential between test charges

$$V_j(R) \;=\; -\frac{1}{\beta}\,\ln P_j(R) \;=\; \frac{1}{2}\,g^2 j^2\,\left[1+\frac{1}{2\beta_t}(1-\beta_t^{-1}D_{n_1})\right]\,D(R)\;.$$

The perturbative coefficients of the Polyakov loop correlations behave qualitatively and quantitatively similar to those of the two-point correlation function of the $2d$ XY model.

2. ’t Hooft line

$$D_\pi(R) \;=\; \exp\left[-\frac{1}{2}B(R)\right]$$

At zero temperature and for $R = (x_0^2 + x_1^2 + x_2^2)^{1/2} \gg 1$

$$B(R) \; = \; G_0/(g^2 a) - \frac{1}{2\pi g^2 a R}$$

At high temperature and for $R = (x_1^2 + x_2^2)^{1/2} \gg 1$, $\tau = 0$

$$B(R, 0) \; = \; \frac{1}{\pi g^2 \beta} \; \ln R$$

At high temperature and for $R = 0$, $\tau = a_t x_0$

$$B(0, \tau) \; = \; \frac{\beta}{g^2 a_s^2} \left(\frac{\tau}{\beta} \right)$$

VI. Effective vortex model at finite temperatures

The effective monopole action at finite temperatures and in the presence of sources for the Wilson and 't Hooft loops

$$Z_{\eta s} \;=\; Z_{\text{sw}}(\eta, s)\; Z_{\text{mon}}(\eta, s)\;.$$

Effective monopole action

$$S_{\text{mon}} \;=\; -\pi^2 \sum_{x,x'} m(x) G_{xx'} m(x') - \pi \sum_x h(x) m(x)$$

To derive effective vortex model at high temperatures one uses the asymptotics of Green function at $\beta \to 0$

$$G_x \;=\; \frac{1}{g^2\beta} G_x^{2d} + \frac{\beta}{g^2 a_s^2} B_2(\tau/\beta)\delta_{x,0} + \frac{\beta^3}{6g^2 a_s^4} B_4(\tau/\beta)\Delta_x + \mathcal{O}(\beta^5)\;,$$

where $\tau = a_t x_0$, G_x^{2d} is the Green function of the $2d$ model, $B_n(z)$ are the Bernoulli polynomials and Δ_x is the $2d$ Laplace operator

$$\Delta_x \;=\; \sum_{n=1}^{2} \left[\delta_{x,0} - \frac{1}{2}(\delta_{x+e_n,0} + \delta_{x-e_n,0})\right]$$

Effective $2d$ vortex model takes the form, $x = (x_1, x_2)$

$$Z_{\text{vor}}(\eta, s) = \sum_{m(x)=-\infty}^{\infty} \delta\left[\sum_x m(x)\right]$$
$$\exp\left[-\frac{\pi^2}{g^2\beta}\sum_{x,x'} m(x) G^{2d}_{xx'} m(x') + \kappa \ln W[m(x)] - \pi \sum_x h(x) m(x)\right]$$

The weight is given by

$$W[m(x)] = \sum_{m(x,t)=-\infty}^{\infty} \delta\left[m(x) - \sum_{t=0}^{N_t-1} m(x,t)\right]$$
$$\exp\left[-\pi^2 \sum_{x,x'}\sum_{t,t'} m(x,t)(G_{xx',tt'} - \frac{1}{g^2\beta} G^{2d}_{xx'}) m(x',t')\right]$$

$\kappa = 0$ corresponds to $2d$ XY model; $\kappa = 1$ corresponds to $3d$ $U(1)$ gauge model at finite temperature

In the leading order $\tau/\beta \ll 1$ effective $2d$ vortex model takes the form at zero sources, $x = (x_1, x_2)$

$$Z_{\text{vor}} = \sum_{m(x)=-\infty}^{\infty} \delta\left[\sum_x m(x)\right]$$
$$\exp\left[-\frac{\pi^2}{g^2\beta}\sum_{x,x'} m(x) G_{xx'} m(x') - \kappa_0 \sum_x m^2(x) - \kappa_1 m(x) \Delta_{xx'} m(x')\right]$$

$$\kappa_0 = \kappa\,\frac{\pi^2\beta}{6g^2a_s^2}$$

$$\kappa_1 = \kappa\,\frac{\pi^2\beta^3}{180g^2a_s^4}$$

The vortex model can be mapped onto the model of the sine-Gordon type

$$Z_{\text{vor}} = \int \prod_x d\alpha_x \ \exp\left[-\sum_{x,x'} \alpha_x B_{xx'} \alpha_{x'} + y \sum_x \cos \alpha_x\right]$$

$$B_{xx'} = \frac{g^2\beta}{4\pi} \Delta_{xx'} + \kappa \frac{g^2\beta^5}{720 a_s^4} \Delta_{xy} \Delta_{yy'} \Delta_{y'x'}$$

$$y = 2 \exp\left[-\frac{\gamma\pi^2}{g^2\beta} + \kappa \, \frac{\pi^2\beta}{6 g^2 a_s^2}\right]$$

Sine-Gordon model can be analyzed by the conventional RG methods where terms proportional to κ are treated perturbatively.

Such calculations give XY critical indices $\eta(T_c) = 1/4$ and $\nu = 1/2$.

VII. Conclusions

- At finite temperature a deconfinement phase transition takes place to a phase where the potential between test charges grows logarithmically
- In the limit $\beta_s = 0$ this is BKT phase transition which belongs to the XY model universality class
- At high temperatures Polyakov loop correlation and 't Hooft line behave qualitatively and quantitatively similar to 2-point function and disorder operator of $2d$ XY model correspondingly
- Effective vortex model predicts critical behaviour similar to that of XY model but this has to be confirmed numerically.
- MC simulations: next talk by M. Gravina